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SOBOLEV ORTHOGONAL POLYNOMIALS: INTERPOLATION AND APPROXIMATION *

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Abstract. In this paper, we study orthogonal polynomials with respect to the bilinear form

$$(f,g)_S = (f(c_0), f(c_1), \ldots, f(c_{N-1}))\mathbf{A} \begin{pmatrix} g(c_0) \\ g(c_1) \\ \vdots \\ g(c_{N-1}) \end{pmatrix} + \langle u, f^{(N)} g^{(N)} \rangle,$$

where u is a quasi-definite (or regular) linear functional on the linear space $\mathbb{P}$ of real polynomials, $c_0, c_1, \ldots, c_{N-1}$ are distinct real numbers, N is a positive integer number, and $\mathbf{A}$ is a real $N \times N$ matrix such that each of its principal submatrices are nonsingular. We show a connection between these non-standard orthogonal polynomials and some standard problems in the theory of interpolation and approximation.

Key words. Sobolev orthogonal polynomials, classical orthogonal polynomials, interpolation, approximation.

AMS subject classifications. 33C45, 42C05.

1. Introduction. During the past few years, orthogonal polynomials with respect to an inner product involving derivatives (so–called Sobolev orthogonal polynomials) have been the object of increasing number of works (see, for instance [1], [5], [6], [4], [7], [8]). Recurrence relations, asymptotics, algebraic, differentiation properties and zeros for various families of polynomials have been studied. In this paper we study a connection between a particular case of non–standard orthogonal polynomials and standard problems in the theory of interpolation and approximation.

In Section 2, we give a description of the monic polynomials $\{Q_n\}_n$ which are orthogonal with respect to

$$(f,g)_S = (f(c_0), f(c_1), \ldots, f(c_{N-1}))\mathbf{A} \begin{pmatrix} g(c_0) \\ g(c_1) \\ \vdots \\ g(c_{N-1}) \end{pmatrix} + \langle u, f^{(N)} g^{(N)} \rangle, \tag{1.1}$$

where u is a regular (or quasi-definite) linear functional on the linear space $\mathbb{P}$ of real polynomials, $c_0, c_1, \ldots, c_{N-1}$ are distinct real numbers, N is a positive integer, and $\mathbf{A}$ is a real $N \times N$ matrix such that each of its principal submatrices are nonsingular. Let $\{P_n\}_n$ be the monic polynomials orthogonal with respect to the functional u. If $n \geq N$, we have $Q_n(c_i) = 0, \quad i = 0, 1, \ldots, N-1$, and $Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x)$, while $\{Q_n\}_{n=0}^{N-1}$ are orthogonal with respect to the discrete part of the symmetric bilinear form (1.1).

In Section 3, we give some examples of monic orthogonal polynomial sequences (in short MOPS) which are orthogonal with respect to the bilinear form (1.1), using the Laguerre and Jacobi linear functionals.

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In Section 4, we show that the MOPS with respect to (1.1) can be expressed as the interpolation error of an N–th primitive of $\{P_{n-N}\}_{n\geq N}$, where $\{P_n\}_n$ is the MOPS associated with the regular linear functional u.

The final section of this paper is devoted to establish the relation between this kind of discrete–continuous Sobolev orthogonality and a problem of simultaneous polynomial interpolation and approximation, in the case when (1.1) is an inner product.

2. The Sobolev discrete–continuous bilinear form. Let $\mathbb{P}$ be the linear space of real polynomials, u a regular linear functional on $\mathbb{P}$ (see [2]), N a positive integer number, and $\mathbf{A}$ a quasi-definite, symmetric and real matrix, that is, a symmetric and real matrix such that all its principal minors are different from zero. The expression

$$(f,g)_S = (f(c_0), f(c_1), \ldots, f(c_{N-1}))\mathbf{A}\begin{pmatrix} g(c_0) \\ g(c_1) \\ \vdots \\ g(c_{N-1}) \end{pmatrix} + \langle u, f^{(N)}g^{(N)}\rangle, \tag{2.1}$$

where $c_0, c_1, \ldots, c_{N-1}$ are distinct real numbers, defines a symmetric bilinear form on $\mathbb{P}$.

Since expression (2.1) involves derivatives, this bilinear form is non-standard, and by analogy with the usual terminology we call it a *discrete–continuous Sobolev bilinear form.*

Let

$$w_N(x) = \prod_{i=0}^{N-1}(x-c_i).$$

In the linear space of real polynomials, we can consider the basis given by

$$B = \left\{ \{l_i(x)\}_{i=0,1,\ldots,N-1}, \left\{w_N(x)x^j\right\}_{j\geq 0} \right\},$$

where

$$l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{N-1} \frac{x-c_j}{c_i-c_j}, \quad i = 0,1,\ldots,N-1,$$

are Lagrange polynomials.

For $n \leq N-1$, the associated Gram matrix $\mathbf{G}_n$ is given by the n-th order principal submatrix of the matrix $\mathbf{A}$. For $n \geq N$, the associated Gram matrix is given by

$$\mathbf{G}_n = \begin{pmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B}_{n-N} \end{pmatrix},$$

where $\mathbf{B}_{n-N}$ is the Gram matrix associated with the quasi-definite linear functional u in the basis $\tilde{B} = \left\{D^{(N)}[w_N(x)x^j], j\geq 0\right\}$. In both cases, $\mathbf{G}_n$ is quasi-definite and therefore, the discrete–continuous Sobolev bilinear form (2.1) is quasi-definite. Thus, we can assure the existence of a sequence of monic polynomials, denoted by $\{Q_n\}_n$, which is orthogonal with respect to (2.1). These polynomials will be called *Sobolev orthogonal polynomials.*

THEOREM 2.1. *Let $\{Q_n\}_n$ be the MOPS with respect to the Sobolev discrete–continuous form (2.1) and let $\{P_n\}_n$ be the MOPS associated with the regular linear functional u.*

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i) The polynomials $\{Q_n\}_{n=0}^{N-1}$ *are orthogonal with respect to the discrete bilinear form*

$$(f,g)_D = (f(c_0), f(c_1), \ldots, f(c_{N-1}))\mathbf{A} \begin{pmatrix} g(c_0) \\ g(c_1) \\ \vdots \\ g(c_{N-1}) \end{pmatrix}, \tag{2.2}$$

ii) If $n \geq N$*, then*

$$Q_n(c_i) = 0, \quad i = 0, 1, \ldots, N-1, \tag{2.3}$$

$$Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x). \tag{2.4}$$

Proof. i) If $0 \leq n, m < N$, then $Q_n^{(N)}(x) = Q_m^{(N)}(x) = 0$, and obviously

$$(Q_n, Q_m)_S = (Q_n, Q_m)_D = (Q_n(c_0), Q_n(c_1), \ldots, Q_n(c_{N-1}))\mathbf{A} \begin{pmatrix} Q_m(c_0) \\ Q_m(c_1) \\ \vdots \\ Q_m(c_{N-1}) \end{pmatrix}.$$

ii) For $n \geq N$, from the orthogonality of the polynomial Q_n, we deduce

$$0 = (Q_n, l_i)_S = (Q_n, l_i)_D = (Q_n(c_0), Q_n(c_1), \ldots, Q_n(c_{N-1}))\mathbf{A} \begin{pmatrix} l_i(c_0) \\ l_i(c_1) \\ \vdots \\ l_i(c_{N-1}) \end{pmatrix}$$

$$= (Q_n(c_0), Q_n(c_1), \ldots, Q_n(c_{N-1}))\mathbf{A} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

for $0 \leq i \leq N-1$. Thus, the vector

$$(Q_n(c_0), Q_n(c_1), \ldots, Q_n(c_{N-1})),$$

is the only solution of a homogeneous linear system with N equations and N unknowns, whose coefficient matrix $\mathbf{A}$ is regular. We conclude that $Q_n(c_i) = 0, \quad i = 0, 1, \ldots, N-1$, i.e., Q_n contains the factor $(x-c_0)(x-c_1)\cdots(x-c_{N-1})$.

In this way, if $n, m \geq N$,

$$(Q_n, Q_m)_S = \langle u, Q_n^{(N)} Q_m^{(N)} \rangle = \tilde{k}_n \delta_{n,m}, \quad \tilde{k}_n \neq 0.$$

Thus, the polynomials $\{Q_n^{(N)}\}_{n \geq N}$ are orthogonal with respect to the linear functional u, and equality (2.4) follows from a simple inspection of the leading coefficients. □

Conversely, we are going to show that a system of monic polynomials $\{Q_n\}_n$ satisfying equations (2.3) and (2.4) is orthogonal with respect to some discrete–continuous Sobolev form like (2.1). This result could be considered a *Favard-type theorem*.

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THEOREM 2.2. *Let $\{P_n\}_n$ be the MOPS associated with a regular linear functional u and $N \geq 1$ be a given integer. Let $\{Q_n\}_n$ be a sequence of monic polynomials satisfying*
i) $\deg Q_n = n, \quad n = 0, 1, \ldots,$
ii) $Q_n(c_i) = 0, \quad 0 \leq i \leq N-1, \quad n \geq N,$
iii) $Q_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x), \quad n \geq N.$

Then, there exists a quasi-definite and symmetric real matrix $\mathbf{A}$, of order N, such that $\{Q_n\}_n$ is the monic orthogonal polynomial sequence associated with the Sobolev bilinear form defined by (2.1).

Proof. Obviously the polynomial Q_n, with $n \geq N$, is orthogonal to every polynomial of degree less than or equal to $n-1$ with respect to a Sobolev bilinear form like (2.1), containing an arbitrary matrix $\mathbf{A}$ in the discrete part and the functional u in the second part.

Next, we will show that we can recover the matrix $\mathbf{A}$ from the N first polynomials $Q_k, k = 0, 1, \ldots, N-1$.

Introduce

$$\mathbf{Q} = \begin{pmatrix} Q_0(c_0) & Q_0(c_1) & \ldots & Q_0(c_{N-1}) \\ Q_1(c_0) & Q_1(c_1) & \ldots & Q_1(c_{N-1}) \\ \vdots & \vdots & & \vdots \\ Q_{N-1}(c_0) & Q_{N-1}(c_1) & \ldots & Q_{N-1}(c_{N-1}) \end{pmatrix}.$$

The matrix $\mathbf{Q}$ is regular since the system of linearly independent polynomials $\{Q_n\}_{n=0}^{N-1}$ satisfies the Haar condition (see [3]).

Let $\mathbf{D}$ be a diagonal regular matrix. Define

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T.$$

Obviously $\mathbf{A}$ is symmetric and quasi-definite and since

$$\mathbf{Q}\mathbf{A}\mathbf{Q}^T = \mathbf{D},$$

the polynomials $Q_0, Q_1, \ldots, Q_{N-1}$ are orthogonal with respect to the bilinear form (2.1), with the matrix $\mathbf{A}$ in the discrete part. Moreover, the diagonal entries of $\mathbf{D}$ are $(Q_k, Q_k)_S$ for $k = 0, 1, \ldots, N-1$. □

REMARK. Observe that the matrix $\mathbf{A}$ is not unique, because its construction depends on the arbitrary regular matrix $\mathbf{D}$.

3. Examples.

3.1. Laguerre case. Let $\alpha \in \mathbb{R}$, and introduce the monic generalized Laguerre polynomials, cf. [8, p. 102],

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{j=0}^{n} frac(-1)^j j! \begin{pmatrix} n+\alpha \\ n-j \end{pmatrix} x^j, \quad n \geq 0,$$

where $\begin{pmatrix} a \\ k \end{pmatrix}$ denotes the generalized binomial coefficient

$$\begin{pmatrix} a \\ k \end{pmatrix} = \frac{(a-k+1)_k}{k!}$$

and $(b)_k$ denotes the *Pochhammer's symbol* defined by

$$(b)_0 = 1, (b)_n = b(b+1)\ldots(b+n-1), \quad b \in \mathbb{R}, \quad n \geq 0.$$

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When α is not a negative integer, Laguerre polynomials are orthogonal with respect to a regular linear functional $u^{(\alpha)}$. This linear functional is positive definite for $\alpha > -1$.

We know that the derivatives of Laguerre polynomials are again Laguerre polynomials

$$\frac{d}{dx} L_n^{(\alpha)}(x) = n L_{n-1}^{(\alpha+1)}(x), \quad n \geq 1.$$

Let $\{Q_n\}_n$ be the sequence of monic polynomials given by

$$Q_n(x) = L_n^{(\alpha-N)}(x), \quad n = 0, 1, \ldots, N-1, \tag{3.1}$$

$$Q_n(x) = L_n^{(\alpha-N)}(x) - \sum_{i=0}^{N-1} L_n^{(\alpha-N)}(c_i) l_i(x), \quad n \geq N, \tag{3.2}$$

where $l_i(x),\ i = 0 \ldots N-1$, are the Lagrange polynomials. It follows from Theorem 2.2 that the sequence $\{Q_n\}_n$ is orthogonal with respect to the Sobolev bilinear form

$$(f, g)_S = (f(c_0), f(c_1), \ldots, f(c_{N-1}))\mathbf{A} \begin{pmatrix} g(c_0) \\ g(c_1) \\ \vdots \\ g(c_{N-1}) \end{pmatrix} + \langle u^{(\alpha)}, f^{(N)} g^{(N)} \rangle,$$

where $c_0, c_1, \ldots, c_{N-1}$ are distinct real numbers and the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{Q}^{-1} \mathbf{D} (\mathbf{Q}^{-1})^T,$$

$\mathbf{Q}$ is the matrix of Laguerre polynomials $\{L_n^{(\alpha-N)}\}_{n=0}^{N-1}$ evaluated at $c_0, c_1, \ldots, c_{N-1}$, i.e.,

$$\mathbf{Q} = \left(L_n^{(\alpha-N)}(c_i) \right)_{i,n=0,\ldots,N-1}$$

and $\mathbf{D}$ is an arbitrary regular diagonal matrix.

3.2. Jacobi case. For α and β real numbers, the generalized Jacobi polynomials can be defined by means of their explicit representation

$$P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} \left(\frac{x-1}{2}\right)^{n-m} \left(\frac{x+1}{2}\right)^{m}, \quad n \geq 0,$$

see [8, p. 68].

When α and β are nonnegative integers, Jacobi polynomials are orthogonal with respect to a regular linear functional $u^{(\alpha,\beta)}$. This linear functional is positive definite for $\alpha, \beta > -1$.

Let $\tilde{P}_n^{(\alpha,\beta)}(x)$, $n \geq 0$, be monic Jacobi polynomials. We know that the derivatives of Jacobi polynomials are again Jacobi polynomials

$$\frac{d}{dx} \tilde{P}_n^{(\alpha,\beta)}(x) = n \tilde{P}_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1.$$

Let $\{Q_n\}_n$ be the sequence of monic polynomials given by

$$Q_n(x) = \tilde{P}_n^{(\alpha-N,\beta-N)}(x), \quad n = 0, 1, \ldots, N-1, \tag{3.3}$$

$$Q_n(x) = \tilde{P}_n^{(\alpha-N,\beta-N)}(x) - \sum_{i=0}^{N-1} \tilde{P}_n^{(\alpha-N,\beta-N)}(c_i) l_i(x), \quad n \geq N, \tag{3.4}$$

where α, β and $\alpha+\beta-2N+1$ are not negative integers. It follows from Theorem 2.2 that the sequence $\{Q_n\}_n$ is orthogonal with respect to the Sobolev bilinear form

$$(f,g)_S = (f(c_0), f(c_1), \ldots, f(c_{N-1}))\mathbf{A}\begin{pmatrix} g(c_0) \\ g(c_1) \\ \vdots \\ g(c_{N-1}) \end{pmatrix} + \langle u^{(\alpha,\beta)}, f^{(N)}g^{(N)}\rangle,$$

where $c_0, c_1, \ldots, c_{N-1}$ are distinct real numbers and the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{Q}^{-1}\mathbf{D}(\mathbf{Q}^{-1})^T,$$

where

$$\mathbf{Q} = \left(\tilde{P}_n^{(\alpha-N,\beta-N)}(c_i)\right)_{i,n=0,\ldots,N-1}$$

and $\mathbf{D}$ is an arbitrary regular diagonal matrix.

REMARK. Jacobi polynomials $\{\tilde{P}_n^{(-1,-1)}\}_{n\geq 2}$ contain for $n \geq 2$, the factor x^2-1. Therefore, for $\alpha=\beta=1$, $N=2$ and $c_0=1$, $c_1=-1$, Theorem (2.2) provides Sobolev orthogonality for these polynomials (see [6]).

4. Sobolev Orthogonal Polynomials and Interpolation. Let $\{Q_n\}_n$ be the MOPS with respect to the Sobolev discrete–continuous form (2.1) and let $\{P_n\}_n$ be the MOPS associated with the regular linear functional u. Then the polynomials $\{Q_n\}_n$ can be expressed as the interpolation error of a N–th primitive of $\{P_{n-N}\}_{n\geq N}$

THEOREM 4.1. *Let the MOPS $\{Q_n\}_n$ and $\{P_n\}$ be defined as above, and let $\{R_n\}_{n\geq N}$ be a sequence of N-th monic primitives of the polynomials $\{P_{n-N}\}_{n\geq N}$. Then*

$$Q_n(x) = R_n[c_0, c_1, \ldots, c_{N-1}, x]\prod_{i=0}^{N-1}(x-c_i), \quad n \geq N,$$

where $R_n[c_0, c_1, \ldots, c_{N-1}, x], \quad n \geq N$, denotes the usual divided difference.

Proof. Integrating in (2.4) N times, we obtain

$$Q_n(x) = R_n(x) + \sum_{i=0}^{N-1} A_i l_i(x), \quad n \geq N,$$

where $l_i(x),\ i=0,\ldots,N-1$, are the Lagrange polynomials. Using (2.3), we deduce

$$A_i = -R_n(c_i), \quad i=0,\ldots,N-1.$$

Hence

$$Q_n(x) = R_n(x) - \sum_{i=0}^{N-1} R_n(c_i)l_i(x), \quad n \geq N,$$

i.e., for $n \geq N,\ Q_n(x)$ for $n \geq N$ is the error of interpolation of the polynomial $R_n(x)$ at $c_0, c_1, \ldots, c_{N-1}$ (see [10], p. 49), and therefore

$$Q_n(x) = R_n[c_0, c_1, \ldots, c_{N-1}, x]\prod_{i=0}^{N-1}(x-c_i), \quad n \geq N,$$

where $R_n[c_0, c_1, \ldots, c_{N-1}, x], \quad n \geq N$, are the divided differences. □

REMARK. In section 3 we observe that $Q_n, \quad n \geq 0$, given by (3.1) and (3.2), is the interpolation error of Laguerre polynomials $L_n^{(\alpha-N)}$ at $c_0, c_1, \ldots, c_{N-1}$. Analogously $Q_n, \quad n \geq 0$, given by (3.3) and (3.4), is the interpolation error of Jacobi polynomials $P_n^{(\alpha-N,\beta-N)}$ at $c_0, c_1, \ldots, c_{N-1}$.

THEOREM 4.2. *Let $\{R_n\}_n$ be a sequence of monic polynomials such that $\deg R_n = n, \quad n = 0, 1, \ldots$, and let $\{Q_n\}_n$ be the sequence of polynomials determined by*

$$Q_n(x) = R_n(x), \quad n = 0, 1, \ldots, N-1, \tag{4.1}$$

and let

$$Q_n(x) = R_n(x) - \sum_{i=0}^{N-1} R_n(c_i) l_i(x), \quad n \geq N, \tag{4.2}$$

where $c_0, c_1, \ldots, c_{N-1}$ are distinct real numbers.

If $\{R_n^{(N)}\}_{n\geq N}$ is an orthogonal polynomial sequence with respect to some regular linear functional u, then there exists a quasi-definite and symmetric real matrix $\mathbf{A}$, of order N, such that $\{Q_n\}_{n\geq N}$ is the MOPS associated with the Sobolev bilinear form defined by (2.1).

Proof. By (4.1) and (4.2) we have $\deg Q_n = \deg R_n = n$, and for $n \geq N$ we have

$$Q_n(c_j) = R_n(c_j) - \sum_{i=0}^{N-1} R_n(c_i) l_i(c_j) = R_n(c_j) - \sum_{i=0}^{N-1} R_n(c_i)\delta_{ij} = 0.$$

Moreover,

$$Q_n^{(N)}(x) = R_n^{(N)}(x) = \frac{n!}{(n-N)!} P_{n-N}(x), \quad n \geq N,$$

where $\{P_n\}_n$ is the MOPS associated with u.
From Theorem (2.2) it follows that $\{Q_n\}_n$ is the MOPS with respect to the bilinear form defined by (2.1), where

$$\mathbf{A} = \mathbf{R}^{-1}\mathbf{D}(\mathbf{R}^{-1})^T$$

$$\mathbf{R} = \begin{pmatrix} R_0(c_0) & R_0(c_1) & \ldots & R_0(c_{N-1}) \\ R_1(c_0) & R_1(c_1) & \ldots & R_1(c_{N-1}) \\ \vdots & \vdots & & \vdots \\ R_{N-1}(c_0) & R_{N-1}(c_1) & \ldots & R_{N-1}(c_{N-1}) \end{pmatrix}$$
$$= \begin{pmatrix} Q_0(c_0) & Q_0(c_1) & \ldots & Q_0(c_{N-1}) \\ Q_1(c_0) & Q_1(c_1) & \ldots & Q_1(c_{N-1}) \\ \vdots & \vdots & & \vdots \\ Q_{N-1}(c_0) & Q_{N-1}(c_1) & \ldots & Q_{N-1}(c_{N-1}) \end{pmatrix},$$

and $\mathbf{D}$ is an arbitrary regular diagonal matrix. □

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5. Sobolev Orthogonal Polynomials and Approximation. This kind of discrete–continuous Sobolev orthogonality can be related to simultaneous polynomial interpolation and approximation when (2.1) is an inner product. Assume that u is positive definite and that $\mathbf{A}$ is a positive definite, symmetric and real matrix. Since u is positive definite, there exists a positive definite Borel measure μ satisfying

$$\langle u, f\rangle = \int_{\mathbb{R}} f(x)d\mu(x)$$

(see [2, p. 57]), and the discrete–continuous Sobolev inner product (2.1) can be written as

$$(5.1)\quad (f,g)_S = (f(c_0), f(c_1), \ldots, f(c_{N-1}))\mathbf{A}\begin{pmatrix} g(c_0) \\ g(c_1) \\ \vdots \\ g(c_{N-1}) \end{pmatrix} + \int_{\mathbb{R}} f^{(N)}(x)g^{(N)}(x)d\mu(x).$$

Let I the convex hull of the set supp$(\mu) \cup \{c_i\}_{i=0}^{N-1}$, and introduce the Sobolev space

$$W_2^N[I, d\mu] = \{f : I \longrightarrow \mathbb{R};\quad f \in C^{N-1}(I),\quad f^{(N)} \in L_\mu^2(I)\}.$$

Define the norm $|f|_S = \sqrt{(f,f)_S}$ in $W_2^N[I, d\mu]$; thus $W_2^N[I, d\mu]$ becomes a normed linear space (see [3], p. 160). This space is strictly convex (see [3], p. 141). Therefore the problem of best approximation in $W_2^N[I, d\mu]$ has a unique solution.

We want to compute the best approximation of $f \in W_2^N[I, d\mu]$ related to $\mathbb{P}_n$. It is well know that $v \in \mathbb{P}_n$ is the best approximation of $f \in W_2^N[I, d\mu]$ if and only if $f - v$ is orthogonal to $\mathbb{P}_n$.

THEOREM 5.1. *Let $f \in W_2^N[I, d\mu]$. The best approximation of f in $(\mathbb{P}_n, (\cdot,\cdot)_S)$ is the N–th primitive of the best approximation of $f^{(N)}$ in $(\mathbb{P}_{n-N}, d\mu)$ that interpolates f at $c_0, c_1, \ldots, c_{N-1}$.*

Proof. Let w be the best approximation of $f^{(N)}$ in $(\mathbb{P}_{n-N}, d\mu)$. Let v be the N–th order primitive of w that interpolates f at $c_0, c_1, \ldots, c_{N-1}$. Therefore

$$\begin{aligned}(f-v,q)_S &= (f-v,q)_D + \int_{\mathbb{R}} (f^{(N)} - v^{(N)})q^{(N)}d\mu \\ &= (f-v,q)_D + \int_{\mathbb{R}} (f^{(N)} - w)q^{(N)}d\mu = 0, \quad \forall q \in \mathbb{P}_n.\end{aligned}$$

Thus, v is the best approximation of f in $(\mathbb{P}_n, (\cdot,\cdot)_S)$. □

Let $\{Q_n\}_n$ be the MOPS with respect to the Sobolev discrete–continuous inner product (5.1). Let v be the best approximation of $f \in W_2^N[I, d\mu]$ in $(\mathbb{P}_n, (\cdot,\cdot)_S)$.

We know that

$$v = \sum_{i=0}^{n} \frac{(f,Q_i)_S}{\| Q_i \|_S^2} Q_i,$$

where $\dfrac{(f,Q_i)_S}{\| Q_i \|_S^2}$ are the Fourier coefficients of v.

THEOREM 5.2. *Let $\{Q_n\}_n$ and v be defined as above.*

i) If $n \leq N-1$, *then*

$$v = \sum_{i=0}^{n} \frac{(f, Q_i)_D}{\| Q_i \|_D^2} Q_i.$$

ii) If $n \geq N$, *then*

$$v = \sum_{i=0}^{N-1} \frac{(f, Q_i)_D}{\| Q_i \|_D^2} Q_i + \sum_{i=N}^{n} \frac{(i-N)!}{i!} \frac{\langle u, f^{(N)} P_{i-N} \rangle}{\langle u, P_{i-N}^2 \rangle} Q_i.$$

REMARK. We observe that the coefficients

$$\frac{\langle u, f^{(N)} P_{i-N} \rangle}{\langle u, P_{i-N}^2 \rangle}$$

are the Fourier coefficients of the best approximation of $f^{(N)}$ in $(\mathbb{P}_{n-N}, d\mu)$.

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