**Electronic Transactions on Numerical Analysis.**
**Volume 6, pp. 78-96, December 1997.**

**ISSN 1068-9613.**

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**

# THE ANALYSIS OF INTERGRID TRANSFER OPERATORS AND MULTIGRID METHODS FOR NONCONFORMING FINITE ELEMENTS*

ZHANGXIN CHEN†

**Abstract.** In this paper we first analyze intergrid transfer operators and their iterates for some nonconforming finite elements used for discretizations of second- and fourth-order elliptic problems. Then two classes of multigrid methods using these elements are considered. The first class is the usual one, which uses discrete equations on all levels which are defined by the same discretization, while the second one is based on the Galerkin approach where quadratic forms over coarse grids are constructed from the quadratic form on the finest grid and the iterates of intergrid transfer operators, which we call the Galerkin multigrid method. The properties of these intergrid transfer operators are utilized for the analysis of the first class, while the properties of their iterates are exploited for the second one. Convergence results available for these two classes of multigrid methods are summarized here.

**Key words.** multigrid methods, nonconforming and mixed finite elements, second and fourth-order problems, intergrid operators.

**AMS subject classifications.** 65N30, 65N22, 65F10.

**1. Introduction.** The study of multigrid methods for nonconforming finite elements started in the later 1980s. Multigrid methods using the $P_1$-nonconforming element for second-order problems (i.e., the Crouzeix-Raviart element [28]) have been considered in [7, 12, 16, 19, 22, 25, 32, 35, 49, 50], while these methods for the rotated $Q_1$-nonconforming element [18, 41] for the same differential problems have been analyzed in [2, 16, 26]. Multigrid methods for the Morley nonconforming element [34] for the biharmonic equation have been developed in [13, 16, 29, 31, 38, 39, 42, 49], and for the plate bending problems using the Zienkiewicz [5] and Adini [1] nonconforming elements have been described in [36, 40, 44, 48, 51]. Finally, these methods for the $P_1$ and rotated $Q_1$-nonconforming divergence-free elements for the stationary Stokes problem have been studied in [14, 15, 45]. In all these earlier papers except in [26], only the $\mathcal{W}$-cycle multigrid methods have been shown to converge under the assumption that the number of smoothing iterations on all levels is sufficiently large. The methodology developed for the multigrid methods of conforming finite elements in [4] has been extensively employed to analyze the nonconforming multigrid methods; the convergence study is based on establishment of the so-called smoothing and approximation properties and analysis of a two-level scheme.

Multigrid methods for nonconforming finite elements have the feature that the multilevel finite element spaces are nonnested and the quadratic forms defined on these spaces are non-inherited. Consequently, the convergence proof of the conforming multigrid methods introduced in [6] does not apply to the nonconforming case since coarse-to-fine intergrid transfer operators for nonconforming finite elements do not preserve the energy norm. That is why the approach in [4] has been mainly exploited in the analysis of the nonconforming multigrid methods in the last decade. In multigrid methods for nested conforming finite elements the multilevel finite element spaces are nested and the quadratic forms are inherited.

The purpose of this paper is to analyze intergrid transfer operators and their iterates for some nonconforming finite elements used for discretizations of second- and fourth-order elliptic problems and to discuss convergence of two classes of multigrid methods using these

* Received May 17, 1997. Accepted August 17, 1997. Communicated by I. Yavneh. This research was supported by National Science Foundation grant DMS-9626179.

† Department of Mathematics, Box 750156, Southern Methodist University, Dallas, Texas 75275–0156 U.S.A, (zchen@dragon.math.smu.edu)

elements. The first class is the usual one, which uses discrete equations on all levels which are defined by the same discretization. The methodology developed in [11] where nonnested spaces and non-inherited quadratic forms are allowed shall be applied to analyze this class of nonconforming multigrid methods. Toward that end, we shall need to find the lower and upper bounds of the energy norm of the usual coarse-to-fine intergrid transfer operators for the nonconforming elements considered here. In [26], it has been shown that the bound of the energy norm of the edge averaging intergrid transfer operators for the rotated $Q_1$-nonconforming element is not bigger than two. As a result of this, the theory of [11] shows the convergence of the $\mathcal{W}$-cycle multigrid methods with any number of smoothing iterations for this element. In this paper, we shall discuss the applicability of this result to the $P_1$, Morley, Zienkiewicz, and Adini nonconforming elements.

The second class of multigrid methods was recently introduced in [20] and is based on the "Galerkin approach" where quadratic forms over coarse grids are constructed from the quadratic form on the finest grid and iterated coarse-to-fine intergrid operators, which we call the Galerkin multigrid method. This approach automatically leads to the case where the coarse-to-fine intergrid transfer operators preserve the energy norm. However, to apply the convergence theory of the conforming multigrid methods [6, 8], a key ingredient is to prove upper bounds of the iterated intergrid transfer operators in terms of the energy norm. These bounds have been shown for the $P_1$ element in [35] and for the rotated $Q_1$ element in [26]. Here we shall discuss them for the Morley, Zienkiewicz, and Adini elements. The convergence of both the $\mathcal{V}$-cycle and $\mathcal{W}$-cycle multigrid methods with any number of smoothing steps for these nonconforming elements using the second approach is considered. Convergence results for partial differential problems with less than full elliptic regularity and without any elliptic regularity are considered. Problems related to the discontinuity in the coefficient of differential problems are not discussed here.

In recent years, the study of multigrid methods for mixed finite element methods, which are popular in the simulation of fluid flow in porous media [21], has been quite active; see [2, 19, 31, 43, 46, 47], for example. However, due to the equivalence between nonconforming and mixed finite element methods [2, 3, 17, 19, 23], the analysis for the nonconforming finite methods directly applies to the mixed methods. Thus all the results derived here carry over to the mixed methods. Also, the present techniques can be used to analyze other nonconforming elements.

The rest of the paper is organized as follows. In the next section we analyze the coarse-to-fine intergrid operators and their iterates; the above mentioned nonconforming elements are treated there. Then in the third section we analyze the two approachs for defining multigrid methods; partial differential problems with less than full elliptic regularity and without elliptic regularity are handled in this section.

## 2. Analysis of Intergrid Transfer Operators.

In this section we analyze the usual coarse-to-fine intergrid transfer operators and their iterates for the $P_1$, rotated $Q_1$, Morley, Zienkiewicz, and Adini nonconforming finite elements.

### 2.1. The $P_1$-nonconforming element.

In this subsection we consider the numerical solution of the model problem

$$\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

using the $P_1$-nonconforming finite element method, where $\Omega \subset \mathbb{R}^2$ is a simply connected bounded polygonal domain with the boundary $\partial\Omega$, $f \in L^2(\Omega)$, and the symmetric coefficient

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



$\mathcal{A} \in (L^\infty(\Omega))^{2\times 2}$ satisfies

$$\xi^t \mathcal{A}(x)\xi \geq a_0 \xi^t \xi, \qquad x \in \Omega,\, \xi \in \mathbb{R}^2, \tag{2.2}$$

with a fixed constant $a_0 > 0$.

Problem (2.1) is recast in weak form as follows. The quadratic form $a(\cdot,\cdot)$ is defined by

$$a(v,w) = (\mathcal{A}\nabla v, \nabla w), \qquad v,w \in H^1(\Omega),$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ or $(L^2(\Omega))^2$ inner product, as appropriate. Then the weak form of (2.1) is, find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall\, v \in H_0^1(\Omega). \tag{2.3}$$

For $0 < h < 1$, let $\mathcal{E}_h$ be a triangulation of $\Omega$ into triangles $\{E\}$ of diameters $h_E$, which are not bigger than $h$, and define the $P_1$-nonconforming finite element space [28]

$$V_h = \{v \in L^2(\Omega):\ v|_E \text{ is linear for all } E \in \mathcal{E}_h, v \text{ is continuous at the midpoints of interior edges, and } v \text{ vanishes at the midpoints of edges on } \partial\Omega\}.$$

Note that $V_h \not\subset H_0^1(\Omega)$. Associated with $V_h$, we introduce a quadratic form on $V_h \oplus H_0^1(\Omega)$ by

$$a_h(v,w) = \sum_{E\in\mathcal{E}_h} (\mathcal{A}\nabla v, \nabla w)_E, \qquad v,w \in V_h \oplus H_0^1(\Omega),$$

where $(\cdot,\cdot)_E$ is the $L^2(E)$ inner product. Then the $P_1$-nonconforming finite element discretization of (2.1) is, find $u_h \in V_h$ such that

$$a_h(u_h,v) = (f,v), \qquad \forall\, v \in V_h. \tag{2.4}$$

To apply the multigrid methods introduced in the next section for solving (2.4), we assume a structure to our family of partitions. Let $h_0$ and $\mathcal{E}_{h_0} = \mathcal{E}_0$ be given. For each integer $1 \leq k \leq K$, let $h_k = 2^{-k}h_0$ and $\mathcal{E}_{h_k} = \mathcal{E}_k$ be constructed by connecting the midpoints of the edges of the triangle in $\mathcal{E}_{k-1}$, and let $\mathcal{E}_h = \mathcal{E}_K$ be the finest grid. In this and the following sections, we shall replace subscript $h_k$ simply by subscript $k$.

Since $V_{k-1} \not\subset V_k$ (i.e., nonnested), we need to introduce intergrid transfer operators to connect them. Following [7, 12], the coarse-to-fine intergrid transfer operator $I_k : V_{k-1} \to V_k$ for $k = 1,\ldots,K$ is defined as follows. For $v \in V_{k-1}$, let $q$ be a midpoint of an edge of a triangle in $\mathcal{E}_k$; then we define $I_k v$ by

$$(I_k v)(q) = \begin{cases} 0 & \text{if } q \in \partial\Omega, \\ v(q) & \text{if } q \notin \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \frac{1}{2}\{v|_{E_1}(q) + v|_{E_2}(q)\} & \text{if } q \in \partial E_1 \cap \partial E_2 \text{ for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases}$$

We also define the iterates of $I_k$ by

$$H_k^K = I_K \cdots I_{k+1} : V_k \to V_K. \tag{2.5}$$

We now state the boundedness of the operators $I_k$ and $H_k^K$, which will be used in the next section and was shown in [7, 12] and [35], respectively. Below $C$ (with or without a subscript) denotes a generic positive constant, which may take on different values in different

**ETNA**
Kent State University
etna@mcs.kent.edu



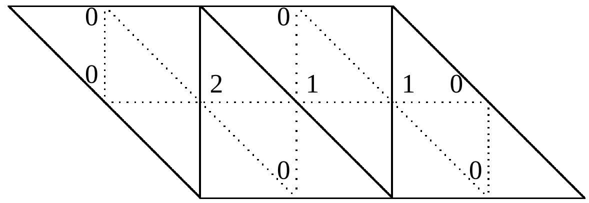


FIG. 1. *The definition of the function $v$ in Example 1.*

occurrences. For the inequality (2.7) below, we assume that there meet at most six edges at each interior vertex in $\mathcal{E}_0$ and four edges at each boundary vertex. This is easily satisfied.

PROPOSITION 2.1. *There exist constants $C$ independent of $k$ such that*

$$a_k(I_k v, I_k v) \le C a_{k-1}(v,v), \quad \forall\, v \in V_{k-1}, \tag{2.6}$$

*and*

$$a_K(H_k^K v, H_k^K v) \le C a_k(v,v), \quad \forall\, v \in V_k. \tag{2.7}$$

Inequalities (2.6) and (2.7) will be used in the analysis of the first and second classes of multigrid methods considered in the next section, respectively. While the value of $C$ in (2.7) is not important for analyzing the second class, the value in (2.6) is critical in applying the theory of [11] to the first one. Different values yield different consequences for the convergence of the $\mathcal{V}$- and $\mathcal{W}$-cycle multigrid methods (see the next section). We here show, via the following example, that the constant $C$ in (2.6) is generally bigger than two for the $P_1$ element.

*Example* 1. Let $\Omega$ be given as in Figure 1 and $v$ be in $V_0$ with the nodal values determined in this figure, where the dotted lines indicate refinement. Then with $\mathcal{A} = I$ it can be checked that

$$a_0(v,v) = 16,$$

and

$$a_1(I_1 v, I_1 v) = 32.5.$$

Consequently,

$$a_1(I_1 v, I_1 v) > 2 a_0(v,v).$$

*Example* 2. We report numerical results to illustrate the behavior of the energy norm of the iterates $H_0^K$,

$$\beta_K = \sup_{\phi_0 \in V_0} \frac{a_K(H_0^K \phi_0, H_0^K \phi_0)}{a_0(\phi_0, \phi_0)},$$

over the basis functions $\phi_0 \in V_0$. The results are given in Table 1, where $\mathcal{A} = I$ and $\Omega = (0,1)^2$ are taken in (2.1). From the table, we see numerical evidence to the fact that $\beta_K$ is uniformly bounded for the $P_1$ element. This agrees with (2.7). For details on the numerical results of $\beta_K$ reported in this paper, see [20].

**ETNA**
Kent State University
etna@mcs.kent.edu



| $K$ | $\beta_K$ | $\beta_K/\beta_{K-1}$ |
|---|---|---|
| 1 | 0.16875E+01 | 1.6875 |
| 2 | 0.21250E+01 | 1.2593 |
| 3 | 0.24141E+01 | 1.1360 |
| 4 | 0.26066E+01 | 1.0797 |
| 5 | 0.27358E+01 | 1.0496 |
| 6 | 0.28228E+01 | 1.0318 |

Table 1. The $P_1$ element.

**2.2. The rotated $Q_1$-nonconforming element.** We now consider the rotated $Q_1$-nonconforming element for (2.1). For this, let $\mathcal{E}_{h_0} = \mathcal{E}_0$ be a triangulation of $\Omega$ into rectangles having maximum diameter $h_0$ and oriented along the coordinate axes. For each integer $1 \leq k \leq K$, let $h_k = 2^{-k}h_0$ and $\mathcal{E}_{h_k} = \mathcal{E}_k$ be constructed by connecting the midpoints of the edges of the rectangle in $\mathcal{E}_{k-1}$, and let $\mathcal{E}_h = \mathcal{E}_K$ be the finest grid. For each $k$, the rotated $Q_1$ nonconforming space is defined by, see [18, 41],

$$V_k = \Big\{ v \in L^2(\Omega) : \ v|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4 (x^2 - y^2), \ a_E^i \in \mathbb{R}, \ \forall E \in \mathcal{E}_k; \\ \text{if } E_1 \text{ and } E_2 \text{ share an edge } e, \text{ then } \int_e v|_{\partial E_1} \mathrm{d}s = \int_e v|_{\partial E_2} \mathrm{d}s; \\ \text{and } \textstyle\int_{\partial E \cap \partial \Omega} v|_{\partial\Omega} \mathrm{d}s = 0 \Big\}.$$

Since $V_k \not\subset H_0^1(\Omega)$ and $V_{k-1} \not\subset V_k$, following [2, 18], we define the coarse-to-fine intergrid transfer operators $I_k : V_{k-1} \to V_k$ as follows. If $v \in V_{k-1}$ and $e$ is an edge of a rectangle in $\mathcal{E}_k$, then $I_k v \in V_k$ is defined by

$$\int_e I_k v \mathrm{d}s = \begin{cases} 0 & \text{if } e \subset \partial\Omega, \\ \displaystyle\int_e v \mathrm{d}s & \text{if } e \not\subset \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \displaystyle\frac{1}{2}\int_e (v|_{E_1} + v|_{E_2}) \mathrm{d}s & \text{if } e \subset \partial E_1 \cap \partial E_2 \\ & \ \text{for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases} \tag{2.8}$$

Their iterates are defined as in (2.5). Also, we have the following boundedness of $I_k$ and $H_k^K$, which was proven in [2] and [26], respectively. Equation (2.10) below was shown for square partitions of a square. Extensions to other domains and triangulations were discussed in [26]; it holds for polygonal domains if their initial triangulation into quadrilaterals is topologically equivalent to a uniform square partition of $\Omega = (0, 1)^2$, for example. Hence, whenever (2.10) is used below, this condition is assumed.

PROPOSITION 2.2. *There are constants $C$ independent of $k$ such that*

$$a_k(I_k v, I_k v) \leq C a_{k-1}(v, v), \quad \forall\, v \in V_{k-1}, \tag{2.9}$$

*and*

$$a_K(H_k^K v, H_k^K v) \leq C a_k(v, v), \quad \forall\, v \in V_k. \tag{2.10}$$

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



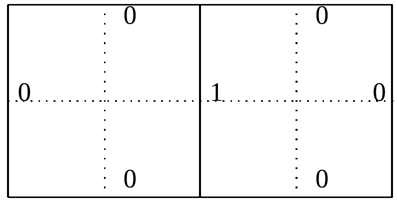


FIG. 2. *The definition of the function* $v$ *in Example 3.*

We now consider a simple case of the model problem (2.1) where the coefficient $\mathcal{A}$ is constant; i.e., $\mathcal{A} = I$. In this case we shall show, by the next example, that the constant $C$ in (2.9) is generally bigger than one. However, it is not bigger than two, as stated in Proposition 2.3 below (see its proof in [26]).

*Example* 3. Let $\Omega$ be as in Figure 2 and $v$ be in $V_0$ with the integral averaging values over edges given in this figure. Then with $\mathcal{A} = I$ it can be shown that

$$a_0(v, v) = 5,$$

and

$$a_1(I_1 v, I_1 v) = 201/32.$$

Hence we find that

$$a_1(I_1 v, I_1 v) > a_0(v, v).$$

PROPOSITION 2.3. *With* $\mathcal{A} = I$*, it holds that*

$$a_k(I_k v, I_k v) \le 2 a_{k-1}(v, v), \quad \forall\, v \in V_{k-1}. \tag{2.11}$$

*Example* 4. As for the $P_1$ element, here we report numerical results on $\beta_K$. The same data are taken as in Example 2 except that now $\mathcal{E}_K$ is a square partition. From Table 2, we also see numerical evidence that $\beta_K$ is uniformly bounded for the rotated $Q_1$ element, which agrees with (2.10).

| $K$ | $\beta_K$ | $\beta_K/\beta_{K-1}$ |
|---|---|---|
| 1 | 0.11875E+01 | 1.1875 |
| 2 | 0.13393E+01 | 1.1278 |
| 3 | 0.14249E+01 | 1.0639 |
| 4 | 0.14719E+01 | 1.0330 |
| 5 | 0.14970E+01 | 1.0171 |
| 6 | 0.15103E+01 | 1.0089 |

Table 2. The rotated $Q_1$ element.

We end with a remark that the rotated $Q_1$ element also can be defined with degrees of freedom given by the values at the midpoints of edges of the elements. However, (2.10) and (2.11) do not hold with this definition [26].

**2.3. The Morley nonconforming element.** In this and the next two subsections, we consider the numerical solution of the fourth-order problem

$$\begin{aligned} \triangle^2 u &= f \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.12}$$

using the Morley, Zienkiewicz, and Adini nonconforming finite element methods, respectively. Now the quadratic form $a(\cdot,\cdot)$ is given by

$$a(v,w) = (v_{xx}, w_{xx}) + 2(v_{xy}, w_{xy}) + (v_{yy}, w_{yy}), \quad v,\, w \in H^2(\Omega).$$

The weak form of (2.12) is, find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall\, v \in H_0^2(\Omega). \tag{2.13}$$

It has a unique solution [27].

Let $\{\mathcal{E}_k\}_{k=0}^K$ be the family of dyadically refined triangulations of $\Omega$ into triangles as defined in §2.1. For each $k$, we define the Morley element, see [34],

$$\begin{aligned} V_k = \{v \in L^2(\Omega) :\ & v|_E \in P_2(E) \text{ for all } E \in \mathcal{E}_k;\ v \text{ is continuous at the} \\ & \text{vertices and vanishes at the vertices on } \partial\Omega;\ \text{and} \\ & \partial v/\partial\nu \text{ is continuous at the midpoints of interior} \\ & \text{edges and vanishes at the midpoints of edges on } \partial\Omega\}. \end{aligned}$$

Note that $V_k \not\subset C^0(\bar{\Omega})$. Associated with $V_k$, $a_k(\cdot,\cdot)$ is defined by

$$a_k(v,w) = \sum_{E\in\mathcal{E}_k} \big\{(v_{xx}, w_{xx})_E + 2(v_{xy}, w_{xy})_E + (v_{yy}, w_{yy})_E\big\}, \quad v,\, w \in V_k.$$

Then the approximate method for (2.12) using the Morley element is determined as in (2.4).

The coarse-to-fine intergrid transfer operator $I_k : V_{k-1} \to V_k$ for $k = 1, \ldots, K$ is again the usual averaging operator, which is given as follows. For $v \in V_{k-1}$, let $q$ be a vertex of a triangle and $\bar{q}$ the midpoint of an edge of a triangle in $\mathcal{E}_k$; then we define $I_k v$ by [13, 39]

$$(I_k v)(q) = \begin{cases} 0 & \text{if } q \in \partial\Omega, \\ v(q) & \text{if } q \text{ is also a vertex in } \mathcal{E}_{k-1}, \\ \frac{1}{2}\{v|_{E_1}(q) + v|_{E_2}(q)\} & \text{if } q \text{ is not a vertex in } \mathcal{E}_{k-1}, \end{cases}$$

and

$$\frac{\partial}{\partial\nu}(I_k v)(\bar{q}) = \begin{cases} 0 & \text{if } \bar{q} \in \partial\Omega, \\ \frac{\partial v}{\partial\nu}(\bar{q}) & \text{if } \bar{q} \notin \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \frac{1}{2}\left\{\frac{\partial v|_{E_1}}{\partial\nu}(\bar{q}) + \frac{\partial v|_{E_2}}{\partial\nu}(\bar{q})\right\} & \text{if } \bar{q} \in \partial E_1 \cap \partial E_2 \\ & \quad \text{for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases}$$

The iterates $H_k^K$ of $I_k$ are defined as in (2.5).

We have the following result for the boundedness of the operator $I_k$, c.f. [13, 39]. Note that we cannot control the growth of the energy norm of $H_k^K$. In fact, the energy norm grows exponentially with the number of grid levels, as is demonstrated numerically in Example 6 below.

PROPOSITION 2.4. *There is a constant $C$ independent of $k$ such that*

$$a_k(I_k v, I_k v) \le C a_{k-1}(v,v), \quad \forall\, v \in V_{k-1}. \tag{2.14}$$

ETNA
Kent State University
etna@mcs.kent.edu



We now show, via the next example, that the constant $C$ in (2.14) is generally bigger than two.

*Example* 5. Let $\Omega$ be as in Figure 1 and $v \in V_0$ such that $v$ is zero at all the vertices and $\partial v/\partial \nu$ has the values at the midpoints as displayed in this figure. Then we see that

$$a_0(v,v) = 28 - 6\sqrt{2}$$

and

$$a_1(I_1 v, I_1 v) = \frac{231}{4} - \frac{147}{16}\sqrt{2}.$$

Thus, we have

$$a_1(I_1 v, I_1 v) > 2a_0(v,v).$$

*Example* 6. Numerical results for the $\beta_K$ with $\Omega = (0,1)^2$ are presented in Table 3 for the Morley element.

| $K$ | $\beta_K$ | $\beta_K/\beta_{K-1}$ |
|---|---|---|
| 1 | 0.19375E+01 | 1.9375 |
| 2 | 0.34297E+01 | 1.7702 |
| 3 | 0.63681E+01 | 1.8568 |
| 4 | 0.12549E+02 | 1.9706 |
| 5 | 0.25969E+02 | 2.0694 |
| 6 | 0.55608E+02 | 2.1413 |

Table 3. The Morley element.

**2.4. The Zienkiewicz element.** We now turn to the Zienkiewicz nonconforming element. For this, we define

$$a(v,w) = (\Delta v, \Delta w) + (1-\sigma)\{2(v_{xy}, w_{xy}) - (v_{xx}, w_{yy}) - (v_{yy}, w_{xx})\}, \qquad v,\ w \in H^2(\Omega),$$

where $0 < \sigma < 1/2$ is the Poisson ratio [27]. Then the weak form of (2.12) for the Zienkiewicz method is, find $u \in H_0^2(\Omega)$ such that (2.13) holds [27].

Let $\{\mathcal{E}_k\}_{k=0}^K$ again be the family of dyadically refined triangulations of $\Omega$ into triangles as defined in §2.1. For each $k$, we define the Zienkiewicz element, see [5],

$$V_k = \{v : v|_E \in P_3(E),\ v(q_E^c) = \tfrac{1}{3}\textstyle\sum_{i=1}^3 v(q_E^i) - \tfrac{1}{6}\sum_{i=1}^3 (q_E^i - q_E^c)\cdot \nabla v(q_E^i),$$
$$\text{for all } E \in \mathcal{E}_k;\ v,\ v_x, \text{ and } v_y \text{ are continuous at the vertices of } \mathcal{E}_k \text{ and vanish at the vertices on } \partial\Omega\},$$

where the $q_E^i$ are the vertices of $E$ and $q_E^c$ is the centroid of $E \in \mathcal{E}_k$. Note that $V_k \subset C^0(\bar{\Omega})$, but $V_k \not\subset C^1(\bar{\Omega})$. For each $V_k$, we define

$$a_k(v,w) = \sum_{E\in\mathcal{E}_k} \big\{(\Delta v, \Delta w)_E + (1-\sigma)\{2(v_{xy}, w_{xy})_E - (v_{xx}, w_{yy})_E - (v_{yy}, w_{xx})_E\}\big\}, \quad v,\ w \in V_k.$$

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



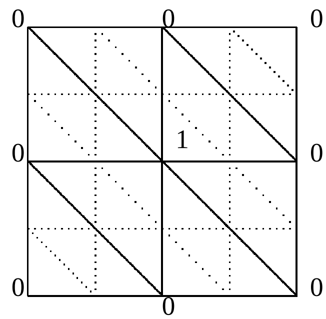


FIG. 3. *The definition of the function $v$ in Example 7.*

With this the Zienkiewicz nonconforming method is defined as in (2.4).

The intergrid transfer operators $I_k : V_{k-1} \to V_k$ is described as follows. For $v \in V_{k-1}$, if $q$ is a vertex of a triangle in $\mathcal{E}_{k-1}$ and $\bar{q}$ is the midpoint of an edge of a triangle in $\mathcal{E}_{k-1}$, then $I_k v \in V_k$ is determined by

$$
\begin{aligned}
&(I_k v)(q) = v(q), \quad \nabla(I_k v)(q) = \nabla v(q),\\
&(I_k v)(\bar{q}) = \begin{cases} 0 & \text{if } \bar{q} \in \partial\Omega,\\ v(\bar{q}) & \text{if } \bar{q} \notin \partial\Omega, \end{cases}\\
&\nabla(I_k v)(\bar{q}) = \begin{cases} 0 & \text{if } \bar{q} \in \partial\Omega,\\ \frac{1}{2}\left\{\nabla v|_{E_1}(\bar{q}) + \nabla v|_{E_2}(\bar{q})\right\} & \text{if } \bar{q} \in \partial E_1 \cap \partial E_2\\ & \quad \text{for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases}
\end{aligned}
$$

The inequality (2.15) below regarding the boundedness of $I_k$ can be seen in [40]. The constant $C$ in this inequality is generally bigger than one, as shown in Example 7 below. However, we have numerically observed that it is not bigger than two. A theoretical proof of this fact is yet to be given. Numerical evidence of the boundedness of the iterates $H_k^K$ can be seen in Example 8 below.

PROPOSITION 2.5. *There exists a constant $C$ independent of $k$ such that*

$$
a_k(I_k v, I_k v) \le C a_{k-1}(v, v), \quad \forall\, v \in V_{k-1}. \tag{2.15}
$$

*Example* 7. Let $\Omega = (0,1)^2$ be determined as in Figure 3 and $v \in V_0$ such that $\partial v/\partial x$ and $\partial v/\partial y$ are zero at all the vertices and $v$ has the nodal values at the vertices as determined by this figure. Then we find that

$$
a_0(v, v) = 192,
$$

and

$$
a_1(I_1 v, I_1 v) = 317.58 - 44.5\sigma.
$$

Thus we observe that

$$
a_1(I_1 v, I_1 v) > a_0(v, v) \quad \text{for} \quad 0 < \sigma < 1/2.
$$

*Example* 8. Numerical results for the $\beta_K$ for the Zienkiewicz element with $\Omega = (0,1)^2$ are described in Table 4.

ETNA
Kent State University
etna@mcs.kent.edu



| $K$ | $\beta_K$ | $\beta_K/\beta_{K-1}$ |
|---|---|---|
| 1 | 0.16541E+01 | 1.6541 |
| 2 | 0.20824E+01 | 1.2589 |
| 3 | 0.23053E+01 | 1.1070 |
| 4 | 0.24088E+01 | 1.0449 |
| 5 | 0.24539E+01 | 1.0187 |
| 6 | 0.24729E+01 | 1.0077 |

Table 4. The Zienkiewicz element.

**2.5. The Adini nonconforming element.** We now consider the Adini nonconforming element. The quadratic form $a(\cdot,\cdot)$ is defined as in §2.4. Let $\{\mathcal{E}_k\}_{k=0}^K$ be the family of dyadically refined triangulations of $\Omega$ into rectangles as defined in §2.2. For each $k$, we define the Adini element, see [1],

$$\begin{aligned} V_k = \{v \in L^2(\Omega) : v|_E \in P_3(E) \oplus \{x^3y\} \oplus \{xy^3\} \text{ for all } E \in \mathcal{E}_k;\\ v,\ v_x, \text{ and } v_y \text{ are continuous at the vertices}\\ \text{of } \mathcal{E}_k \text{ and vanish at the vertices on } \partial\Omega\}. \end{aligned}$$

Again, $V_k \subset C^0(\bar{\Omega})$, but $V_k \not\subset C^1(\bar{\Omega})$. The quadratic form $a_k(\cdot,\cdot)$ is given as in §2.4, and the Adini nonconforming method is defined as in (2.4).

The intergrid transfer operator $I_k : V_{k-1} \to V_k$ is modified as follows. For $v \in V_{k-1}$, if $q$ is a vertex of a rectangle in $\mathcal{E}_{k-1}$, $\bar{q}$ is the midpoint of an edge of a rectangle in $\mathcal{E}_{k-1}$, and $q^c$ is the center of a rectangle in $\mathcal{E}_{k-1}$, then $I_k v \in V_k$ is determined by

$$\begin{aligned} &(I_kv)(q) = v(q), \qquad \nabla(I_kv)(q) = \nabla v(q),\\ &(I_kv)(q^c) = v(q^c), \quad \nabla(I_kv)(q^c) = \nabla v(q^c),\\ &(I_kv)(\bar{q}) = \begin{cases} 0 & \text{if } \bar{q} \in \partial\Omega,\\ v(\bar{q}) & \text{if } \bar{q} \notin \partial\Omega, \end{cases}\\ &\nabla(I_kv)(\bar{q}) = \begin{cases} 0 & \text{if } \bar{q} \in \partial\Omega,\\ \frac{1}{2}\left\{\nabla v|_{E_1}(\bar{q}) + \nabla v|_{E_2}(\bar{q})\right\} & \text{if } \bar{q} \in \partial E_1 \cap \partial E_2\\ & \quad \text{for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases} \end{aligned}$$

Similar properties for $I_k$ and $H_k^K$ to those for the Zienkiewicz element have been observed for the Adini element; see Proposition 2.6 [36] and Examples 9 and 10 below.

PROPOSITION 2.6. *There is a constant $C$ independent of $k$ such that*

$$a_k(I_kv, I_kv) \le Ca_{k-1}(v,v), \quad \forall\, v \in V_{k-1}. \tag{2.16}$$

*Example* 9. Let $\Omega = (0,1)^2$ be given as in Figure 4 and $v \in V_0$ such that $v$ and $\partial v/\partial y$ are zero at all the vertices and $\partial v/\partial x$ has the nodal values at the vertices as determined by this figure. Then we have

$$a_0(v,v) = (176 - 16\sigma)/30,$$

and

$$a_1(I_1v, I_1v) = (193 - 8\sigma)/30,$$

**ETNA**
Kent State University
etna@mcs.kent.edu



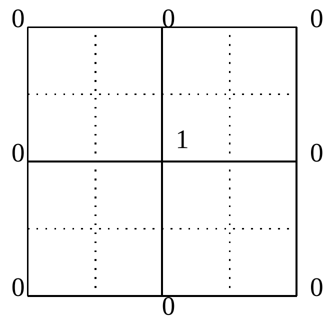


FIG. 4. *The definition of the function $v$ in Example 9.*

so that

$$a_1(I_1 v, I_1 v) > a_0(v, v) \quad \text{for} \quad 0 < \sigma < 1/2.$$

*Example* 10. Numerical results for the $\beta_K$ for the Adini element with $\Omega = (0,1)^2$ are displayed in Table 5.

| $K$ | $\beta_K$ | $\beta_K/\beta_{K-1}$ |
|---|---|---|
| 1 | 0.10966E+01 | 1.0966 |
| 2 | 0.11767E+01 | 1.0730 |
| 3 | 0.12088E+01 | 1.0273 |
| 4 | 0.12189E+01 | 1.0084 |
| 5 | 0.12219E+01 | 1.0025 |
| 6 | 0.12228E+01 | 1.0007 |

Table 5. The Adini element.

For all the nonconforming elements tested here except for the Adini elements, numerical results for $\beta_K$ were also reported in [37].

**3. Analysis of Multigrid Methods.** In this section, we apply the results of the previous section to derive convergence of multigrid methods. We state several theorems to illustrate the type of convergence results available utilizing the estimates on the intergrid transfer operators and their iterates. We first state convergence results in a general setting. Two approaches of defining multigrid methods are then discussed. Partial differential problems with less than full elliptic regularity and without elliptic regularity are considered.

**3.1. Multigrid methods.** We assume that we are given a sequence of nonconforming finite element spaces

$$V_0,\ V_1, \ldots, V_K,$$

along with the nonsingular coarse-to-fine grid operators $I_k : V_{k-1} \to V_k$ for $k = 1, \ldots, K$. In addition, assume that we are given symmetric positive definite quadratic forms $a_k(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ over $V_k \times V_k$ for $k = 0, \ldots, K$. Finally, suppose that we are given another family of symmetric positive definite quadratic forms $b_k(\cdot,\cdot)$ over $V_k \times V_k$ for $k = 0, \ldots, K$ such that $b_K(\cdot,\cdot) = a_K(\cdot,\cdot)$. On all lower levels, $b_k(\cdot,\cdot)$ may be different from $a_k(\cdot,\cdot)$. The norms corresponding to $(\cdot,\cdot)_k$ and $b_k(\cdot,\cdot)$ will be denoted by $||\cdot||_k$ and $||\cdot||_{1,k}$, respectively. Examples of spaces, operators, and quadratic forms will be given later in this section.

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



Given $f \in V_K$, the multigrid methods will be designed for the solution of the problem: Find $u_K \in V_K$ such that

$$a_K(u_K, v) = (f, v)_K, \qquad \forall v \in V_K. \tag{3.1}$$

To introduce them, we define the discretization operator $A_k : V_k \to V_k$ on level $k$ given by

$$(A_k v, w)_k = b_k(v, w), \qquad \forall\, w \in V_k,\ k = 0, \ldots, K. \tag{3.2}$$

Note that the operator $A_k$ is clearly symmetric (in both the $b_k(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ inner products) and positive definite. Also, we define the operators $P_{k-1} : V_k \to V_{k-1}$ and $P^0_{k-1} : V_k \to V_{k-1}$ by

$$b_{k-1}(P_{k-1}v, w) = b_k(v, I_k w), \qquad \forall\, w \in V_{k-1},\ k = 1, \ldots, K, \tag{3.3}$$

and

$$\left(P^0_{k-1}v, w\right)_{k-1} = (v, I_k w)_k, \qquad \forall\, w \in V_{k-1},\ k = 1, \ldots, K.$$

It is obvious that $I_k P_{k-1}$ is a symmetric operator with respect to the $b_k$ form. Finally, let $R_k : V_k \to V_k$ for $k = 1, \ldots, K$ be the linear operators associated with the point Jacobi or Gauss-Seidel smoothing procedures, let $R^t_k$ denote the adjoint of $R_k$ with respect to the $(\cdot,\cdot)_k$ inner product, and define

$$R_k^{(l)} = \begin{cases} R_k & \text{if } l \text{ is odd,} \\ R_k^t & \text{if } l \text{ is even.} \end{cases}$$

On $V_0$, let $R_0 = A_0^{-1}$; i.e., we solve exactly on the coarsest level. Following [11], the multigrid operator $B_k : V_k \to V_k$ is defined recursively as follows:

MULTIGRID METHOD 3.1. Let $1 \le k \le K$ and $p$ be a positive integer. Let $B_0 = A_0^{-1}$. Assume that $B_{k-1}$ has been defined and define $B_k g$ for $g \in V_k$ as follows:

1. Let $x^0 = 0$ and $z^0 = 0$.
2. Define $x^l$ for $l = 1, \ldots, m(k)$ by
$$x^l = x^{l-1} + R_k^{(l+m(k))}(g - A_k x^{l-1}).$$
3. Define $y^{m(k)} = x^{m(k)} + I_k z^p$, where $z^i$ for $i = 1, \ldots, p$ is defined by
$$z^i = z^{i-1} + B_{k-1}\left[P^0_{k-1}\left(g - A_k x^{m(k)}\right) - A_{k-1} z^{i-1}\right].$$
4. Define $y^l$ for $l = m(k)+1, \ldots, 2m(k)$ by
$$y^l = y^{l-1} + R_k^{(l+m(k))}\left(g - A_k y^{l-1}\right).$$
5. Let $B_k g = y^{2m(k)}$.

In the Multigrid Method (MG) 3.1, $m(k)$ gives the number of pre- and post-smoothing iterations and can vary as a function of $k$. The values $p = 1$ and $p = 2$ yield the so-called $\mathcal{V}$- and $\mathcal{W}$-cycle multigrid methods, respectively. A variable $\mathcal{V}$-cycle method is one in which the number of smoothings $m(k)$ increases exponentially as $k$ decreases (i.e., $p = 1$ and $m(k) = 2^{K-k}$). Other versions of multigrid methods without pre- or post-smoothing iterations can be analyzed similarly.

To apply the convergence theory developed in [11] for analyzing MG 3.1, we need the following two estimates:

$$b_k(I_k v, I_k v) \le C_* b_{k-1}(v,v), \qquad \forall\, v \in V_{k-1}, \tag{3.4}$$

and

$$|b_k\left((I - I_k P_{k-1})\, v, v\right)| \le C_\alpha \left(\frac{\|A_k v\|_k^2}{\lambda_k}\right)^{\alpha/l} b_k(v,v)^{1-(\alpha/l)}, \qquad \forall\, v \in V_k, \tag{3.5}$$

for $k = 1, \ldots, K$, where $C_*$ and $C_\alpha$ are constants independent of $k$, $\lambda_k$ is the largest eigenvalue of $A_k$, $0 < \alpha \le 1$, $l = 1$ for second-order problems, and $l = 2$ for forth-order problems. The convergence rate for MG 3.1 on the $k$th level is measured by a convergence factor $\delta_k$ satisfying

$$|b_k\left((I - B_k A_k) v, v\right)| \le \delta_k b_k(v,v), \qquad \forall\, v \in V_k,\ k = 0, \ldots, K. \tag{3.6}$$

THEOREM 3.1. *Assume that* (3.4) *with* $C_* = 1$ *and* (3.5) *are satisfied. Then we have the following cases:*

(i) *Define* $B_k$ *by* $p = 1$ *and* $m(k) = m$ *for all* $k$ *in MG* 3.1. *Then inequality* (3.6) *holds with*

$$\delta_k = \frac{C k^{(l-\alpha)/\alpha}}{C k^{(l-\alpha)/\alpha} + m^{\alpha/l}}.$$

(ii) *Define* $B_k$ *by* $p = 2$ *and* $m(k) = m$ *for all* $k$ *in MG* 3.1. *Then* (3.6) *holds with* $\delta_k = \delta$ *(independent of* $k$*) given by*

$$\delta = \frac{C}{C + m^{\alpha/l}}.$$

(iii) *Define* $B_k$ *by* $p = 1$ *and* $m(k) = 2^{K-k}$ *for* $k = 1, \ldots, K$ *in MG* 3.1. *Then* (3.6) *holds with* $\delta_k$ *determined by*

$$\delta_k = \frac{C}{C + m(k)^{\alpha/l}}.$$

The constant $C$ in Theorem 3.1 depends on $C_*$, $C_\alpha$, and the estimate on the smoothing operator $R_k$, but is independent of $k$.

THEOREM 3.2. *Assume that* (3.4) *and* (3.5) *are satisfied. Then*

(i) *for* $m$ *big enough (independent of* $k$*), the above result for the* $\mathcal{W}$*-cycle holds.*

(ii) *there are* $\theta_0$, $\theta_1 > 0$, *independent of* $k$, *such that the variable* $\mathcal{V}$*-cycle multigrid operator* $B_k$ *satisfies*

$$\theta_0 b_k(v,v) \le b_k(B_k A_k v, v) \le \theta_1 b_k(v,v), \qquad \forall v \in V_k,$$

*where*

$$\theta_0 \ge \frac{m(k)^{\alpha/l}}{C + m(k)^{\alpha/l}} \quad \textit{and} \quad \theta_1 \le \frac{C + m(k)^{\alpha/l}}{m(k)^{\alpha/l}}.$$

**ETNA**
**Kent State University**
**etna@mcs.kent.edu**



When the "$m$ big enough" in the above theorem is replaced by $C_* = 2$ in (3.4), we have the next result, which is slightly stronger than Theorem 3.2 for the $\mathcal{W}$-cycle.

THEOREM 3.3. *Assume that* (3.4) *with* $C_* = 2$ *and* (3.5) *are satisfied. Then*
(i) *the same result as in Theorem* 3.1 *for the* $\mathcal{W}$*-cycle holds.*
(ii) *the same result as in Theorem* 3.2 *for the variable* $\mathcal{V}$*-cycle holds.*

The validity of inequality (3.5) requires the elliptic regularity property of solutions of partial differential equations. An alternative hypothesis without requiring such a property can be provided with an appropriate choice of the quadratic forms $b(\cdot,\cdot)_k$ such that

$$b_{k-1}(v,w) = b_k(I_k v, I_k w), \quad \forall v,\, w \in V_{k-1},\, k = 1,\dots,K. \tag{3.7}$$

THEOREM 3.4. *Assume that* (3.7) *is satisfied and that there exist linear operators* $Q_K^k : V_K \to V_k, k = 0,\dots,K$*, with* $Q_K^K = I$*, such that*

$$\begin{aligned} &\|(Q_K^k - I_k Q_K^{k-1})v\|_k^2 \le C\lambda_k^{-1} b_K(v,v), \quad && k = 1,\dots,K, \\ &b_k(Q_K^k v, Q_K^k v) \le C b_K(v,v), && k = 0,\dots,K-1. \end{aligned} \tag{3.8}$$

*Then inequality* (3.6) *with* $k = K$ *holds with one smoothing iteration per level for both the* $\mathcal{V}$*- and* $\mathcal{W}$*-cycle multigrid methods with*

$$\delta_K = 1 - \frac{1}{CK},$$

*where* $C$ *is independent of* $K$.

For the proof of the first three theorems, we refer to [11]. For the proof of Theorem 3.4 in the conforming case, see [10], and for the nonconforming case, consult [20]. Condition (3.8) and thus Theorem 3.4 can be verified without any elliptic regularity assumption for the underlying partial differential equations, as mentioned above. For numerical results on the discontinuity in the coefficient of differential problems for the second class of multigrid methods defined in §3.3 below, see [24].

Note that we have uniform convergence estimates for the $\mathcal{W}$-cycle and variable $\mathcal{V}$-cycle methods in Theorem 3.1–3.3. However, the convergence rate for the multigrid $\mathcal{V}$-cycle methods in Theorems 3.1 and 3.4 deteriorates with the number of grid levels. We shall now state a uniform convergence rate for the $\mathcal{V}$-cycle methods with one smoothing on each level. For this, define $\Pi_K^k : V_K \to V_k$ by

$$b_k(\Pi_K^k v, w) = b_K(v, H_k^K w), \quad v \in V_K, w \in V_k,$$

for $k = 0,\dots,K-1$ and $\Pi_K^K = I$ for $k = K$; i.e., $\Pi_K^k$ is the adjoint operator of $H_k^K$ with respect to $b_k(\cdot,\cdot)$.

THEOREM 3.5. *Assume that* (3.7) *and the following condition are satisfied:*

$$\lambda_k \|(\Pi_K^k - I_k \Pi_K^{k-1})v\|_k^2 \le C \|(\Pi_K^k - I_k \Pi_K^{k-1})v\|_{1,k}^2, \quad \forall v \in V_K. \tag{3.9}$$

*Then inequality* (3.6) *with* $k = K$ *holds with one smoothing iteration for both the* $\mathcal{V}$*- and* $\mathcal{W}$*-cycle multigrid methods with* $\delta < 1$ *independent of* $K$.

The proof of this theorem can be found in [20].

**3.2. The first class of multigrid methods.** The first class of multigrid methods is the usual one, which uses discrete equations on all levels which are defined by the same discretization. That is, the quadratic forms $b_k(\cdot,\cdot)$ are given by

$$b_k(v,w) = a_k(v,w), \quad v,\, w \in V_k,\, k = 0,\dots,K,$$

where $a_k(\cdot,\cdot)$ for each of the nonconforming elements considered here are defined as in §2. In this case we have the next results for our nonconforming finite elements.

**3.2.1. The $P_1$-nonconforming element.** For the $P_1$-nonconforming element, the quadratic forms $(\cdot,\cdot)_k$ are defined by

$$(v,w)_k = h_k^2 \sum_q v(q)w(q), \qquad v, w \in V_k,$$

where the summation is taken over all the midpoints $q$ in $\mathcal{E}_k$. The regularity and approximation property (3.5) has been shown in [16] under the following elliptic regularity on the solution of (2.1),

$$\|u\|_{1+\alpha} \le C\|f\|_{-1+\alpha}, \quad 0 < \alpha \le 1, \tag{3.10}$$

where $\|\cdot\|_{1+\alpha}$ denotes the Sobolev norm $\|\cdot\|_{H^{1+\alpha}(\Omega)}$. Consequently, due to (2.6) and Example 1, only Theorem 3.2 applies to this element.

**3.2.2. The rotated $Q_1$-nonconforming element.** The quadratic forms $(\cdot,\cdot)_k$ are determined as follows. Let $\{\phi_k^j\}$ be the basis functions of $V_k$ such that the edge average of $\phi_k^j$ equals one at exactly one edge and zero at all other edges. Then each $v \in V_k$ has the representation

$$v = \sum_j v^j \phi_k^j.$$

Now, for $v,\, w \in V_k$ we define

$$(v,w)_k = h_k^2 \sum_j v^j w^j.$$

By the uniform $L^2$-stability of the basis functions, we can easily show that the norm induced by $(\cdot,\cdot)_k$ is equivalent to the standard $L^2(\Omega)$ norm $\|\cdot\|$.

The regularity and approximation property (3.5) can be seen as in the $P_1$ element [2]. Now, thanks to (2.9), (2.11), and Example 3, Theorem 3.2 can be applied to the rotated $Q_1$ element for a general $\mathcal{A}$ in (2.1), while Theorem 3.3 holds when $\mathcal{A} = I$ in (2.1).

**3.2.3. The Morley element.** For the Morley element, the quadratic forms $(\cdot,\cdot)_k$ are given by

$$(v,w)_k = h_k^2 \sum_q v(q)w(q) + h_k^4 \sum_{\bar q} \frac{\partial v}{\partial \nu}(\bar q)\frac{\partial w}{\partial \nu}(\bar q), \qquad v,\, w \in V_k,$$

where the summations are taken over all the vertices $q$ and midpoints $\bar q$ in $\mathcal{E}_k$, respectively. The property (3.5) can be shown in a similar fashion as for the $P_1$ element [16] under the following elliptic regularity on the solution of (2.12):

$$\|u\|_{2+\alpha} \le C\|f\|_{-2+\alpha}, \quad 0 < \alpha \le 1. \tag{3.11}$$

Thus, by (2.14) and Example 5, only Theorem 3.2 applies to the Morley element.

**3.2.4. The Zienkiewicz element.** The forms $(\cdot,\cdot)_k$ are defined by

$$(v,w)_k = h_k^2 \sum_q v(q)w(q) + h_k^4 \sum_q \big(v_x(q)w_x(q) + v_y(q)w_y(q)\big), \quad v,\, w \in V_k,$$

where the summation is taken over all the vertices $q$ in $\mathcal{E}_k$. For the Zienkiewicz nonconforming element, the property (3.5) can be shown under (3.11). Hence it follows from (2.15) and Example 7 that Theorem 3.2 applies to this element. As mentioned before, numerical evidence suggests that Theorem 3.3 may apply to it.

**3.2.5. The Adini element.** The quadratic forms $(\cdot,\cdot)_k$ are defined as in the case of the Zienkiewicz element, and the property (3.5) also follows from an analogous argument under (3.11). Therefore, by (2.16) and Example 9, we see that similar convergence results to those for the Zienkiewicz element hold for the Adini element.

In summary, Theorem 3.2 applies to the $P_1$ and Morley elements, while Theorem 3.3 applies to the rotated $Q_1$ element (with $\mathcal{A} = I$) and possibly to the Zienkiewicz and Adini elements. Namely, we have shown that the $\mathcal{W}$-cycle multigrid methods converge for the $P_1$ and Morley elements with a sufficiently large number of smoothing iterations on all levels (which is well known), and for the rotated $Q_1$ element and possibly (based on numerical evidence) for the Zienkiewicz and Adini elements with one smoothing iteration per level (which is less known), and that the variable $\mathcal{V}$-cycle multigrid methods provide a uniform condition number estimate for all these nonconforming elements. As a matter of fact, for the Morley element the $\mathcal{W}$-cycle methods diverge unless the number of smoothing iterations on all levels is sufficiently large [31]. For the $P_1$ element we have not numerically observed this fact; in fact, numerical evidence suggests that the $\mathcal{V}$- and $\mathcal{W}$-cycle methods converge with one smoothing for this element [20]. Finally, Theorem 3.1 does not apply to any of these elements; i.e, we do not have any result for the standard $\mathcal{V}$-cycle methods. It is for this reason that we shall consider the second class of multigrid methods in the next subsection.

**3.3. The second class of multigrid methods.** The second class of multigrid methods is determined by

$$b_k(v,w) = a_K(H_k^K v, H_k^K w), \qquad \forall v,\, w \in V_k,\ k = 0, \ldots, K-1, \tag{3.12}$$

where we recall that the iterates $H_k^K$ of $I_k$ are defined as in (2.5) and on the finest level $b_K(\cdot,\cdot) = a_K(\cdot,\cdot) = a_h(\cdot,\cdot)$, which is determined from the continuous problem as in the last section. For each of the nonconforming elements under consideration, the quadratic forms $(\cdot,\cdot)_k$ can be defined as in §3.2. It follows from (3.12) that (3.4) automatically holds with $C_* = 1$. Consequently, it suffices to show (3.5). The ideas presented in [20] indicate that the proof of (3.5) depends on the boundedness of the energy norm of $H_k^K$. In fact, the regularity and approximation assumption (3.5) was shown for the $P_1$ and rotated $Q_1$ elements; see [20]. Also, it is mentioned in [20] that (3.5) possibly holds for the Zienkiewicz and Adini elements. As a consequence, Theorem 3.1 applies to the $P_1$ and rotated $Q_1$ elements; i.e., both the $\mathcal{V}$- and $\mathcal{W}$-cycle multigrid methods with any number of smoothing iterations converge with the convergence rate given as in this theorem for these elements when $b_k(\cdot,\cdot)$ is defined by (3.12). For the Morley element, due to the fact that we cannot control the growth of the energy norm of $H_k^K$ (see §2.3), Theorem 3.1 does not apply. Since the energy norm of $H_k^K$ grows exponentially with the number of grid levels, it is not appropriate to employ the second approach to define the multigrid methods for this element.

Note that (3.12) also implies (3.7), so we now consider Theorems 3.4 and 3.5 for the $P_1$, rotated $Q_1$, Zienkiewicz, and Adini elements. Theorem 3.5 was proven in [20] for the former two elements under a full elliptic regularity assumption on the solution of (2.1) (i.e., $\alpha = 1$ in

(3.10)), and its extension to the latter two elements is possible (based on numerical evidence). For Theorem 3.4, we need the operators $Q_K^k$, which are constructed as follows.

**3.3.1. The $P_1$ element.** Following [20, 37], we define the fine-to-coarse intergrid transfer operators $T_{k-1} : V_k \to V_{k-1}$ as follows. If $v \in V_k$ and $q$ is the midpoint of an edge $e$ of a triangle in $\mathcal{E}_{k-1}$, $T_{k-1}v \in V_{k-1}$ is given by

$$(T_{k-1}v)(q) = \frac{1}{2}(v(q_1) + v(q_2)),$$

where $q_1$ and $q_2$ are the respective midpoints of the edges $e_1$ and $e_2$ in $\mathcal{E}_k$, which form the edge $e$ in $\mathcal{E}_{k-1}$. Note that the definition of $T_{k-1}$ automatically preserves the zero nodal values on the boundary. We now introduce the iterated transfer operators

$$Q_K^k = T_k \cdots T_{K-1} : V_K \to V_k, \quad k = 0, \dots, K. \tag{3.13}$$

With $Q_K^k$ we can show (3.8); see [20], so Theorem 3.4 holds for the $P_1$ element.

**3.3.2. The rotated $Q_1$ element.** The operators $T_{k-1} : V_k \to V_{k-1}$ are defined similarly. If $v \in V_k$ and $e$ is an edge of an element in $\partial\mathcal{E}_{k-1}$, $T_{k-1}v \in V_{k-1}$ is given by [26]

$$\frac{1}{|e|}\int_e T_{k-1}v\mathrm{d}s = \frac{1}{2}\left\{\frac{1}{|e_1|}\int_{e_1} v\mathrm{d}s + \frac{1}{|e_2|}\int_{e_2} v\mathrm{d}s\right\},$$

where $e_1$ and $e_2$ in $\partial\mathcal{E}_k$ form the edge $e \in \partial\mathcal{E}_{k-1}$. Note that the definition of $T_{k-1}$ also automatically preserves the zero average values on boundary edges. The iterates $Q_K^k$ of $T_k$ are given as in (3.13), and also satisfy (3.8); see [20]. Hence Theorem 3.4 applies to the rotated $Q_1$ element.

**3.3.3. The Zienkiewicz and Adini elements.** For the Zienkiewicz and Adini elements, if $v \in V_k$ and $q$ is a vertex of a triangle in $\mathcal{E}_{k-1}$, then $T_{k-1}v \in V_{k-1}$ is defined by, see [20],

$$(T_{k-1}v)(q) = v(q), \quad \nabla(T_{k-1}v)(q) = \nabla v(q),$$

which has the zero nodal values on the boundary and leads to $Q_K^k$ as in (3.13). Condition (3.8) could be shown similarly if the energy norm of $H_k^K$ would be uniformly bounded. However, the boundedness of $H_k^K$ has not been proved yet.

In summary, exploiting the second approach of defining multigrid methods for the $P_1$, rotated $Q_1$, Zienkiewicz, and Adini nonconforming elements, Theorems 3.1, 3.4, and 3.5 can be appled to the first two methods. Numerical evidence suggests that they may also be applied to the last two methods. This approach is not suitable for the Morley element.

REFERENCES

[1] A. ADINI AND R. CLOUGH, *Analysis of plate bending by the finite element method*, NSF Report G. 7337, 1961.
[2] T. ARBOGAST AND Z. CHEN, *On the implementation of mixed methods as nonconforming methods for second order elliptic problems*, Math. Comp., 64 (1995), pp. 943–972.
[3] D. N. ARNOLD AND F. BREZZI, *Mixed and nonconforming finite element methods: implementation, post-processing and error estimates*, RAIRO Model. Math. Anal. Numer., 19 (1985), pp. 7–32.
[4] R. BANK AND T. DUPONT, *An optimal order process for solving finite element equations*, Math. Comp., 36 (1981), pp. 35–51.
[5] G. BAZELEY, Y. CHEUNG, B. IRONS, AND O. ZIENKIEWICZ, *Triangular elements in bending conforming and nonconforming solutions*, Proceedings of the Conference on Matrix Methods in Structural Mechanics, Wright Patterson A.F.B., Ohio, 1965.

[6] D. BRAESS AND W. HACHBUSCH, *A new convergence proof for the multigrid method including the V-cycle*, SIAM J. Numer. Anal., 20 (1983), pp. 967–975.
[7] D. BRAESS AND R. VERFÜRTH, *Multigrid methods for nonconforming finite element methods*, SIAM J. Numer. Anal., 27 (1990), pp. 979–986.
[8] J. BRAMBLE, *Multigrid Methods*, Pitman Research Notes in Math., vol. 294, Longman, London, 1993.
[9] J. BRAMBLE AND J. PASCIAK, *New estimates for multilevel algorithms including the V-cycle*, Math. Comp., 60 (1993), pp. 447–471.
[10] J. BRAMBLE, J. PASCIAK, J. WANG, AND J. XU, *Convergence estimates for multigrid algorithms without regularity assumptions*, Math. Comp., 57 (1991), pp. 23–45.
[11] J. BRAMBLE, J. PASCIAK, AND J. XU, *The analysis of multigrid algorithms with non-nested spaces or non-inherited quadratic forms*, Math. Comp., 56 (1991), pp. 1–34.
[12] S. BRENNER, *An optimal-order multigrid method for P1 nonconforming finite elements*, Math. Comp., 52 (1989), pp. 1–15.
[13] S. BRENNER, *An optimal-order nonconforming multigrid method for the biharmonic equation*, SIAM J. Numer. Anal., 26 (1989), pp. 1124–1138.
[14] S. BRENNER, *Multigrid methods for nonconforming finite elements*, Proceedings of Fourth Copper Mountain Conference on Multigrid Methods, J. Mandel, et. als., eds., SIAM, Philadelphia, 1989, 54–65.
[15] S. BRENNER, *A nonconforming multigrid method for the stationary Stokes equations*, Math. Comp., 55 (1990), pp. 411–437.
[16] S. BRENNER, *Convergence of nonconforming multigrid methods without full elliptic regularity*, Preprint, 1995.
[17] Z. CHEN, *Analysis of mixed methods using conforming and nonconforming finite element methods*, RAIRO Math. Model. Numer. Anal., 27 (1993), pp. 9–34.
[18] Z. CHEN, *Projection finite element methods for semiconductor device equations*, Comput. Math. Appl., 25 (1993), pp. 81–88.
[19] Z. CHEN, *Equivalence between and multigrid algorithms for mixed and nonconforming methods for second order elliptic problems*, East-West J. Numer. Math., 4 (1996), pp. 1–33.
[20] Z. CHEN, *On the convergence of Galerkin-multigrid methods for nonconforming finite elements*, Technical Report # 96-02, Southern Methodist University, Texas, 1996.
[21] Z. CHEN AND R. E. EWING, *From single-phase to compositional flow: applicability of mixed finite elements*, Transport in Porous Media, 27 (1997), pp. 225–242.
[22] Z. CHEN, R. E. EWING, Y. KUZNETSOV, R. LAZAROV, AND S. MALIASSOV, *Multilevel preconditioners for mixed methods for second order elliptic problems*, J. Numer. Linear Algebra Appl., 30 (1996), pp. 427–453.
[23] Z. CHEN, R. E. EWING, AND R. LAZAROV, *Domain decomposition algorithms for mixed methods for second order elliptic problems*, Math. Comp., 65 (1996), pp. 467–490.
[24] Z. CHEN AND D. Y. KWAK, *Convergence of multigrid methods for nonconforming finite elements without regularity assumptions*, Comput. Appl. Math., 1998, to appear.
[25] Z. CHEN, D. Y. KWAK, AND Y. J. YON, *Multigrid algorithms for nonconforming and mixed methods for nonsymmetric and indefinite problems*, SIAM J. Sci. Comput., 1998, to appear.
[26] Z. CHEN AND P. OSWALD, *Multigrid and multilevel methods for nonconforming rotated Q1 elements*, Math. Comp., 1998, to appear.
[27] P. G. CIARLET, *The finite Element Method for Elliptic Problems*, North–Holland, Amsterdam, 1978.
[28] M. CROUZEIX AND P. RAVIART, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations I*, RAIRO, 3 (1973), pp. 33–75.
[29] Q. DENG AND X. FENG, *Optimal order nonnested multigrid methods for the biharmonic equation*, Technical Report, Department of Mathematics, University of Tennessee, June, 1995.
[30] W. HACKBUSCH, *Multigrid Methods and Applications*, Springer-Verlag, Berlin-Heidelberg-New York, 1985.
[31] M. HANISCH, *Multigrid preconditioning for the biharmonic Dirichlet problem*, SIAM J. Numer. Anal., 30 (1993), pp. 184–214.
[32] C. LEE, *A nonconforming multigrid method using conforming subspaces*, in Proceedings of the Sixth Copper Mountain Conference on Multigrid Methods, N. Melson et al., eds., NASA Conference Publication 3224, Part 1 (1993), pp. 317–330.
[33] S. MCCORMICK (eds.), *Multigrid Methods*, SIAM Frontiers in Applied Mathematics 3, SIAM, Philadelphia, 1987.
[34] L. MORLEY, *The triangular equilibrium problem in the solution of plate bending problems*, Aero. Quart., 19 (1968), pp. 149–169.
[35] P. OSWALD, *On a hierarchical basis multilevel method with nonconforming P1 elements*, Numer. Math., 62 (1992), pp. 189–212.
[36] P. OSWALD, *Multilevel preconditioners for discretizations of the biharmonic equation by rectangular finite elements*, J. Numer. Linear Algebra Appl., 2 (1995), pp. 487–505.
[37] P. OSWALD, *Intergrid transfer operators and multilevel preconditioners for nonconforming discretizations*,

Appl. Numer. Math., to appear.

[38] P. PEISKER, *A multilevel algorithm for the biharmonic problem*, Numer. Math., 46 (1985), pp. 623–634.

[39] P. PEISKER AND D. BRAESS, *A conjugate gradient method and a multigrid algorithm for Morley's finite element approximation of the biharmonic equation*, Numer. Math., 50 (1987), pp. 567–586.

[40] P. PEISKER, W. RUST, AND E. STEIN, *Iterative solution methods for plate bending problems: multigrid and preconditioned cg algorithm*, SIAM J. Numer. Anal., 27 (1990), pp. 1450–1465.

[41] R. RANNACHER AND S. TUREK, *Simple nonconforming quadrilateral Stokes element*, Numer. Methods Partial Differential Equations, 8 (1992), pp. 97–111.

[42] P. SCHREIBER AND S. TUREK, *Multigrid results for the nonconforming Morley element*, Preprint, 1994.

[43] V. V. SHAIDUROV, *Multigrid Methods for Finite Elements*, Kluwer Academic Publishers, the Netherlands, 1989.

[44] Z. SHI, X. YU, AND Z. XIE, *A multigrid method for Bergan's energy-orthogonal plate element*, Preprint, 1994.

[45] S. TUREK, *Multigrid techniques for a divergence-free finite element discretization*, East-West J. Numer. Math., 2 (1994), pp. 229–255.

[46] R. VERFÜRTH, *A multilevel algorithm for mixed problems*, SIAM J. Numer. Anal., 21 (1984), pp. 264–271.

[47] R. VERFÜRTH, *Multilevel algorithms for mixed problems II. Treatment of the mini-element*, SIAM J. Numer. Anal., 25 (1988), pp. 285–293.

[48] M. WANG, *The multigrid method for TRUNC plate element*, J. Comput. Math., 11 (1993), pp. 178–187.

[49] M. WANG, *The W-cycle multigrid method for finite elements with nonnested spaces*, Adv. in Math., 23 (1994), pp. 238–250.

[50] J. XU, *Convergence estimates for some multigrid algorithms, Decomposition Methods for Partial Differential Equations*, T. F. Chan, et. al., ed., SIAM, Philadelphia, 1990, pp. 174–187.

[51] S. ZHOU AND G. FENG, *Multigrid methods for the Zienkiewicz element approximation of the biharmonic equation* (in Chinese), J. Hunan Univ., 20 (1993), pp. 1–6.