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ON WEIGHTED LACUNARY INTERPOLATION*

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Abstract. In this paper the regularity of a special lacunary interpolation problem is investigated, where for a given r ($r \geq 2, r \in \mathbb{N}$) the derivatives up to the r-2nd order together with the weighted rth derivative are prescribed at the nodes. Sufficient conditions on the nodes and the weight function, for the problem to be regular, are derived. Under these conditions a method to construct the explicit formulae for the fundamental polynomials of the regular weighted lacunary interpolation is discussed. Examples are presented using the roots of the classical orthogonal polynomials.

Key words. Birkhoff interpolation, lacunary interpolation, Hermite interpolation, weighted $(0,2)$-interpolation, weighted $(0,1,3)$-interpolation, regularity, explicit formulae

AMS subject classifications. 41A05

1. Introduction. The special lacunary interpolation problem studied in this paper is called weighted $(0, 1, \ldots, r-2, r)$-interpolation for a given r ($r \geq 2$, $r \in \mathbb{N}$), where the derivatives up to the r-2nd order together with the weighted rth derivative are prescribed at the nodes. On a finite or infinite interval $[a, b]$ for $n \in \mathbb{N}$, let $\{x_i\}_{i=1}^n$ be a set of distinct points, the nodes, and let $w \in C^r(a, b)$ be a given function, the weight function. Furthermore let $y_i^{(l)}$ ($l = 0, 1, \ldots, r-2, r;\ i = 1, \ldots, n$) be arbitrary given real numbers. Find a polynomial R_n of degree less than rn such that

$$(1.1)\quad R_n^{(l)}(x_i) = y_i^{(l)}, \qquad (wR_n)^{(r)}(x_i) = y_i^{(r)}, \qquad (l = 0, 1, \ldots, r-2;\ i = 1, \ldots, n),$$

(for the sake of simplicity we omit double indices, so we'll write $x_i = x_{i,n}$ and $y_i^{(l)} = y_{i,n}^{(l)}$). The weighted $(0, 2)$-interpolation problem ($r = 2$) was studied originally by J. Balázs [1] as a generalization of the $(0, 2)$-interpolation problem initiated by P. Turán [6].

The weighted $(0, 1, \ldots, r-2, r)$-interpolation is called regular at the nodes $\{x_i\}_{i=1}^n$ with respect to the weight function w, if for any choice of $y_i^{(l)}$ there exists a unique polynomial R_n of degree less than rn which satisfies the conditions (1.1). The problem is not regular in general, because in some cases such a polynomial R_n does not exist (see, e.g., J. Balázs [1] for $r = 2$ and A. Krebsz [2] for $r = 3$), or if it exists, the uniqueness might fail. Furthermore, in order to prove convergence theorems in the regular cases, the explicit formulae for the interpolation polynomial R_n are also needed. Several authors investigated the problem for $r = 2$ and $r = 3$ and found regular solutions and explicit formulae by prescribing special additional conditions to (1.1). In these cases the degree of the interpolation polynomial R_n was increased by the number of the additional conditions. For a general approach to the special cases when $r = 2$ or $r = 3$ we refer to M. Lénárd [4] and A. Krebsz and M. Lénárd [3] and to the references therein.

In Section 2, sufficient conditions on the nodes and the weight function are given for the problem to be regular. In Section 3, a method is presented to construct the explicit formulae for the fundamental polynomials of the regular weighted $(0, 1, \ldots, r-2, r)$-interpolation problem under these conditions. In Section 4, examples are given for regular cases on the roots of the classical orthogonal polynomials.

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2. Regularity. In order to find regular cases with their explicit formulae for the weighted $(0, 1, \ldots, r-2, r)$-interpolation, we extend the problem (1.1) with additional Hermite-type conditions.

For given $n, m \in \mathbb{N}$, on a finite or infinite interval $[a, b]$ let $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^m$ be disjoint sets of distinct points, the nodes, and let $w \in C^r(a, b)$ be a given function, the weight function. Furthermore, let $y_i^{(l)}$ $(l = 0, 1, \ldots, r-2, r;\ i = 1, \ldots, n)$ and $\bar{y}_i^{(j)}$ $(j = 0, \ldots, j_i - 1;\ i = 1, \ldots, m)$ be arbitrary given real numbers, $M = j_1 + \cdots + j_m$ and $N = rn + M$. Find a minimal degree polynomial R_N of degree less than N satisfying weighted $(0, 1, \ldots, r-2, r)$-interpolation conditions

$$R_N^{(l)}(x_i) = y_i^{(l)}, \qquad (wR_N)^{(r)}(x_i) = y_i^{(r)}, \qquad (l = 0, 1, \ldots, r-2;\ i = 1, \ldots, n), \tag{2.1}$$

with additional Hermite-type interpolation conditions on $\{\bar{x}_i\}_{i=1}^m$,

$$R_N^{(j)}(\bar{x}_i) = \bar{y}_i^{(j)}, \qquad (j = 0, \ldots, j_i - 1;\ i = 1, \ldots, m). \tag{2.2}$$

(For $m = 0$ the problem is the weighted $(0, 1, \ldots, r-2, r)$-interpolation.) This extended interpolation problem is also not regular in general, as it is shown for $r = 2$ and $r = 3$ in [4] and [3]. Hence we study the interpolation problem (2.1)-(2.2) with further additional conditions. P. Mathur and S. Datta [5] discussed a special case when $m = 1$, $j_1 = r - 1$, and the additional condition is $R_N^{(r-1)}(\bar{x}_1) = \bar{y}_1^{(r-1)}$.

In what follows, let p_n and q be polynomials of degree n and M, respectively, associated with the interpolation conditions (2.1)-(2.2), that is,

$$\begin{aligned} p_n(x_i) &= 0, \qquad (i = 1, \ldots, n), \\ q^{(j)}(\bar{x}_i) &= 0, \qquad (j = 0, \ldots, j_i - 1;\ i = 1, \ldots, m). \end{aligned} \tag{2.3}$$

If only weighted interpolation conditions are prescribed, let $q(x) \equiv 1$ and $m = 0$. Furthermore, let

$$\ell_k(x) = \frac{p_n(x)}{p_n'(x_k)(x - x_k)}, \qquad (k = 1, \ldots, n),$$

denote the fundamental polynomials of Lagrange-interpolation, that is, $\ell_k(x_i) = \delta_{k,i}$ for $i, k = 1, \ldots, n$. Using induction on r it is easy to verify that, for $i = 1, \ldots, n$,

$$(p_n^r)^{(l)}(x_i) = \begin{cases} 0, & \text{for } l < r, \\ r!(p_n')^r(x_i), & \text{for } l = r, \\ \dfrac{r}{2}(r+1)!\left((p_n')^{r-1}p_n''\right)(x_i), & \text{for } l = r+1, \end{cases} \tag{2.4}$$

and

$$(\ell_k^r)^{(l)}(x_i) = \begin{cases} 0, & \text{for } i \neq k,\ \ l < r, \\ r!(\ell_k')^r(x_i), & \text{for } i \neq k,\ \ l = r. \end{cases} \tag{2.5}$$

Studying the regularity of the problem let us consider the homogeneous case, when $y_i^{(l)} = 0$ $(l = 0, 1, \ldots, r-2, r;\ i = 1, \ldots, n)$ and $\bar{y}_i^{(j)} = 0$ $(j = 0, \ldots, j_i - 1;\ i = 1, \ldots, m)$. It is obvious that every polynomial $\bar{R}_N$ which satisfies the conditions,

$$\begin{aligned} \bar{R}_N^{(j)}(x_i) &= 0, \qquad (i = 1, \ldots, n;\ \ j = 0, 1, \ldots, r-2), \\ \bar{R}_N^{(j)}(\bar{x}_i) &= 0, \qquad (i = 1, \ldots, m;\ \ j = 0, 1, \ldots, j_i - 1), \end{aligned} \tag{2.6}$$

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can be written in the form

$$\bar{R}_N(x) = \left(qp_n^{r-1}Q\right)(x), \tag{2.7}$$

where p_n and q are defined in (2.3) and Q is a polynomial. Furthermore, on using (2.4) we obtain

$$\left(w\bar{R}_N\right)^{(r)}(x_i) = \left(wqp_n^{r-1}Q\right)^{(r)}(x_i) = \left(wqp_n^{r-1}\right)^{(r)}(x_i)Q(x_i) + r!\left(wq(p_n')^{r-1}Q'\right)(x_i)$$

for $i = 1,\ldots,n$. Therefore, if

$$\left(wqp_n^{r-1}\right)^{(r)}(x_i) = 0 \qquad \text{and} \qquad w(x_i) \neq 0, \qquad (i = 1,\ldots,n), \tag{2.8}$$

then

$$\left(w\bar{R}_N\right)^{(r)}(x_i) = 0 \qquad \text{if and only if} \qquad Q'(x_i) = 0, \qquad (i = 1,\ldots,n). \tag{2.9}$$

Thus $Q'(x) = p_n(x)\lambda(x)$, where λ is a polynomial. Since we are looking for a minimal degree polynomial $\bar{R}_N$ in the form (2.7) which fulfills (2.6) and (2.9) with k_0 ($k_0 \in \mathbb{N}$) additional homogeneous conditions, the degree of the polynomial Q must be less than $n+k_0$. For the sake of simplicity, in what follows, we will prescribe only one or two conditions at one or two points ($k_0 = 1$ or 2). In these cases we obtain either

$$Q(x) = c \qquad \text{or} \qquad Q(x) = c\int_{x_0}^{x} p_n(t)dt + d,$$

where the parameters c and d are to be determined from these additional conditions.

If, for example, the additional homogeneous condition is

$$\left(w\bar{R}_N\right)^{(r)}(x_0) = 0,$$

then

$$\left(w\bar{R}_N\right)^{(r)}(x_0) = \left(cwqp_n^{r-1}\right)^{(r)}(x_0) = c\left(wqp_n^{r-1}\right)^{(r)}(x_0) = 0.$$

Hence, the condition of regularity is

$$\left(wqp_n^{r-1}\right)^{(r)}(x_0) \neq 0.$$

Other cases can be discussed in a similar way and we list some of them in the following statement.

THEOREM 2.1. *If at the nodes $\{x_i\}_{i=1}^n$ and $\{\bar{x}_i\}_{i=1}^m$ the weight function w satisfies (2.8), then the interpolation problem (2.1)–(2.2) is regular under the additional condition(s)* (i)-(v) *if and only if the corresponding condition in the third column of Table 2.1 is fulfilled.*

REMARK 2.2. The modified weighted $(0,1,\ldots,r-2,r)$-interpolation studied in [5] by P. Mathur and S. Datta, corresponds to the special case when $m = 1$, $x_0 = \bar{x}_1$, $q(x) = (x-x_0)^{r-1}$ and the additional condition is (i) in Table 2.1 with $j = r-1$.

3. The fundamental polynomials. In this section, we first construct polynomials $A_{j,k}$ which satisfy the following weighted $(0,1,\ldots,r-2,r)$-interpolation conditions at the nodes $\{x_i\}_{i=1}^n$

$$\left.\begin{aligned} A_{j,k}^{(l)}(x_i) &= \delta_{jl}\delta_{ki} \\ (wA_{j,k})^{(r)}(x_i) &= 0 \end{aligned}\right\} \qquad \begin{aligned} &(i,k = 1,\ldots,n; \\ &j,l = 0,1,\ldots,r-2), \end{aligned} \tag{3.1}$$

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TABLE 2.1

	Additional Interpolatory Condition(s)	Condition for Regularity
(i)	$R_N^{(j)}(x_0) = y_0^{(j)}$	$\left(qp_n^{r-1}\right)^{(j)}(x_0) \neq 0$
(ii)	$(wR_N)^{(r)}(x_0) = y_0^{(r)}$	$\left(wqp_n^{r-1}\right)^{(r)}(x_0) \neq 0$
(iii)	$R_N^{(r-2)}(x_0) = y_0^{(r-2)}$, $(wR_N)^{(r)}(x_0) = y_0^{(r)}$	$\left(qp_n^{r-1}\right)^{(r-2)}(x_0)\left(wqp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r)}(x_0) -$ $\left(wqp_n^{r-1}\right)^{(r)}(x_0)\left(qp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r-2)}(x_0) \neq 0$
(iv)	$R_N^{(j)}(x_0) = y_0^{(j)}$, $R_N^{(j)}(x_{n+1}) = y_{n+1}^{(j)}$	$(qp_n^{r-1})^{(j)}(x_{n+1})\left(qp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(j)}(x_0) -$ $(qp_n^{r-1})^{(j)}(x_0)\left(qp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(j)}(x_{n+1}) \neq 0$
(v)	$(wR_N)^{(r)}(x_0) = y_0^{(r)}$, $(wR_N)^{(r)}(x_{n+1}) = y_{n+1}^{(r)}$	$\left(wqp_n^{r-1}\right)^{(r)}(x_{n+1})\left(wqp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r)}(x_0) -$ $\left(wqp_n^{r-1}\right)^{(r)}(x_0)\left(wqp_n^{r-1}\int_{x_0}^x p_n(t)dt\right)^{(r)}(x_{n+1}) \neq 0$

and

$$\left.\begin{aligned} A_{r,k}^{(l)}(x_i) &= 0 \\ (wA_{r,k})^{(r)}(x_i) &= \delta_{ki} \end{aligned}\right\} \qquad \begin{aligned} (i,k &= 1,\dots,n; \\ l &= 0,1,\dots,r-2), \end{aligned} \tag{3.2}$$

with Hermite-type interpolation conditions

$$A_{j,k}^{(l)}(\bar{x}_i) = 0, \quad (i=1,\dots,m;\ k=1,\dots,n;\ j=0,\dots,r-2,r;\ l=0,\dots,j_i-1). \tag{3.3}$$

Let p_n and q be the polynomials defined in (2.3) and let, for $k = 1,\dots,n$,

$$A_{r,k}(x) = \frac{(qp_n^{r-1})(x)}{r!\,(wq(p_n')^{r-1})\,(x_k)}\left\{c_{r,k} + \int_{x_0}^x [\ell_k(t) + b_{r,k}p_n(t)]dt\right\} \tag{3.4}$$

and, recursively, for $j = r-2, r-3, \dots, 1, 0$,

$$A_{j,k}(x) = \frac{q(x)}{j!q(x_k)}\left\{(x-x_k)^j\ell_k^r(x) + p_n^{r-1}(x)Q_{j,k}(x)\right\} + \sum_{l=j+1}^{r-2} d_{j,k}^{[l]}A_{l,k}(x), \tag{3.5}$$

where

$$Q_{j,k}(x) = \frac{1}{(p_n')^{r-1}(x_k)}\left\{c_{j,k} + \int_{x_0}^x\left[\frac{q_{j,k}(t)}{(t-x_k)^{r-j-1}} + a_{j,k}\ell_k(t) + b_{j,k}p_n(t)\right]dt\right\}, \tag{3.6}$$

$$q_{j,k}(x) = \ell_k(x)\left[\ell_k'(x_k) + \sum_{l=1}^{r-j-2}\gamma_{l,k}(x-x_k)^l\right] - \ell_k'(x), \tag{3.7}$$

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$$\gamma_{l,k} = \frac{1}{l!}\left\{\ell_k^{(l+1)}(x_k) - \ell_k'(x_k)\ell_k^{(l)}(x_k) - \sum_{s=1}^{l-1}\gamma_{s,k}\binom{l}{s}s!\ell_k^{(l-s)}(x_k)\right\}, \tag{3.8}$$

$$a_{j,k} = -\frac{(wq\ell_k^r)^{(r-j)}(x_k)}{(r-j)!(wq)(x_k)} - \frac{q_{j,k}^{(r-j-1)}(x_k)}{(r-j-1)!}, \tag{3.9}$$

$$d_{j,k}^{[l]} = -\binom{l}{j}\frac{(q\ell_k^r)^{(l-j)}(x_k)}{q(x_k)}, \tag{3.10}$$

where x_0, $b_{j,k}$, and $c_{j,k}$ $(k=1,\ldots,n;\ \ j=0,1,\ldots,r-2,r)$ are free parameters. (In (3.5), (3.7), and (3.8) the value of $\sum$ is 0 if the upper limit of the summation is less than the lower limit.)

THEOREM 3.1. *If at the nodes $\{x_i\}_{i=1}^n$ the weight function w and the polynomial q satisfy the conditions*

$$\left(wqp_n^{r-1}\right)^{(r)}(x_i) = 0 \qquad \textit{and} \qquad w(x_i)q(x_i) \neq 0, \qquad (i=1,\ldots,n), \tag{3.11}$$

then the polynomials $A_{j,k}$ $(j=0,1,\ldots,r-2,r;\ \ k=1,\ldots,n)$ defined in (3.4)-(3.10) are of degree at most $nr+M+1$ and fulfill the conditions (3.1)-(3.3).

Proof. Let $k\in\{1,\ldots,n\}$ be fixed. An easy calculation shows that $A_{r,k}$ in (3.4) satisfies the equations (3.2).

Now we are looking for $A_{r-2,k}$ in the form

$$A_{r-2,k}(x) = \frac{q(x)}{(r-2)!q(x_k)}\left\{(x-x_k)^{r-2}\ell_k^r(x) + p_n^{r-1}(x)Q_{r-2,k}(x)\right\},$$

where $Q_{r-2,k}$ is a polynomial of degree at most $n+1$. As $p_n(x_i)=0$ and $\ell_k(x_i)=\delta_{ki}$ $(i=1,\ldots,n)$, we obtain

$$A_{r-2,k}^{(l)}(x_i) = 0, \qquad (l=0,1,\ldots,r-3),$$

and $A_{r-2,k}^{(r-2)}(x_i)=\delta_{ki}$. On using (2.5) and (3.11) we get that $(wA_{r-2,k})^{(r)}(x_i)=0$ for $i\neq k$ if and only if

$$Q_{r-2,k}'(x_i) = \frac{-\ell_k'(x_i)}{(p_n')^{r-1}(x_k)(x_i-x_k)}.$$

Hence the polynomial $Q_{r-2,k}'$ can be defined by

$$Q_{r-2,k}'(x) = \frac{1}{(p_n')^{r-1}(x_k)}\left\{\frac{\ell_k'(x_k)\ell_k(x) - \ell_k'(x)}{x-x_k} + a_{r-2,k}\ell_k(x) + b_{r-2,k}p_n(x)\right\}.$$

From the equation $(wA_{r-2,k})^{(r)}(x_k)=0$ we obtain the parameter

$$a_{r-2,k} = -\frac{(wq\ell_k^r)''(x_k)}{2(wq)(x_k)} - \left(\ell'^2_k(x_k) - \ell_k''(x_k)\right),$$

as defined in (3.9) for $j=r-2$. Hence $Q_{r-2,k}$ is the polynomial in (3.6) for $j=r-2$.

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The polynomials $A_{j,k}$ we construct recursively for $j = r-3, \ldots, 1, 0$ in a similar way. Hence we are looking for $A_{j,k}$ in the form of (3.5), where the $Q_{j,k}$ are polynomials of degree $n+1$ and the $d_{j,k}^{[l]}$ are parameters. It is obvious that

$$\begin{aligned}
A_{j,k}^{(l)}(x_i) &= 0, && (l = 0, 1, \ldots, r-2; \quad i \neq k), \\
A_{j,k}^{(l)}(x_k) &= 0, && (l = 0, 1, \ldots, j-1), \\
A_{j,k}^{(j)}(x_k) &= 1.
\end{aligned}$$

From the conditions

$$A_{j,k}^{(l)}(x_k) = 0, \qquad (l = j+1, \ldots, r-2),$$

we obtain the parameters $d_{j,k}^{[l]}$ as it is given in (3.10). Then the polynomials $Q_{j,k}$ are to be determined from the conditions

$$\left(wA_{j,k}\right)^{(r)}(x_i) = 0, \qquad (i = 1, \ldots, n).$$

On using (2.5) and (3.11), we get that $(wA_{j,k})^{(r)}(x_i) = 0$ for $i \neq k$ if and only if

$$Q'_{j,k}(x_i) = \frac{-1}{(p'_n)^{r-1}(x_k)} \cdot \frac{\ell'_k(x_i)}{(x_i - x_k)^{r-j-1}}.$$

Therefore, let us write the polynomials $Q'_{j,k}$ in the following form,

$$Q'_{j,k}(x) = \frac{1}{(p'_n)^{r-1}(x_k)} \left\{ \frac{q_{j,k}(x)}{(x - x_k)^{r-j-1}} + a_{j,k}\ell_k(x) + b_{j,k}p_n(x) \right\},$$

where $a_{j,k}$ and $b_{j,k}$ are parameters, while $q_{j,k}$ are polynomials which fulfill the conditions

$$q_{j,k}(x_i) = -\ell'_k(x_i), \qquad (i \neq k).$$

Now we are looking for the polynomials $q_{j,k}$ in the form of (3.7), where the parameters $\gamma_{l,k}$ are determined from the conditions

$$q_{j,k}^{(s)}(x_k) = 0, \qquad (s = 1, \ldots, r-j-2).$$

Hence $\gamma_{l,k}$ for $l = 1, \ldots, r-j-2$ in (3.8) assure that $Q_{j,k}$ are polynomials.

Finally, from the equation $\left(wA_{j,k}\right)^{(r)}(x_k) = 0$, using (3.11), we get the parameter $a_{j,k}$ in (3.9). Hence the polynomials $A_{j,k}$ for $j = 0, 1, \ldots, r-2$ defined in (3.5) with (3.6)-(3.10) are of degree at most $nr + M + 1$ and fulfill the equations (3.1). ◻

The following special case illustrates how to apply Theorem 3.1 to construct the fundamental polynomials of regular weighted $(0, 1, \ldots, r-2, r)$-interpolation. Let us consider the additional condition (ii) in Table 2.1 when $q(x) = (x - x_0)^{r-1}$, that is $m = 1$, $\bar{x}_1 = x_0$ and $j_1 = r-1$. Hence we are looking for a polynomial of degree less than $(n+1)r$ which fulfills (2.1) with

$$R_N^{(l)}(x_0) = y_0^{(l)}, \quad (wR_N)^{(r)}(x_0) = y_0^{(r)}, \qquad (l = 0, 1, \ldots, r-2),$$

and the problem is a weighted $(0, 1, \ldots, r-2, r)$-interpolation problem at the nodes $\{x_i\}_{i=0}^n$ with respect to the weight function w. In this case the condition (3.11) can be written as

$$\left((x - x_0)^{r-1} \, w \, p_n^{r-1}\right)^{(r)}(x_i) = 0 \quad \text{and} \quad w(x_i) \neq 0, \qquad (i = 1, \ldots, n), \tag{3.12}$$

and the condition for regularity from Table 2.1 is

$$\left((x - x_0)^{r-1} \, w \, p_n^{r-1}\right)^{(r)} (x_0) \neq 0, \tag{3.13}$$

where $x_0 \neq x_i$ $(i = 1, \ldots, n)$.

THEOREM 3.2. *If at the nodes $\{x_i\}_{i=0}^n$ the weight function w satisfies (3.12) and (3.13), then the weighted $(0, 1, \ldots, r-2, r)$-interpolation is regular at the nodes $\{x_i\}_{i=0}^n$ with respect to the weight function w. Furthermore, the interpolation polynomial is of degree less than $(n+1)r$ and can be written in the form*

$$R_N(x) = \sum_{j=0}^{r-2} \sum_{k=0}^{n} A_{j,k}(x) y_k^{(j)} + \sum_{k=0}^{n} A_{r,k}(x) y_k^{(r)},$$

where the fundamental polynomials $A_{j,k}$ are given explicitly.

Proof. Under the conditions (3.12)-(3.13) the regularity of the problem is a simple corollary of Theorem 2.1. The fundamental polynomials $A_{j,k}$ $(j = 0, 1, \ldots, r-2, r;\ k = 1, \ldots, n)$ are associated with the weighted $(0, 1, \ldots, r-2, r)$-interpolation conditions and they fulfill the equations (3.1)-(3.3). They are defined in (3.4)-(3.10), where

$$b_{j,k} = 0, \qquad (j = 0, 1, \ldots, r-2, r;\ \ k = 1, \ldots, n).$$

and the parameters $c_{j,k}$ are determined recursively for $j = r, r-2, r-3, \ldots, 1, 0$ from the conditions

$$(wA_{j,k})^{(r)}(x_0) = 0, \qquad (k = 1, \ldots, n).$$

Hence, the degree of these polynomials is less than $(n+1)r$.

Next let us construct the fundamental polynomials $A_{j,0}$ for $j = 0, 1, \ldots, r-2$ which are associated with the Hermite-type conditions, so they fulfill the conditions

$$\begin{aligned} A_{j,0}^{(l)}(x_k) &= 0, && (l = 0, 1, \ldots, r-2;\ \ k = 1, \ldots, n), \\ A_{j,0}^{(l)}(x_0) &= \delta_{j,l}, && (l = 0, 1, \ldots, r-2), \\ (wA_{j,0})^{(r)}(x_k) &= 0, && (k = 0, 1, \ldots, n). \end{aligned}$$

We are looking for $A_{j,0}$ in the form

$$A_{j,0}(x) = p_n^r(x)(x - x_0)^j r_j(x) + (x - x_0)^{r-1} p_n^{r-1}(x) Q_j(x), \tag{3.14}$$

where Q_j is a polynomial of degree at most n and

$$r_j(x) = a_0^{(j)} + a_1^{(j)}(x - x_0) + \cdots + a_{r-2-j}^{(j)}(x - x_0)^{r-2-j}. \tag{3.15}$$

It is obvious that

$$\begin{aligned} A_{j,0}^{(l)}(x_i) &= 0, && (l = 0, 1, \ldots, r-2), \\ A_{j,0}^{(l)}(x_0) &= 0, && (l = 0, 1, \ldots, j-1). \end{aligned}$$

The coefficients $a_l^{(j)}$ in (3.15) are determined recursively from the conditions

$$A_{j,0}^{(l)}(x_0) = \delta_{l,j}, \qquad (l = j, j+1, \ldots, r-2),$$

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and we obtain

$$a_0^{(j)} = \frac{1}{j!p_n^r(x_0)},$$

$$a_l^{(j)} = \frac{-1}{p_n^r(x_0)} \sum_{i=0}^{l-1} \frac{(p_n^r)^{(l-i)}(x_0)}{(l-i)!} a_i^{(j)}, \qquad (l = 1, \ldots, r-2-j).$$

Now we construct the polynomial Q_j in (3.14) using the conditions $(wA_{j,0})^{(r)}(x_i) = 0$ for $i = 1, \ldots, n$. Applying (2.4) and (3.12) we get

$$(wA_{j,0})^{(r)}(x_i) = 0 \quad \text{if and only if} \quad Q_j'(x_i) = -\frac{p_n'(x_i)r_j(x_i)}{(x_i - x_0)^{r-1-j}}, \qquad (i = 1, \ldots, n),$$

and hence we obtain

$$Q_j'(x) = \frac{-p_n'(x)r_j(x) + p_n(x)\bar{r}_j(x)}{(x - x_0)^{r-1-j}},$$

where the polynomial $\bar{r}_j$ is of degree $r - 2 - j$ and the coefficients are to be determined uniquely from the equations

$$\left(-p_n'\, r_j + p_n\, \bar{r}_j\right)^{(l)}(x_0) = 0, \qquad (l = 0, 1, \ldots, r-2-j).$$

Therefore,

$$Q_j(x) = c_j + \int_{x_0}^{x} \frac{-p_n'(t)r_j(t) + p_n(t)\bar{r}_j(t)}{(t - x_0)^{r-1-j}}\, dt,$$

where the parameter c_j is to be determined from the condition $(wA_{j,0})^{(r)}(x_0) = 0$.

Finally, it is easy to verify that the polynomial

$$A_{r,0}(x) = \frac{p_n^{r-1}(x)(x - x_0)^{r-1}}{\left((x - x_0)^{r-1} w p_n^{r-1}\right)^{(r)}(x_0)}$$

fulfills the conditions

$$A_{r,0}^{(l)}(x_k) = 0, \qquad (l = 0, 1, \ldots, r-2; \;\; k = 0, 1, \ldots, n),$$

$$(wA_{r,0})^{(r)}(x_k) = \delta_{k,0}, \qquad (k = 0, 1, \ldots, n). \quad \square$$

4. Regular weighted $(0, 1, \ldots, r-2, r)$-interpolation at the roots of the classical orthogonal polynomials. The classical orthogonal polynomials (Jacobi, Hermite, and Laguerre polynomials) fulfill the differential equation

$$(\omega y)'' + f \cdot (\omega y) = 0$$

with some weight function ω and function f. Hence, if p_n is a classical orthogonal polynomial of degree n and $p_n(x_i) = 0$, then

$$(\omega p_n)''(x_i) = 0, \qquad (i = 1, \ldots, n)$$

where ω is given in Table 4.1; cf. Szegő [7].

TABLE 4.1

$p_n(x)$	(a,b)	$\omega(x)$
Hermite $H_n(x)$	$(-\infty,\infty)$	$e^{-\frac{x^2}{2}}$
Laguerre $L_n^{(\alpha)}(x)$ $(\alpha > -1)$	$[0,\infty)$	$e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}}$
Jacobi $P_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta > -1)$	$[-1,1]$	$(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}$

LEMMA 4.1. *If $\omega(x_i) \neq 0$ at the nodes $\{x_i\}_{i=1}^n$, then*

$$(\omega p_n)''(x_i) = 0 \quad \textit{if and only if} \quad \left((\omega p_n)^{r-1}\right)^{(r)}(x_i) = 0, \qquad (i = 1, \ldots, n).$$

Proof. On using (2.4) and the fact that x_i $(i = 1, \ldots, n)$ are distinct roots of the polynomial p_n, we obtain

$$\begin{aligned}
\left((\omega p_n)^{r-1}\right)^{(r)}(x_i) &= \left(\omega^{r-1} \cdot (p_n)^{r-1}\right)^{(r)}(x_i) \\
&= \frac{(r-1)r!}{2}\left(\omega \cdot (p_n')\right)^{r-2}(x_i)\left(\omega p_n'' + 2\omega' p_n'\right)(x_i) \\
&= \frac{(r-1)r!}{2}\left(\omega \cdot (p_n')\right)^{r-2}(x_i)\,(\omega p_n)''(x_i). \quad \square
\end{aligned}$$

Let the nodes $\{x_i\}_{i=1}^n$ be the roots of the classical orthogonal polynomial p_n and let ω be the function associated with p_n in Table 4.1. Furthermore, let us define the weight function w as

$$w(x) = \frac{\omega^{r-1}(x)}{q(x)}, \tag{4.1}$$

where q is the polynomial defined in (2.3). On using Lemma 4.1 it is obvious that the nodes and the weight function defined in (4.1) satisfy the conditions (2.8). Hence, applying Theorems 2.1 and 3.2, we obtain regular weighted $(0, 1, \ldots, r-2, r)$-interpolation on the roots of the classical orthogonal polynomials with respect to the weight function w in (4.1). Now we present only a special case when the nodes are the roots of the Laguerre-polynomial $L_n^{(\alpha)}$ on the interval $[0,\infty)$. Other cases can be discussed in a similar way.

THEOREM 4.2. *If the nodes $\{x_i\}_{i=1}^n$ are the roots of the Laguerre-polynomial $L_n^{(\alpha)}$ $(\alpha > -1)$, then the weighted $(0, 1, \ldots, r-2, r)$-interpolation with respect to the weight function*

$$w(x) = \left(x^{\frac{\alpha-1}{2}} e^{-\frac{x}{2}}\right)^{r-1}$$

is regular for

$$\alpha = \frac{2j}{r-1} - 1, \qquad (j = 1, \ldots, r).$$

Proof. Let $p_n = L_n^{(\alpha)}$ $(\alpha > -1)$, $x_0 = 0$, $q(x) = x^{r-1}$, and $\omega(x) = x^{\frac{\alpha+1}{2}} e^{-\frac{x}{2}}$. Then

$$w(x) = \frac{\omega^{r-1}(x)}{x^{r-1}} = \left(x^{\frac{\alpha-1}{2}} e^{-\frac{x}{2}}\right)^{r-1}.$$

As $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} > 0$ and $\left(L_n^{(\alpha)}\right)'(x) = -L_{n-1}^{(\alpha+1)}(x)$ (c.f. G. Szegő [7]), the sign of $\left(L_n^{(\alpha)}\right)^{(j)}(0)$ is $(-1)^j$ and therefore by induction for all j

$$\left(\left(e^{-\frac{x}{2}} L_n^{(\alpha)}\right)^{r-1}\right)^{(j)}(0) \neq 0.$$

Thus, the condition (3.13) is fulfilled if

$$\frac{\alpha+1}{2}(r-1) = j, \qquad (j = 1, \ldots, r),$$

and we can apply Theorem 3.2. □

On using the notation $L_n^{(-1)}(x) = -\frac{x}{n} L_{n-1}^{(1)}(x)$, in the special case $j = r-1$, we obtain the following theorem.

THEOREM 4.3. *If the nodes are the roots of the polynomial $L_n^{(-1)}$, then the weighted $(0, 1, \ldots, r-2, r)$-interpolation is regular with respect to the weight function $w(x) = e^{-\frac{x}{2}(r-1)}$. The interpolation polynomial is constructed explicitly in Theorem 3.2.*

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