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Conjugacy in Hyper Semi-dynamical Systems

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Abstract

The notion of conjugacy on hyper semi-dynamical systems is studied from algebraic and topological points of views.
Topological join operations for a new approach to topological hypergroups are considered. By using topological join operations, topological conjugacy are studied. Hyper orbits and limit points as two invariant sets under conjugacy are deduced.

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1 Introduction

Theory of hyper-group as a generalization of group theory presented new mathematical research branches [1, 2]. As a result of this theory the notion of generalized dynamical systems deduced in [3].
In fact a generalized dynamical system is a triple (M, D, H), where $(H, \cdot)$ is a hyper-group, M is a non-empty set and D is a set of mappings $h^a : M \to M$

where $a \in H$ with the following property:
If $a, b \in H$, and $m \in M$, then $h^a o h^b(m) \in h^{ab}(m)$, where $h^{ab}(m) = \{h^u(m) : u \in ab\}$.

If the image of join operation contains only singletons, then (M, D, H), will be a semi-dynamical systems [3]. So for more stress on the manner of generalization we rename (M, D, H) by hyper semi-dynamical system.

Definition 1.1 Two hyper semi-dynamical systems $(M, \{h^a\}, H)$ and $(\tilde{M}, \{f^b\}, \tilde{H})$ are called conjugate hyper semi-dynamical systems if there exist one to one and onto maps $\psi : M \to \tilde{M}$ and $\phi : H \to \tilde{H}$ such that the following two axioms hold.
(i) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in H$;
(ii) $\psi o h^a = f^{\phi(a)} o \psi$ for all $a \in H$ (see diagram 1.)

$$\text{(1)} \qquad \begin{array}{ccc} M & \stackrel{\psi}{\to} & \tilde{M} \\ h^a \downarrow & & \downarrow f^{\phi(a)} \\ M & \stackrel{\psi}{\to} & \tilde{M} \end{array}$$

In the next theorem we show that the conjugate relation is an equivalence relation on hyper semi-dynamical systems.

Theorem 1.1 Let (ψ, ϕ) be a conjugate relation between $(M, \{h^a\}, H)$ and $(\tilde{M}, \{f^b\}, \tilde{H})$, and let $(\tilde{\psi}, \tilde{\phi})$ be a conjugate relation between $(\tilde{M}, \{f^b\}, \tilde{H})$ and $(\hat{M}, \{g^c\}, \hat{H})$. Then
(i) (ψ^{-1}, ϕ^{-1}) is a conjugate relation between $(\tilde{M}, \{f^b\}, \tilde{H})$ and $(M, \{h^a\}, H)$.
(ii) $(\tilde{\psi} o \psi, \tilde{\phi} o \phi)$ is a conjugate relation between $(M, \{h^a\}, H)$ and $(\hat{M}, \{g^c\}, \hat{H})$.

Proof. (i) If $\tilde{a}, \tilde{b} \in \tilde{H}$, then $\phi^{-1}(\tilde{a}\tilde{b}) = \phi^{-1}(\phi(ab)) = ab = \phi^{-1}(\phi(a))\phi^{-1}(\phi(b)) = \phi^{-1}(\tilde{a})\phi^{-1}(\tilde{b})$.
For $\tilde{a} \in \tilde{H}$ we have
$\psi o h^{\phi^{-1}(\tilde{a})} o \psi^{-1} = f^{\phi(\phi^{-1}(\tilde{a}))} o \psi o \psi^{-1} = f^{\tilde{a}}$. So $h^{\phi^{-1}(\tilde{a})} o \psi^{-1} = \psi^{-1} o f^{\tilde{a}}$.
(ii) For $a \in H$ we have

$g^{(\tilde{\phi} o \phi)(a)} o(\tilde{\psi} o \psi) = (\tilde{\psi} o f^{\phi(a)}) o \psi = \tilde{\psi} o(\psi o h^a) = (\tilde{\psi} o \psi) o h^a.\Box$

Hyper semi-dynamical systems creates a method for constructing new hyper groups. In fact for $m \in M$ the hyper-orbit of m which is the set $O^H(m) = \{h^a(m) : a \in H\}$ is a hyper group [3]. The next theorem implies that the conjugate relation preserves hyper-orbits.

Theorem 1.2 If $(M, \{h^a\}, H)$ and $(\tilde{M}, \{f^b\}, \tilde{H})$ are hyper conjugate under (ψ, ϕ), then $\psi(O^H(m)) = O^{\tilde{H}}(\psi(m))$.

Proof. If $y \in \psi(O^H(m))$, then

$$y = \psi(h^a(m)) = (f^{\phi(a)} o \psi)(m) = f^{\phi(a)}(\psi(m)) \in O^{\tilde{H}}(\psi(m)).$$

Hence $\psi(O^H(m)) \subseteq O^{\tilde{H}}(\psi(m))$.

Since conjugate relation is an equivalence relation then the first part of the proof shows that $\psi^{-1}(O^{\tilde{H}}(\psi(m))) \subseteq O^H(m)$. Thus $O^{\tilde{H}}(\psi(m)) \subseteq \psi(O^H(m)).\Box$

2 Continuity in Hyper Semi-dynamical Systems

In this section we assume that H is a Hausdorff topological space and $*$ is a join operation on H.

Definition 2.1 $*$ is called a continuous join operation if for given $a, b \in H$, and for all open set W with $W \cap a * b \neq \emptyset$ there exist open neighborhoods U of a and V of b and a subset $Z \subseteq U * V$ such that

i) $W \cap Z \neq \emptyset$, and;

ii) $c * d \cap Z \neq \emptyset$ for all $c \in U$ and $d \in V$.

We ask the reader to pay attention to this point that: if the image of join operation contains only singletons, then the condition (ii) implies that $Z = U * V$, and we will have the definition of topological semigroups. So Definition 2.1 is an extension of the topological semigroups.

With a continuous join operation we can define a topology on $P_*(H)$.
In fact we say that U is open in $P_*(H)$ if $*^{-1}(U)$ is open in $H \times H$ where the topology of $H \times H$ is the product topology.

Example 2.1 let H be the two sphere S^2, and let $\star$ be defined by
$a * b =$ (meridian passing through a) $\cup$(cirquit passing through b)
is a continuous join operation.

Definition 2.2 A hyper semi-dynamical system (M, D, H) is called a topological hyper semi-dynamical system if
i) H is a Hausdorff topological space and the join operation of H is a continuous join operation;
ii) M is a topological space
iv) The members of D are homeomorphims.

If H and I are two hyper group, then $H \times I$ with the join operation $(h, i)(f, g) = (hf) \times (ig)$ is hyper-group.

Theorem 2.1 If (M, D, H) and (N, E, I) are two topological hyper semi-dynamical systems, then $(M \times N, F, H \times I)$ is a topological hyper semi-dynamical system, where
$F = \{g^{(a,b)} : M \times N \rightarrow M \times N : (a, b) \in H \times I$ and $g^{(a,b)}(m, n) = (h^a(m), f^b(n))\}$

Proof. In theorem 3.1 of [3] proved that $(M \times N, F, H \times I)$ is a hyper semi-dynamical system.
Now we show that the join operation of $H \times I$ is a continuous one. Let $(h, i), (f, g) \in H \times I$ be given. Moreover let $W = W_1 \times W_2$ be an open set where $W \cap (h, i)(f, g) \neq \emptyset$ and W_1 and W_2 are open sets in H and I respectively. Then $W_1 \cap hf \neq \emptyset$ and $W_2 \cap ig \neq \emptyset$. So there exist open neighborhoods U_1 of h, V_1 of i, U_2 of f and V_2 of g and the sets $Z_1 \subseteq U_1V_1$ and $Z_2 \subseteq U_2V_2$ such that $W_1 \cap Z_1 \neq \emptyset$, and $c_1d_1 \cap Z_1 \neq \emptyset$ for all $c_1 \in U_1$ and $d_1 \in V_1$ Moreover $W_2 \cap Z_2 \neq \emptyset$ and $c_2d_2 \cap Z_2 \neq \emptyset$ for all $c_2 \in U_2$ and $d_2 \in V_2$.

Thus if $Z = Z_1 \times Z_2$, $U = U_1 \times U_2$ and $V = V_1 \times V_2$, then $(h, i) \in U$, $(f, g) \in V$, $Z \subseteq UV$, $W \cap Z \neq \emptyset$ and $cd \cap Z \neq \emptyset$ for all $c \in U$, and $d \in V$. Hence the join operation of $H \times I$ is a continuous one.
If $g^{(a,b)} \in F$, then since the components of $g^{(a,b)}$ are homeomorphisms, then $g^{(a,b)}$ is a homeomorphisms of the topological space $M \times M$. Moreover the Hausdorff property protect under product topology. So $H \times I$ is also a Hausdorff topological space. Thus the proof is complete.□

3 Topological Conjugacy

We assume that $(M, \{h^a\}, H)$ and $(\tilde{M}, \{f^b\}, \tilde{H})$ are two conjugate topological hyper semi-dynamical systems under conjugate relation (ψ, φ). Then we say that these two hyper semi-dynamical systems are topological conjugate if ψ and φ are two homeomorphisms. In this case (ψ, φ) is called a topological conjugacy.

Definition 3.1 Let $(M, \{h^a\}, H)$ be a topological hyper semi-dynamical system and let $m \in M$. Then the limit set of m is the set of limit points of $O^H(m)$ and denoted by $\Lambda^H(m)$.

Theorem 3.1 Let (ψ, φ) be a topological conjugacy between two topological hyper semi-dynamical system $(M, \{h^a\}, H)$ and $(\tilde{M}, \{f^b\}, \tilde{H})$.
Then $\psi(\Lambda^H(m)) = \tilde{\Lambda}^{\tilde{H}}(\psi(m))$, where $m \in M$.

Proof. If $q \in \psi(\Lambda^H(m))$, then $q = \psi(p)$ where $p \in \Lambda^H(m)$. Hence p is a limit point of $O^H(m)$. Since ψ is a homeomorphism then $q = \psi(p)$ is a limit point of $\psi(O^H(m))$. So theorem 1.2 implies that q is a limit point of $O^{\tilde{H}}(\psi(m))$. Hence $q \in \tilde{\Lambda}^{\tilde{H}}(\psi(m))$. Thus $\psi(\Lambda^H(m)) \subseteq \tilde{\Lambda}^{\tilde{H}}(\psi(m))$.
Similarly by replacing ψ with ψ^{-1} we have $\psi^{-1}(\tilde{\Lambda}^{\tilde{H}}(\psi(m)) \subseteq \Lambda^H(m)$. So $\tilde{\Lambda}^{\tilde{H}}(\psi(m)) \subseteq \psi(\Lambda^H(m))$. Therefore $\tilde{\Lambda}^{\tilde{H}}(\psi(m)) = \psi(\Lambda^H(m))$.□

Conclusion

Beside the results of this paper we have also prepared the means for approaching to stability of hyper semi-dynamical system which is a new research topic.

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