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On the Tricomi's quadrature formula

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Dedicated to Professor Gheorghe Micula on his 60^{th} birthday

Abstract

In this paper we obtain new results concerning the Tricomi's quadrature formula.

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In [10] F. Tricomi introduced a interesting quadrature rule, analogous to famous Richardson's method [5].

The paper [8], [9], [1] complete with new results the Tricomi's paper.

In this article we present some new results concerning the generalization of the Tricomi's quadrature formula, given in [1].

1. Let f be a function from $AC^{n-1}[a,b]$ - the class of functions f whose $(n-1)$-th derivative $f^{(n-1)}$ is absolutely continuous in $[a,b]$.

We consider the elementary quadrature formula

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} \sum_{i=1}^{m} A_{ki} f^{(k)}_{(x_i)} + R(f), \tag{1}$$

with the algebraical degree of exactness $n-1$ and with the remainder given by

$$R(f) = f^{(n)}_{(\xi)} \int_a^b \phi(x)dx, \ \xi \in (a,b), \tag{2}$$

where ϕ is the influence function ([7]).

We divide the interval $[a,b]$ into a certain number p partial intervals $[u_{j-1}, u_j]$, $j = \overline{1,p}$, defined by the points

$$a = u_0 < u_1 < ... < u_{p-1} < u_p = b \tag{3}$$

and we put $d_j = u_j - u_{j-1}$, $j = \overline{1,p}$.

We evaluate the integral

$$I = \int_a^b f(x)dx,$$

using the generalized quadrature formula generated to the elementary formula (1) (see [7]). We obtain

$$I = S_p + R_p \tag{4}$$

with

$$S_p = \sum_{j=1}^{p} \sum_{k=0}^{n-1} \sum_{i=1}^{m} \left(\frac{d_j}{b-a} \right)^{k+1} A_{ki} f^{(k)} \left(u_{j-1} + d_j \frac{x_j - a}{b-a} \right) \tag{5}$$

and the remainder

$$R_p = T \left[\sum_{j=1}^{p} \left(\frac{d_j}{b-a} \right)^{n+1} \right] f^{(n)}_{(\xi_1)}, \ \xi_1 \in [a,b), \tag{6}$$

where

$$T=\int_a^b \phi(x)dx. \tag{7}$$

Now we divide the interval $[a,b]$ into a certain number q, $q>p$, partial intervals $[v_{j-1},v_j]$, $j=\overline{1,q}$, defined by the points

$$a=v_0<v_1<...<v_{q-1}<v_q=b, \tag{8}$$

with $c_j=v_j-v_{j-1}$, $j=\overline{1,q}$.

We evaluate the integral I using the generalized quadrature formula generated to the elementary formula (1) and the partition (8). We find:

$$I=S_q+R_q, \tag{9}$$

with

$$S_q=\sum_{j=1}^{q}\sum_{k=0}^{n-1}\sum_{i=1}^{m}\left(\frac{c_j}{b-a}\right)^{k+1}A_{ki}f^{(k)}\left(v_{j-1}+c_j\frac{x_i-a}{b-a}\right) \tag{10}$$

and the remainder

$$R_q=T\left[\sum_{j=1}^{q}\left(\frac{c_j}{b-a}\right)^{n+1}\right]f^{(n)}(\xi_2),\ \ \xi_2\in(a,b). \tag{11}$$

If the function f is a polynomial of degree $\le n$, then $f^{(n)}(\xi_1)=f^{(n)}(\xi_2)$.

In this situation, if we put

$$\begin{cases} D=\sum\limits_{j=1}^{p}\left(\dfrac{d_j}{b-a}\right)^{n+1} \\ \text{and} \\ C=\sum\limits_{j=1}^{p}\left(\dfrac{c_j}{b-a}\right)^{n+1}, \end{cases} \tag{12}$$

then from (4), (6), (9), (11) it results

$$Tf^{(n)}(\xi_1)=Tf^{(n)}(\xi_2)=\frac{S_q-S_p}{D-C}. \tag{13}$$

Using (9), (11) şi (13), we get

$$I = S_q + \frac{t^n}{1-t^n}(S_q - S_p), \tag{14}$$

where $t = \sqrt[n]{C/D}$.

Now, for an arbitrary function $f \in AC^{n-1}[a,b]$, we consider the quadarture formula

$$I = S_q + \frac{t^n}{1-t^n}(S_q - S_p) + R_{p,q}, \tag{15}$$

where S_q, S_p and t are the above.

From (15), (4) and (9), we have

$$R_{p,q} = \frac{1}{1-t^n}(R_q - t^n R_p),$$

whence, using (6) and (11), we find the following forms for the remainder of the formula (15):

$$R_{p,q} = \frac{TC}{1-t^n}[f^{(n)}_{(\xi_2)} - f^{(n)}_{(\xi_1)}], \tag{16}$$

$$R_{p,q} = \frac{TDt^n}{1-t^n}[f^{(n)}_{(\xi_2)} - f^{(n)}_{(\xi_1)}], \tag{17}$$

$$R_{p,q} = \frac{TCD}{D-C}[f^{(n)}_{(\xi_2)} - f^{(n)}_{(\xi_1)}], \tag{18}$$

$$\xi_1,\ \xi_2 \in (a,b)$$

In view of (16), (17) and (18), it results

$$|R_{p,q}| \le \frac{|T|C}{1-t^n}\Omega^{(n)}, \tag{19}$$

$$|R_{p,q}| \le \frac{|T|Dt^n}{1-t^n}\Omega^{(n)}, \tag{20}$$

$$|R_{p,q}| \leq \frac{|T|CD}{D-C}\Omega^n, \tag{21}$$

where $\Omega^{(n)}$ is the oscillation of the function $f^{(n)}$ on $[a,b]$.

If we take into account that for T usually have a relation by form

$$|T| = |\int_a^b \phi(x)dx| = |\alpha||b-a|^{n+1},$$

where α is a constant, then from (9) - (21), we have

$$|R_{p,q}| \leq \frac{|\alpha|\left(\sum_{j=1}^{p} C_j^{n+1}\right)\Omega^{(n)}}{1-t^n}, \tag{22}$$

$$|R_{p,q}| \leq \frac{|\alpha|\left(\sum_{j=1}^{p} d_j^{n+1}\right)\Omega^{(n)}}{1-t^n}, \tag{23}$$

$$|R_{p,q}| \leq \frac{|\alpha|\left(\sum_{j=1}^{p} d_j^{n+1}\right)\left(\sum_{j=1}^{q} C_j^{n+1}\right)}{\sum_{j=1}^{p} d_j^{n+1} - \sum_{j=1}^{n} c_j^{n+1}} \cdot \Omega^{(n)}. \tag{24}$$

2. Particular cases.

2.1 If we consider

$$\begin{cases} d_1 = d_2 = ... = d_b = \dfrac{b-a}{p} = \omega_p \\ \text{and} \\ c_1 = c_2 = ... = c_q = \dfrac{b-a}{q} = \omega_q, \end{cases} \tag{25}$$

then $t = p/q$.

In this case, the formula (15) presents a generalization of the results from [8] for the quadrature formulas which they contain the values of the derivative the function f on the nodes.

For the remainder, from (23) we found

$$|R_{p,q}| \leq |\alpha| p \omega_p^{n+1} \frac{t^n}{1-t^n} \Omega^{(n)}. \tag{26}$$

For $q = 2p$ we obtain a generalization of the Tricomi's results.

2.2 If (25) are satisfied and the elementary quadrature formula (1) contains only the values of the function f on the nodes, then (15) and (22) - (24) give the results by [8]. For $q = 2p$ we obtain the Tricomi's results ([10]).

Examples. In this section we present some examples of quadrature formula by Tricomi's type obtained by the particularizing of the elementary rule (1).

3.1. Assume that (1) is the rectangular formula ([7]). We find the following Tricomi's quadrature formula:

$$I = S_q + \frac{t^2}{1-t^2}(S_q - S_p) + R_{p,q}$$

with

$$S_p = \sum_{j=1}^{p} d_j (a + d_1 + d_2 + ... + d_{j-1} + \frac{d_j}{2}),$$

$$S_q = \sum_{j=1}^{q} c_j (a + c_1 + c_2 + ... + c_{j-1} + \frac{c_j}{2}),$$

$$t = \sqrt{\left(\sum_{j=1}^{q} c_j^3\right) / \left(\sum_{j=1}^{q} d_j^3\right)},$$

$$|R_{p,q}| \leq \frac{1}{24}\left(\sum_{j=1}^{p} d_j^3\right) \cdot \frac{t^2}{1-t^2}\Omega^{(2)},$$

where $\Omega^{(2)}$ is the oscillation of the function f'' on $[a,b]$.

From here, for $d_1 = ... = d_p = \dfrac{b-a}{b} = \omega_p$, $c_1 = c_2 = ... = c_q = \dfrac{b-a}{q} = \omega_q$ we find the results by [8].

3.2 We consider (1) the Simpson's formula.

It results the following Tricomi's quadrature formula

$$I = S_q + \frac{t^4}{1-t^4}(S_q - S_p) + R_{p,q},$$

with

$$S_p = \frac{d_j}{2}f(a) + \sum_{j=1}^{p-1} \frac{d_j + d_{j+1}}{6}(a + d_1 + ... + d_j) +$$

$$+ \sum_{j=1}^{p} \frac{2d_j}{3} f\left(a + d_1 + ... + d_{j-1} + \frac{d_j}{3}\right) + \frac{d_p}{6} f(b),$$

$$S_q = \frac{c_1}{6}f(a) + \sum_{j=1}^{q-1} \frac{c_j + c_{j+1}}{6} f\left(a + c_1 + ... + c_j\right) +$$

$$+ \sum_{j=1}^{q} \frac{2c_j}{3} f\left(a + c_1 + ... + c_{j+1} + \frac{c_j}{3}\right) + \frac{C_q}{6} f(b),$$

$$t = \sqrt[4]{\left(\sum_{j=1}^{q} c_j^5\right) / \left(\sum_{j=1}^{p} d_j^5\right)},$$

$$|R_{p,q}| \leq \frac{1}{2880}\left(\sum_{j=1}^{p} d_j^5\right) \frac{t^4}{1-t^4}\Omega^{(4)},$$

where $\Omega^{(4)}$ is the oscillation of the function $f^{(4)}$ on $[a,b]$.

From here, for $d_1 = d_2 = ... = d_p = 2\dfrac{b-a}{2p} = 2\omega_{2p}$ and $c_1 = c_2 = ... = c_q = \dfrac{2(b-a)}{2q}$, we obtain the results by [8]. If $q = 2p$ then we find the case studied by Tricomi's ([10]).

3.3. Finally, we assume that (1) is Newton's quadrature formula ([2], [3]).

We obtain the following Tricomi's quadrature formula:

$$I = S_q + \frac{t^4}{1-t^4}(S_q - S_p) + R_{p,q}$$

with

$$S_p = \frac{d_i}{8} f(a) + \sum_{j=1}^{p-1} \frac{d_j + d_{j+1}}{8} f(a + d_1 + ... + d_j) +$$

$$+ \frac{d_p}{8} f(b) + \sum_{j=1}^{p} \frac{3d_j}{8} f\left(a + d_1 + ... + d_{j-1} + \frac{d_j}{3}\right) +$$

$$+ \sum_{j=1}^{p} \frac{3d_j}{8} f\left(a + d_1 + ... + d_{j-1} + \frac{2d_j}{3}\right),$$

$$S_q = \frac{c_i}{8} f(a) + \sum_{j=1}^{q-1} \frac{c_j + c_{j+1}}{8} f(a + c_1 + ... + c_j) +$$

$$+ \frac{c_q}{8} f(b) + \sum_{j=1}^{q} \frac{3c_j}{8} f\left(a + c_1 + ... + c_{j-1} + \frac{c_j}{3}\right) +$$

$$+ \sum_{j=1}^{q} \frac{3c_j}{8} f\left(a + c_1 + ... + c_{j-1} + \frac{2c_j}{3}\right),$$

$$t = \sqrt[4]{\left(\sum_{j=1}^{q} c_j^5\right) / \left(\sum_{j=1}^{p} d_j^5\right)},$$

$$|R_{p,q}| \le \frac{1}{6480} \left(\sum_{j=1}^{p} d_j^5\right) \frac{t^4}{1-t^4} \Omega^{(4)},$$

where $\Omega^{(4)}$ is the oscillation of the function $f^{(4)}$ on the $[a, b]$.

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