General Mathematics Vol. 12, No. 4 (2004), 49–56

Inequalities concerning starlike functions and their n-th root

Maria Pettineo

Abstract

If A is the class of all analytic functions in the complex unit disc Δ, of the form:

$$f(z) = z + a_2 z^2 + \cdots$$

and if $f \in A$ satisfies in Δ the condition:

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|$$

then $\mathrm{Re}\,\sqrt[n]{f(z)/z} \geq (n+1)/(n+2)$. We show also that if f is starlike in Δ (**i.e.** $\mathrm{Re}\,zf'(z)/f(z) > 0$ in Δ), then $\mathrm{Re}\,\sqrt[n]{f(z)/z} > n/(n+2)$.

Mathematical Subject Classification: Primary: 30C45

Keywords: starlike function, uniformly starlike function, uniformly convex function.

1 Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and let A be the set of all analytic functions in Δ, having the power series development:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

The subclass **ST** of A consists of functions f which satisfy in Δ the condition:

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \quad z \in \Delta$$

ST is named the class of **starlike functions** in A

The subclass **CV** of **ST**(named the class of **convex functions**in A) consists of functions $f \in A$ which satisfy in Δ the condition:

$$\mathrm{Re}\left[1 + \frac{zf'(z)}{f'(z)}\right] > 0 \;\; z \in \Delta$$

We denote also by **QUST**(**quasi-uniformly starlike functions**) the subclass of **ST** which contains the functions satisfying the condition:

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|$$

and by **UCV** (**uniformly convex functions**) the subclass of **CV** which contains functions satisfying the condition:

$$\text{(1)} \qquad \mathrm{Re}[1 + \frac{zf'(z)}{f'(z)}] \geq \left|\frac{zf'(z)}{f'(z)}\right|, \; z \in \Delta\}$$

We mention here that the class **UCV** was introduced (together with the class **UST** of **uniformly starlike functions** in A) by **A.W.Goodman** ([2], [3]) who defined the uniformly starlike and convex functions as functions in A with the property that the image of every circular arc contained in Δ, having the center $\zeta \in \Delta$ is starlike with respect to ζ(respectively convex). These properties are expressed by using two complex variables. In 1993, **Frode Ronning** shoved (in [8]) that $f \in$ **UCV** if and only if relation (1) holds. By using the theory of differential subordinations (**S.S. Miller** and **P.T. Mocanu**, [6], [7]), **A. Mannino** showed in 2004 ([4]) that every function $f \in$ **QUST** satisfies the property:

$$\mathrm{Re}\sqrt{\frac{f(z)}{z}} \geq \frac{2}{3} \quad in\ \Delta$$

(the root is considered with the principal determination). The purpose of this paper is to generalize this last result, together with the former result of **A. Marx** ([5])(with states that the principal determination of the square root of $f' \in A$ is greater than $1/2$ if f is convex in Δ). We will show that:

$$f \in \mathrm{C}\mathbf{V} \ \ implies \ \ \mathrm{Re}\, \sqrt[n]{f'(z)} \geq \frac{n}{n+2}$$

and

$$f \in \mathrm{Q}\mathbf{UST} \ \ implies \ \ \mathrm{Re}\, \sqrt[n]{\frac{f(z)}{z}} \geq \frac{n}{n+1}.$$

2 Preliminaries

For proving our principal result we will need the following definitions and results:

Definition 1. *A function is said to be in the class* $\mathbf{ST}(\alpha)$ *if and only if* f *is in* A *and* $\mathrm{Re}\, zf'(z)/z > \alpha$ *in* Δ.

Lemma 1. [1] *A function* $f \in A$ *is convex in* Δ *if and only if the function* $zf'(z)$ *is starlike in* Δ

Lemma 1 is well-known as "Alexanderş duality theorem" and has a very simple proof based on the characterization of starlike and convex functions in the unit disc.

Lemma 2. [6] *Let* a *be a complex number with* $\mathrm{Re}\, a > 0$ *and let* $\psi : \mathbb{C} \times \Delta \longrightarrow \mathbb{C}$ *a function satisfying:*

$$\mathrm{Re}\, \psi(\mathrm{i}x, y; z) \leq 0 \ \ in\ \Delta \ \ and \ \ for\ all\ x\ and\ y, \ \ with \ \ y \leq -\frac{|a - \mathrm{i}x|^2}{2\,\mathrm{Re}\, a}.$$

If

$$p(z) = a + p_1 z + p_2 z^2 + \cdots \ \ is\ analytic\ in\ \Delta, \ \ then:$$

$$[\mathrm{Re}\, \psi(p(z), zp'(z); z) > 0 \ \ for\ all\ z \in \Delta] \ \ implies \ \ \mathrm{Re}\, p(z) > 0 \ \ in\ \Delta.$$

Proofs of more general forms of **Lemma 2** can be found in [6] and in [7].

3 Main result

Theorem 1. *If* $f \in \mathbf{ST}$ *then:*

$$\operatorname{Re} \sqrt[n]{\frac{f(z)}{z}} > \frac{n}{n+2}$$

where the determination of the n-th root is the principal one.

Proof. Let

$$p(z) = \sqrt[n]{\frac{f(z)}{z}} - \frac{n}{n+2}$$

We have that $p(0) = 2/(n+2) > 0$ and: $f(z)/z = [p(z) = n/(n+2)]^n$. Thus:

$$\frac{zf'(z)}{f(z)} = 1 + n\frac{zp'(z)}{p(z) + \frac{n}{n+2}}$$

Denote by $\psi : \mathbb{C} \times \Delta \to \mathbb{C}$, $\psi(\alpha, \beta; z) = 1 + n\frac{\alpha}{\beta + n/(n+2)}$. Because $f \in \mathbf{ST}$, we have that $\operatorname{Re} \psi[p(z), zp'(z); z] > 0$ in Δ. In order to prove that this relation implies that $\operatorname{Re} p(z) > 0$ in Δ, we will use **Lemma 2** with $a = 2/(n+2)$.

$$\operatorname{Re} \psi(\mathrm{i}x, y; z) = \operatorname{Re}\left[1 + n\frac{y}{\mathrm{i}x + \frac{n}{n+2}}\right] = 1 + \frac{n^2(n+2)y}{n^2 + (n+2)^2x^2}$$

If

$$y \leq -\frac{|\operatorname{Re} p(0) - \mathrm{i}x|}{2\operatorname{Re} p(0)} = -\frac{4 + (n+2)^2x^2}{4(n+2)}$$

we have:

$$\operatorname{Re} \psi(\mathrm{i}x, y; z) \leq 1 - n^2\frac{4 + (n+2)^2x^2}{4[n^2 + (n+2)^2x^2]}$$

But:

$$\frac{4 + (n+2)^2x^2}{n^2 + (n+2)^2x^2} \geq \frac{4}{n^2}$$

because the minimum of the real function $g : [0, \infty) \to \mathbb{R}$

$$g(t) = \frac{4 + (n+2)^2t}{n^2 + (n+2)^2t}$$

is $4/n^2$. It follows that $\operatorname{Re}\psi(\mathrm{i}x, y; z) \leq 0$ for all real x and $y \leq -\frac{|\operatorname{Re} p(0)-\mathrm{i}x|}{2\operatorname{Re} p(0)}$. By **Lemma 2** we have that $\operatorname{Re}\psi[p(z), zp'(z); z] > 0$ in Δ implies $\operatorname{Re} p(z) > 0$ in Δ, which is equivalent to: $f \in \mathbf{ST}$ implies that $\operatorname{Re}\sqrt[n]{\frac{f(z)}{z}} > n/(n+2)$ in Δ and the theorem is proved.

Theorem 2. *Let* $f \in \mathbf{QUST}$. *Then we have:*

$$\operatorname{Re}\sqrt[n]{\frac{f(z)}{z}} > \frac{n}{n+1} \quad in \ \ \Delta$$

where the n-th root is considered with the principal determination.

Proof. Let $f \in \mathbf{QUST}$. we put, like in **Theorem 1**,

$$p(z) = \sqrt[n]{\frac{f(z)}{z}} - \frac{n}{n+1}$$

It follows easily that $p(0) = 1/(n+1) > 0$ and:

$$\frac{zf'(z)}{f(z)} = 1 + n(n+1)\frac{zp'(z)}{n + (n+1)p(z)}$$

Let:

$$\psi(\alpha, \beta; z) = 1 + n(n+1)\frac{\beta}{n + (n+1)\alpha} - \left| n(n+1)\frac{\beta}{n + (n+1)\alpha} \right|$$

$f \in \mathbf{QUST}$ is equivalent to:

$$\operatorname{Re}\psi[p(z), zp'(z); z] \geq 0 \ \ \text{in} \ \ \delta$$

We will apply **Lemma 2** for proving that this relation implies $\operatorname{Re} p(z) > 0$ in Δ, which is equivalent to: $\operatorname{Re}\sqrt[n]{f(z)/z} > n/(n+1)$. In order to apply **Lemma 2** we have to compute $\operatorname{Re}\psi(\mathrm{i}x, y; z)$ and to show that this number is less or equal to zero for all real x and

(2) $$y \leq -\frac{|\operatorname{Re} p(0) - \mathrm{i}x|}{2\operatorname{Re} p(0)} = -\frac{1 + (n+1)^2x^2}{2(n+1)}$$

A simple calculation shows that:

$$\operatorname{Re}\psi(\mathrm{i}x,y;z)=1+n^2(n+1)\frac{y}{n^2+(n+1)^2x^2}-$$

$$-n(n+1)\frac{|y|}{\sqrt{n^2+(n+1)^2x^2}}$$

From (2) we have that $y=|y|\leq-\frac{1+(n+1)^2x^2}{2(n+1)}$ and thus:

$$\operatorname{Re}\psi(\mathrm{i}x,y;z)\leq 1-n(n+1)\frac{1+(n+1)^2x^2}{2(n+1)[n^2+(n+1)^2x^2]}- \tag{3}$$

$$-n(n+1)\frac{1+(n+1)^2x^2}{2(n+1)\sqrt{n^2+(n+1)^2x^2}}$$

Let $g_1,\ g_2:[0,\infty)\to\mathbb{R}$ given by:

$$g_1(t)=\frac{n^2}{2}\frac{1+(n+1)^2t}{n^2+(n+1)^2t}$$

$$g_2(t)=\frac{n}{2}\frac{1+(n+1)^2t}{\sqrt{n^2+(n+1)^2t}}$$

From (3) it is easy to see that $\operatorname{Re}\psi(\mathrm{i}x,y;z))\leq 1-g_1(x^2)-g_2(x^2)$. But g_1 and g_2 are increasing functions on $[0,\infty)$ and thus, $g_1(x^2)\geq g_1(0)=1/2$ and $g_2(x^2)\geq g(0)=1/2$ for all real x. It follows that

$$\operatorname{Re}\psi(\mathrm{i}x,y;z)\leq 1-1/2-1/2=0$$

and the theorem is proved by applying **Lemma 2**.

4 A particular case

If we consider in **Theorem 1** and in **Theorem 2**, $f(z)=zg'(z)$, then the starlikeness of f is equivalent (by **Lemma 1**) with the convexity of g and a simple calculation shows also that $f\in$**QUST** if and only if $g\in$**UCV**. We can then apply **Theorem 1** and **Theorem 2** to the function $zg'(z)$ and obtain the following result:

Corolary 1. *If* $g \in \mathbf{CV}$*, then we have:*

$$\mathrm{Re}\,\sqrt[n]{g'(z)} \geq \frac{n}{n+2}$$

and if $g \in \mathbf{UCV}$ *we have*

$$\mathrm{Re}\,\sqrt[n]{g'(z)} \geq \frac{n}{n+1}$$

where the n-th roots are considered with their principal determinations.

References

[1] J.W. Alexander, *Functions that map the interior of the unit circle upon simple regions*, Ann. of Math., **17**(1915), pp. 12-22.

[2] A.W. Goodman, *On uniformly starlike functions*, J.Math. Anal. Appl. **155**) (1991), pp. 364-370.

[3] A.W. Goodman, *On uniformly convex functions*, Annales Polonici Mathematici **LVI.1**) (1991), pp. 88-93.

[4] A. Mannino, *Some inequalities concerning starlike and convex functions*, General Mathematics **vol. 12 no. 1** (2004), Sibiu, pp. 5-12.

[5] A. Marx, *Untersuchungen űber schlichte Abbildungen*, Math. Ann. **107** (1932/33), pp. 40-67.

[6] S.S.Miller, P.T.Mocanu, *Differential subordinations and univalent functions*, Michigan Math. Journal **28**, (1981), pp. 151-171.

[7] S.S.Miller, P.T.Mocanu, *The theory and applications of second order differential subordinations*, Studia Univ. Babeş–Bolyai, Math. **34**, 4(1989),3–33.

[8] Frode Ronning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), 189-196.

[9] Frode Ronning, *Some radius results for univalent functions*, J. Math. Anal. Appl. **194**, 1995), pp. 319-327.

Maria Pettineo
Dipartimento di Matematica e Applicazioni
Via Archirafi n^o 34
90123 Palermo, Italia.