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About one fixed point theorem

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Dedicated to Professor D. D. Stancu on his 75th birthday.

Abstract

The aim of this paper is the study of a fixed point property for pairs of classes of topological spaces defined as follows:
for a class of sets φ and a set X we shall denote by

$$\mathcal{C}(X) = \{C \in \mathcal{C} : C \subset X\} \text{ and } \mathcal{C}^*(X) = \{C \in \mathcal{C}(X) : C \neq \varnothing\}.$$

We say that a map $T : X \to Y$ has $\mathcal{C}$ (resp. $\mathcal{C}^*$) values if for each $x \in X$, $T(x) \in \mathcal{C}(X)$ (resp. $T(x) \in \mathcal{C}^*(X)$).

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1 Introduction

In what follows we use the following definition:

Definition 1. *We say that a pair $(\mathcal{T}, \mathcal{C})$ consisting of two classes of compact Hausdorff topological spaces has the fixed point property provided:*
i) $X, Y \in \mathcal{C}$ *implies* $X \times Y \in \mathcal{T}$*;*

ii) $C \in \mathcal{C}(X)$, $D \in \mathcal{C}(Y)$ *implies* $C \times D \in \mathcal{C}(X \times Y)$, *for each* $X, Y \in \mathcal{T}$;
iii) for each $X \in \mathcal{T}$, *any upper semicontinuous map* $T : X \to X$ *with* $\mathcal{C}^*$ *values has a fixed point.*

Example 1. Both $\mathcal{T}$ and $\mathcal{C}$ are the class of all compact convex subsets of all Hausdorff locally convex topological vector spaces. In this case condition (iii) in Definition is satisfied according to the Kakutami - Tan - Glicksberg fixed point theorem (see [2], [3]).

2 Fixed points

Let $\{X_i\}_{1 \le i \le n}$ be a finite family of sets ($n \ge 2$). Let

$$X = \prod_{i=1}^{n} X_i \text{ and } X^i = \prod_{\substack{j=1 \\ j \ne 1}}^{n} X_j.$$

Any $x = (x_1, x_2, ..., x_n) \in X$ can be expressed as $x = (x^i, x_i)$ for any $i \in \{1, 2, ..., n\}$, where x^i denotes the canonical projection of x on X^i.

Theorem 1. *Let* $X_i \in \mathcal{T}(1 \le i \le n)$ *and for each* $i \in \{1, 2, ..., n\}$ *let* $T_i : X \to X_i$ *be an upper semicontinuous map with* $\mathcal{C}^*$ *values. Then there exists an* $\widehat{x} \in X$ *such that* $\widehat{x}_i \in T_i(\widehat{x})$ *for each* $i \in \{1, 2, ..., n\}$.

The proof can be found in [1].

Theorem 2. *Let* $X_i \in \mathcal{T}(1 \le i \le n)$ *and* $S_i : X_i \to X_{i+1}(1 \le i \le n-1)$, $S_n : X_n \to X_1$ *be upper semicontinuous map with* $\mathcal{C}^*$ *values. Then the composite* $S_n \circ ... \circ S_1$ *has a fixed point.*

Proof. Let $T_i : X \to X_i(1 \le i \le n)$ be the maps defined by

$$T_1(x_1, x_2, ..., x_n) = S_n(x_n)$$

$$T_i(x_1, x_2, ..., x_n) = S_{i-1}(x_{i-1}) \text{ for } 2 \leq i \leq n.$$

By Theorem 1 there exists $\widehat{x} = (\widehat{x}_1, ..., \widehat{x}_n) \in X$ such that $\widehat{x}_1 \in T_1(\widehat{x}) = S_n(\widehat{x}_n)$, $\widehat{x}_2 \in T_2(\widehat{x}) = S_1(\widehat{x}_1)$, ..., $\widehat{x}_1 \in T_n(\widehat{x}) = S_{n-1}(\widehat{x}_{n-1})$, whence $\widehat{x}_1 \in (S_n \circ ... \circ S_1)(\widehat{x}_1)$.

Theorem 1 can be reformulated to the form of quasi-equilibrium theorem as follows:

Theorem 3. *For each $i \in \{1, 2, ..., n\}$ let $X_i \in \mathcal{T}$, $S_i : X \to X_i$ closed map and $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ upper semicontinuous functions. Suppose that for each $i \in \{1, 2, ..., n\}$ the following conditions are satisfied:*
a) $g_i(x) \leq f_i(x)$ for each $x \in X$;
b) the function M_i defined on X by $M_i(x) = \max\limits_{y \in S_i(x)} g_i(x_i, y)$ is lower semicontinuous;
c) for each $x \in X$, $\{y \in S_i(x) : f_i(x^i, y) \geq M_i(x)\} \in \mathcal{C}(X)$. Then there exists an $\widehat{x} \in X$ such that for each $i \in \{1, 2, ..., n\}$, $\widehat{x}_i \in S_i(\widehat{x})$ and $f_i(\widehat{x}_i, \widehat{x}_i) \geq M_i(\widehat{x})$.

Proof. For each $i \in \{1, 2, ..., n\}$ define a map $T_i : X \to X_i$ by $T_i(x) = \{y \in S_i(x) : f_i(x^i, y) \geq M_i(x)\}$ for $x \in X$.

Note that each $T_i(x)$ is non-empty by (i), since $S_i(x)$ is compact and $g_i(x^i, \cdot)$ is upper semicontinuous on $S_i(x)$. We prove that the graph of T_i is closed in $X \times X_i$. In view of this let $(x_\alpha, y_\alpha) \in$ graph T_i such that $(x_\alpha, y_\alpha) \to (x, y)$.

Then

$$f_i(x^i, y) \geq \limsup_{\alpha} f_i(x^i_\alpha, y_\alpha) \geq \limsup_{\alpha} M_i(x_\alpha) \geq \liminf_{\alpha} M_i(x_\alpha) \geq M_i(x)$$

and, since graph S_i is closed in $X \times X_i$, $y_\alpha \in S_i(x_\alpha)$ implies $y \in S_i(x)$. Hence $(x, y) \in$ graph T_i. Since X_i is compact, T_i is upper semicontinuous. By Theorem 1, there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for each

$i \in \{1, 2, ..., n\}$; that is $\widehat{x}_i \in S_i(\widehat{x})$ and $f_i(\widehat{x}^i, \widehat{x}_i) \geq M_i(\widehat{x})$. This completes the proof.

The origin of Theorem 3 is the Nash equilibrium theorem [4].

References

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