General Mathematics Vol. 12, No. 1 (2004), 13–22

Connected topological generalized groups

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Abstract

In this paper, connected topological generalized groups are studied. We are going to show that: topological generalized groups with e-generalized subgroups are connected topological generalized groups. Connected factor spaces and stable connected component under identity are considered.

2000 Mathematical Subject Classification: 22A15, 22A20

Key Words: Semigroup admitting relative inverse, Topological generalized group, Completely simple semigroup.

1 Introduction

Generalized groups as an algebraic structure were deduced from a geometrical problem in 1998 ([8]). We recall that a generalized group is a non-empty

set G admitting an operation called multiplication, which satisfies the set of conditions given below:
(i) $(xy)z = x(yz)$ for all x, y, z in G;
(ii) for each x in G there exists a unique z in G such that $xz = zx = x$ (we denote z by $e(x)$);
(iii) for each x in G there exists y in G such that $xy = yx = e(x)$.
This structure has different meaning from Vagner (Wagner) generalized groups ([11]) and semigroups admitting relative inverse ([2,3]). In [1] Araújo and Konieczny by applying Rees theorem ([5,6,10]) proved that the notion of generalized groups is equivalent to the notion of completely simple semigroup. Properties of this structure from topological point of view was presented in [4] and [9]. We recall that ([9]) a topological generalized group G is a semigroup which satisfies the following conditions:
1. For each $x \in G$ there is a unique $e(x) \in G$ such that $xe(x) = e(x)x = x$;
2. For each $x \in G$ there exists $x^{-1} \in G$ such that $x(x^{-1}) = (x^{-1})x = e(x)$;
3. G is a Hausdorff topological space;
4. The mappings

$$\begin{aligned} m_1 : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

and

$$\begin{aligned} m_2 : G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

are continuous mappings.

If $a \in G$ then $G_a = e^{-1}(\{e(a)\})$ with the product of G is a topological group, and G is disjoint union of such topological groups.

Example 1.1. *The set $G = \mathbf{R} \times (\mathbf{R} - \{0\})$ with the topology induced by a Euclidean metric, and with operation $(a,b)(c,d) = (bc, bd)$ is a topological generalized group.*

Theorem 1.1. *Let G be a topological group, and let $a^2 = e$ for all $a \in G$. Then the space G with the product $a * b = aba$ is a topological generalized group.*

Proof. The condition $a^2 = e$, for all $a \in G$, implies that G is an Abelian group.

Let a, b and c belonging to G be given, then

$$\begin{aligned}(a * b) * c &= (aba) * c = ab(aca)ba \\ &= abcba = a(bcb)a = a(b * c)a = a * (b * c) \ .\end{aligned}$$

If $a * b = b * a = a$, then $aba = a$. So $ab = e$. Hence $b = a$. Thus $e(a) = a$.

Similarly $a^{-1} = a$, where a^{-1} is the inverse of a in $(G, *)$.

Therefore $(G, *)$ is a generalized group. The product $(a, b) \mapsto ab$ is a continuous mapping. So $(a, b) \mapsto aba$ is a continuous mapping. Moreover $a \mapsto a$ is also a continuous mapping. Thus $(G, *)$ is a topological generalized group.

A generalized subgroup ([7]) N of a generalized group G is called a generalized normal subgroup of G if there exists a generalized group E and a homomorphism $f : G \to E$ such that for all $a \in G$ we have

$N_a = \emptyset$ or $N_a = ker\ f_a$ where $N_a := N \cap G_a$, $f_a := f|_{G_a}$ and $ker\ f_a = \{x \in G_a : f(x) = f(e(a))\}$.

If N is a normal subgroup of G and $\Gamma_N = \{a \in G \mid N_a \neq \phi\}$, then Γ_N is a generalized subgroup of G.

Theorem 1.2. *Let N be a generalized normal subgroup of the normal generalized group G, then the set $G/N = \bigcup_{a \in G} G_a/N_a$ with the multiplication*

$$\begin{aligned} \cdot : G/N \times G/N &\longrightarrow G/N \\ (xN_a, yN_b) &\longmapsto xyN_{ab} \end{aligned}$$

is a normal generalized group. [7]

Theorem 1.3. *Let N be a closed generalized normal subgroup of G, then G/N is a topological generalized group.* ([9])

2 Properties which make connected topological generalized groups

In this section we shall study connected topological generalized group.

Theorem 2.1. *Let G be a topological generalized group and let $card(e(G)) < \infty$. Then G_a is an open and closed subset of G, where $a \in G$.*

Proof. If $card(e(G)) = 1$, then $G_a = G$ for all $a \in G$. So G_a is a closed and open set for all $a \in G$.

Let $1 < card(e(G)) < \infty$. Since $e : G \to G$ is a continuous map ,

$G_a = e^{-1}(\{e(a)\})$ is a closed subset of G, where $a \in G$. Moreover

$$G_{e(a)} = G - \left(\bigcup_{e(b)\in e(G), e(b)\neq e(a)} G_{e(b)} \right) .$$

So $G_{e(a)}$ is also an open subset of G, for $a \in G$.

Corollary 2.1. *Let G be a topological generalized group and let $1 < card(e(G)) < \infty$, then G is not a connected set.*

Proof. $G = \bigcup_{e(a)\in e(G)} G_{e(a)}$ and $G_{e(a)} \cap G_{e(b)} = \phi$ for $e(a) \neq e(b)$. So corollary follows from Theorem 2.1.

Theorem 2.2. *If G is a topological generalized group, N is an open generalized normal subgroup of G, and $card(e(G)) < \infty$, then N is a closed subset of G.*

Proof. Let $a \in G$ be given, then $N_{e(a)} = N \cap G_{e(a)}$ is an open set in $G_{e(a)}$, and N_a is a normal subgroup of the topological group $G_{e(a)}$. So $G_{e(a)} - N_{e(a)}$ is an open set in $G_{e(a)}$. Thus $N_{e(a)}$ is a closed subgroup of $G_{e(a)}$, where the topology of $G_{e(a)}$ is the relative topology. Hence there exists a closed subset $F_{e(a)}$ in G such that $N_{e(a)} = F_{e(a)} \cap G_{e(a)}$. Thus

$$N = \bigcup_{e(a)\in e(G)} N_{e(a)} = \bigcup_{e(a)\in e(G)} (F_{e(a)} \cap G_{e(a)})$$

is a closed subgroup of G.

Example 2.1. *Let G be the set of non-zero real numbers, then G with the product $a * b = a|b|$ is a topological generalized group. In this case $e(G) = \{1, -1\}$, and G is not a connected set.*

Definition 2.1. *A generalized subgroup H of G is called an e-generalized subgroup of G if $e(G) \subseteq H$, where e is the identity mapping.*

Theorem 2.3. *Let G be a topological generalized group and let G_a be a connected set, where $a \in G$. Suppose further that G has a connected e-generalized subgroup, then G is a connected set.*

Proof. Let $a \in G$ be given, and let N be a connected e-generalized subgroup of G, then $e(a) \in N \cap G_a$. So $N \cap G_a \neq \phi$. Thus $N \cup G_a$ is a connected set for all $a \in G$. Since N is a subset of $(N \cup G_a) \cap (N \cup G_b)$ for all $a, b \in G$, $\bigcup_{a \in G}(N \cup G_a)$ is a connected set. We have $G = \bigcup_{a \in G}(N \cup G_a)$. So G is a connected set.

Example 2.2. *The set $G = \mathbb{R} \times \mathbb{R}$ with the product*

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2),$$

and Euclidean topology is a topological normal generalized group.
If $(x_1, x_2) \in \mathbb{R}^2$, then

$$G_{(x_1,x_2)} = \mathbb{R} \times \{x_2\} \quad , \quad \textit{and} \quad e(G) = \{0\} \times \mathbb{R}$$

are connected sets. So G is a connected set.

3 Connected factor spaces

In this section we are going to consider conditions which imply that a topological factor group is a connected topological factor group.

Proposition 3.1. *Let N be a closed generalized normal subgroup of a topological normal generalized group G, and let Γ_N be a connected space, then G/N is a connected topological generalized group.*

Proof. Since N is a closed generalized normal subgroup of G, G/N is a topological generalized group . Moreover the mapping $\pi : \Gamma_N \to G/N$ defined by $\pi(x) = xN_x$ is a continuous map . So G/N is a connected topological generalized group.

Corollary 3.1. *Let N be a closed e-generalized normal subgroup of a connected topological generalized group G, then G/N with the topology induced by π is a connected generalized group.*

Proof. Since $e(G) \subseteq N$, we have $e(a) \in N \cap G_a$ for all $a \in G$. So $\Gamma_N = G$, and corollary follows from proposition 3.1.

Theorem 3.1. *Let G be a generalized topological group and N be a connected and closed generalized normal subgroup of G containing $e(G)$. Moreover let G_a/N_a and G be connected sets, for every $a \in G$, then G/N and G are connected sets.*

Proof. Case 1. If $card(e(G)) = 1$, then G is a group and theorem follows from topological group theory .

Case 2. If $card(e(G)) > 1$, then the mapping $\pi|_{G_a} : G_a \to G_a/N_a$ is an open and onto mapping.

So G_a is a connected set. Thus theorem 3.1 shows that G is a connected set. The continuity of the mapping $\pi : G \to G/N$ implies that G/N is a connected set.

Remark. *In theorem 3.1 if $1 < card(e(G)) < \infty$, then Theorem 2.1 and Theorem 2.3 show that there is no such N.*

4 Conclusion

A connected subset S of a topological generalized group G is called a stable connected components under identity if it satisfies the following conditions.
(i) $e(S) \subseteq S$;
(ii) If N is a connected subset of G and $S \subset N$, then $S = N$.

Example 4.1. *The non-empty set G with the product $a * b = a$ and discrete topology is a topological generalized group. The set $\{a\}$ is a stable connected components under identity, where $a \in G$.*

Theorem 4.1. *If G is a topological generalized group and S is a stable connected component under identity of G, then S is a closed generalized subgroup of G.*

Proof. $S^{-1} = \{s^{-1} : s \in S\}$ is a connected subset of G, because the mapping $m_1 : G \to G$ in the form $m_1(g) = g^{-1}$ is a connected set, and $S \subseteq S \cup S^{-1}$. Hence $S \cup S^{-1} = S$, as a result $S^{-1} \subseteq S$. Moreover xS is a connected set, for all $x \in G$. Because the mapping $m_2 : G \times G$ in the form $m_2(x, y) = xy$ is a continuous map. If $x \in S$, then $x = xe(x) \in xS$. So $(xS) \cap S \neq \phi$. Hence $(xS) \cup S$ is a connected set and $S \subseteq (xS) \cup S$. Thus $xS \subseteq S$. So $xy \in S$, for all $x, y \in S$. Therefore S is a generalized subgroup of S. $\overline{S}$ is also a connected set and $S \subseteq \overline{S}$. So $\overline{S} = S$.

We shall bring this paper to an end by posing the following problem:
Is every stable connected component under identity of G a normal subgroup of G ?

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