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# ON THE DISCRETE SOLUTION OF THE GENERALIZED STOKES PROBLEM IN ONE TIME-STEP FOR TWO PHASE FLOW *

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*Dedicated to Víctor Pereyra on the occasion of his 70th birthday*

**Abstract.** In this paper, we demonstrate the existence, uniqueness, and uniform stability of the discrete solution obtained with the nonconforming Crouzeix-Raviart/$P_0$ finite element for a generalized Stokes problem of a two-phase flow in one time-step.

**Key words.** two-phase flow, Stokes problem, discrete solution, finite element

**AMS subject classifications.** 15A15, 15A09, 15A23

## 1. Introduction.

In many applications in science and technology, the flow of two immiscible and incompressible fluids in a pipeline, or two-phase flow, plays an important role [22, 23]. In particular, the lubricated transportation technique is frequently used to facilitate the movement of viscous oils through a pipeline lubricated with a low-viscosity liquid such as water. For this process to be successful, the low-viscosity fluid must be introduced and maintained between the viscous oil and the wall of the pipeline, forming a capsule between the wall and the high-viscosity fluid [7]. This flow pattern is called core annular flow, and the proposed physical model for its study assumes that in this configuration both fluids travel adjacently in time and space, the interface being a natural separation surface between the two fluids.

Based on this, Maury et al. in 2002 proposed to study two-dimensional linear two-phase flow ($D = 2$) of a fluid made up of water and petroleum in a horizontal pipeline, governed by the transient Navier-Stokes equations; see [27]. Initially, Maury proposed a mathematical model based on these equations with boundary conditions in an axisymmetrical configuration. This configuration simplified the study, in the sense that the interface between both fluids was represented as a single free boundary, but it added an artificial boundary condition which was located in the middle of the pipeline. This free boundary is an unknown to be determined in the problem, and this, although apparently simple, makes the problem more difficult by adding the transport equation which models the evolution in time and space of the interface as it becomes deformed due to the action of stresses produced by the surface tension between the two fluids. The Arbitrary Lagrangian-Eulerian (ALE) method of approximation, which is based on a grid that moves with the fluid, and a method of characteristics for discretizing the nonlinear convection term on the Navier-Stokes equations were proposed. Thus, a semi-discretization in time of the Navier-Stokes equations was obtained, leading to a generalized Stokes problem with non-standard boundary conditions. In Girault, López, and Maury [17], this problem was discretized in space using the Arnold-Brezzi-Fortin finite element (“mini-element”), and numerical error estimations were established.

Next, Angulo and López presented in [4] a more realistic extension of a two-phase-flow

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two-dimensional problem by considering a non-axisymmetrical configuration. This variant eliminates the artificial boundary and complicates the problem further since it includes two free surfaces to be determined, and these deform in different and arbitrary fashion in space and time. Using the combined ALE-characteristic method, an equivalent variational formulation to the generalized Stokes problem corresponding to the non-axisymmetrical configuration was proposed, and it was demonstrated that this formulation is a well-posed problem.

In this paper, we present a mixed finite element, different from the "mini-element," to approximate the variational formulation proposed in [4], and we demonstrate that the discrete problem is well-posed and that the solution is stable for this mixed finite element. This mixed finite element approximates each velocity component with the Crouzeix-Raviart finite element and the pressure by a constant in each element (Crouzeix-Raviart/$P_0$). Some numerical simulations which verify this heuristically are also presented.

This paper is organized as follows. In Section 2 we introduce the Sobolev spaces and the associated norms used in this article. In Section 3, following [4] very closely, we present the nonlinear equations that model two-phase non-axisymmetrical flow and the generalized Stokes problem that must be solved for each time step. Section 4 is devoted both to the discretization of the generalized Stokes problem using the Crouzeix-Raviart/$P_0$ mixed finite element and to the results on the existence, uniqueness, and stability of the solution to the discrete problem. Finally, in Section 5, we show some numerical simulations for a two-phase flow test problem taken from [7].

**2. Preliminary basics.** In this section we introduce the Sobolev spaces and the associated norms employed in this paper; see [26] and [2] for further details.

Given a domain $\Omega \subset \mathbb{R}^2$ with a boundary $\partial\Omega$, the Sobolev space of functions $H^m(\Omega)$ is defined as:

$$H^m(\Omega) = \left\{ v \in L^2(\Omega) : \ \partial^k v \in L^2(\Omega), \forall\, |k| \leq m \right\},$$

where $|k| = k_1 + k_2$ with $(k_1, k_2)$ a pair of non-negative integers (in two dimensions) and

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}.$$

This space is equipped with the seminorm

$$|v|_{H^m(\Omega)} = \left[ \sum_{|k|=m} \int_\Omega \left| \partial^k v \right|^2 d\Omega \right]^{1/2},$$

and is a Hilbert space for the norm

$$\|v\|_{H^m(\Omega)} = \left[ \sum_{0 \leq k \leq m} |v|^2_{H^k(\Omega)} \right]^{1/2}.$$

The scalar product of $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation.

**3. The nonlinear model for a 2-$D$ two-phase flow.** In a longitudinal section of a piece of the pipeline, the fluid with low viscosity (water) is adjacent to the pipe wall, and it surrounds the fluid with high viscosity (petroleum). Figure 3.1 illustrates this two-phase flow pattern as a domain $\Omega \subset \mathbb{R}^2$ delimited by the piece of horizontal pipe.

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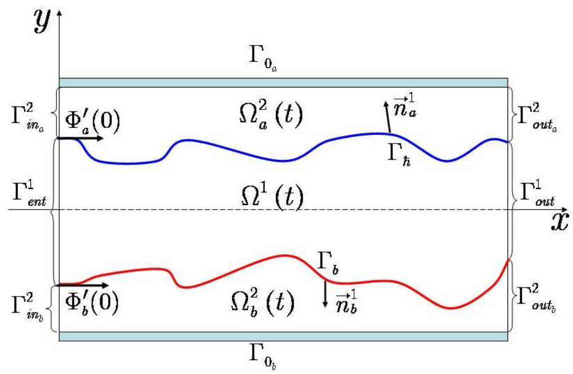

FIGURE 3.1. *Domain $\Omega$ and boundary $\partial\Omega$.*

For each time $t \in [0,T]$, the domain $\Omega$ is decomposed into two subdomains, $\Omega^1(t)$ and $\Omega^2(t)$. Here, $\Omega^1$ is the region occupied by the heavy fluid (the petroleum) and $\Omega^2$ is the region occupied by the water. This latter region, $\Omega^2$, is divided into two subregions, one superior and another inferior, that we shall call $\Omega_a^2$ and $\Omega_b^2$, respectively, such that

$$\Omega^2(t) = \Omega_a^2(t) \cup \Omega_b^2(t).$$

On the other hand, the boundary of $\Omega^i$, $i = 1, 2$, is given by:

$$\partial\Omega^1(t) = \Gamma_{in}^1 \cup \Gamma_{out}^1 \cup \Gamma_a(t) \cup \Gamma_b(t), \tag{3.1}$$

$$\partial\Omega^2(t) = \Gamma_{in}^2 \cup \Gamma_{out}^2 \cup \Gamma_a(t) \cup \Gamma_b(t) \cup \Gamma_{0_a} \cup \Gamma_{0_b}, \tag{3.2}$$

where $\Gamma_{in}^1$ and $\Gamma_{in}^2 = \Gamma_{in_a}^2 \cup \Gamma_{in_b}^2$ represent the inlet boundaries for each $\Omega^i$, and $\Gamma_{out}^1$ and $\Gamma_{out}^2 = \Gamma_{out_a}^2 \cup \Gamma_{out_b}^2$ represent the outlet boundaries for each subdomain $\Omega^i$. We shall then denote by $\Gamma_{in} = \Gamma_{in}^1 \cup \Gamma_{in}^2$ and $\Gamma_{out} = \Gamma_{out}^1 \cup \Gamma_{out}^2$ the inlet and outlet boundaries, respectively, for the entire domain $\Omega$. On the other hand, $\Gamma_{0_a}$ and $\Gamma_{0_b}$ represent the boundaries corresponding to the pipe wall; and, finally, the separation interfaces between both components are given by $\Gamma_a(t) = \overline{\Omega}^1(t) \cap \overline{\Omega}_a^2(t)$ and $\Gamma_b(t) = \overline{\Omega}^1(t) \cap \overline{\Omega}_b^2(t)$ in the upper and lower part, respectively.

It is assumed that the flow is sufficiently smooth, i.e., with a low Reynolds number ($R < 2000$) [29], and this situation holds until a certain time $T$. Therefore, at the initial time the interfaces between the two fluids are considered to be straight lines. It is assumed that these interfaces are never adjacent to the pipe walls, and that they are always separated sufficiently well that there is little possibility that they will collide [14]. Based on these premises, both interfaces can be conveniently parametrized as follows:

$$\Gamma_a : \quad (x,t) \longmapsto \Phi_a(x,t),$$
$$\Gamma_b : \quad (x,t) \longmapsto \Phi_b(x,t),$$

so that, for small quantities $\delta_1$ , $\delta_2$ , $\delta_3 > 0$ and all $x \in [0,L]$, we have

$$\Phi_a(x,t) - \Phi_b(x,t) > \delta_1 > 0, \quad \Phi_b(x,t) > -D + \delta_2, \quad \Phi_a(x,t) < D - \delta_3; \tag{3.3}$$

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the subdomains being defined by:

$$\Omega^1(t) = \{(x,y) \in \Omega : \quad 0 \le x \le L, \;\; \Phi_b(x,t) < y < \Phi_a(x,t)\}\,, \tag{3.4}$$

$$\Omega^2_a(t) = \{(x,y) \in \Omega : \quad 0 \le x \le L, \;\; \Phi_a(x,t) < y \le D\}\,, \tag{3.5}$$

$$\Omega^2_b(t) = \{(x,y) \in \Omega : \quad 0 \le x \le L, \;\; -D \le y < \Phi_b(x,t)\}\,, \tag{3.6}$$

where $D > 0$ is the radius of the pipeline and $L$ is the pipeline length.

To describe the density and the viscosity in all $\Omega$, we introduce the piecewise constant quantities $\rho$ and $\mu$ defined by

$$\rho = \sum_{i=1}^{2} \chi^i \rho^i \quad \text{and} \quad \mu = \sum_{i=1}^{2} \chi^i \mu^i, \tag{3.7}$$

where $\chi^i$ is the characteristic function of subdomain $\Omega^i$, $i = 1, 2$, with $\chi^2$ defined by

$$\chi^2 = \chi_{\Omega^2_a \cup \Omega^2_b} = \chi_{\Omega^2_a} + \chi_{\Omega^2_b}.$$

Here $\rho^i$ and $\mu^i$ are, respectively, the given densities and the viscosities constant in each $\Omega^i$, $i = 1, 2$. The velocity and pressure fields are set forth as follows:

$$\forall\; (\mathbf{x}, t) \in \Omega^i \times [0, T], \quad \mathbf{u} = \mathbf{u}^i(\mathbf{x}, t) = \left(u^i_x(\mathbf{x}, t), u^i_y(\mathbf{x}, t)\right), \;\; p = p^i(\mathbf{x}, t), \;\; i = 1, 2.$$

Then, for each time $t \in [0, T] \subset \mathbb{R}$, the Navier-Stokes problem is written as

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{g} \quad & \text{in each } \Omega^i, \;\; i = 1, 2, \\ \nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \end{aligned} \tag{3.8}$$

where $\mathbf{g}$ is the gravity and

$$\mathbf{u}\cdot\nabla\mathbf{u} = u_x \frac{\partial \mathbf{u}}{\partial x} + u_y \frac{\partial \mathbf{u}}{\partial y}.$$

Togther with this PDE, we have the initial condition

$$\forall \mathbf{x} \in \Omega^i, \quad \mathbf{u}^i(\mathbf{x}, 0) = \mathcal{U}^i_0(\mathbf{x}), \;\; i = 1, 2, \tag{3.9}$$

in which $\mathcal{U}^i_0$ is a smooth function such that $\nabla\cdot\mathcal{U}^i_0 = 0$ in $\Omega^i$, $i = 1, 2$, and such that $\mathcal{U}^i_0(\Gamma_{0_j} \cap \Gamma^2_{in}) = \mathbf{0}$, $j = a, b$. In turn, the boundary conditions are

$$\begin{aligned} \mathbf{u} = \mathbf{U} \quad & \text{on } \Gamma_{in}, \\ \mathbf{u}^2 = \mathbf{0} \quad & \text{on } \Gamma_{0_j} \text{ for } j = a, b, \\ \sigma\cdot\mathbf{n} = -p_{out}\mathbf{n} \quad & \text{on } \Gamma_{out}, \end{aligned} \tag{3.10}$$

and the interface conditions (continuity of the velocity and balance of the normal stress with the surface tension across the interface) are

$$\left[\mathbf{u}^i\right]_{\Gamma_j} = \mathbf{0}, \;\; [\sigma]_{\Gamma_j}\cdot\mathbf{n}^1_j = -\frac{\kappa_j}{R_j}\mathbf{n}^1_j \quad \text{for } i = 1, 2 \text{ and } j = a, b, \tag{3.11}$$

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where $\mathbf{U} = \mathbf{U}^i$ on $\Gamma^i_{in}$ for $i = 1, 2$ denotes the given inlet velocity independent of time, $p_{out}$ is a given exterior pressure on the outlet boundary, $\mathbf{n}$ is the unit exterior normal vector to the boundary of $\Omega^i$, $\mathbf{n}^1_j$ is the normal vector to $\Gamma_j$, $j = a, b$, exterior to $\Omega^1$, and $[\cdot]_{\Gamma_j}$ denotes the jump on $\Gamma_j$ in the direction of $\mathbf{n}^1_j$, $j = a, b$, i.e.,

$$[f]_{\Gamma_j} = f|_{\Omega^1} - f|_{\Omega^2_j}.$$

Physically, the first condition in (3.10) represents the velocity field with which the fluids enter into $\Omega^i$, $i = 1, 2$, through $\Gamma_{in}$. The second condition refers to the non-sliding condition that is greater than the velocity field of fluid 2 since it is in contact with the pipe wall, $\Gamma_{0_j}$, $j = a, b$, as a solid element of subdomain $\Omega^2(t)$. Finally, the third condition represents the balance of normal surface tension forces with respect to the external pressure necessary to establish equilibrium with the stresses exerted by both fluids on the outlet boundary $\Gamma_{out}$. For details on this type of boundary conditions, see [5, 12, 13]. On the other hand, $\kappa_j > 0$, $j = a, b$, are given physical constants depending of the two fluids in contact. In this spirit, as the lubricated fluid (water, top and bottom) is the same, it would be natural to take $\kappa_a = \kappa_b$ =surface tension between water and oil. As the geometry of the interface is involved in the term [10], $R_j$, $j = a, b$, denotes the radius of the curvature with the appropriate sign, i.e., with the convention that $R_j > 0$ if the center of the curvature of $\Gamma_j$ is located in $\Omega^1$, $j = a, b$; and $\sigma$ is the stress tensor given by the Navier-Stokes constitutive equation for incompressible Newtonian fluids:

$$\sigma = \sigma\left(\mathbf{u}, p\right) = \mu A_1\left(\mathbf{u}\right) - p\mathbf{I},$$

where $A_1(\mathbf{u}) = \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)$ is the rate of strain tensor; see [5, 7, 25, 29].

We assume that the inlet velocity $\mathbf{U}$ has the form:

$$\forall y \in (-D, D), \quad \mathbf{U} = -U(y)\mathbf{n} = \left(U(y), 0\right)^t, \quad U(y) > 0, \tag{3.12}$$

i.e., the inlet velocity is parallel to the normal vector $\mathbf{n}$ and is directed *inside* $\Omega$. Moreover, we assume that $U(D) = U(-D) = 0$; thus, $\mathbf{U}$ satisfies the compatibility condition

$$\mathbf{U}^2\left(\Gamma_{0_j} \cap \Gamma^2_{in}\right) = \mathbf{0}, \quad j = a, b. \tag{3.13}$$

Finally, the equation for the motion of the free surface $\Gamma_j$, $j = a, b$, (stating the immiscibility of the fluids) is given by

$$\frac{\partial \Phi_j}{\partial t} + u_x \frac{\partial \Phi_j}{\partial x} = u_y, \quad j = a, b, \tag{3.14}$$

with the following initial and boundary conditions:

$$\begin{aligned} \forall x \in [0, L], \quad \Phi_j(x, 0) = \pm y_0, \quad j = a, b, \\ \forall t \in [0, T], \quad \Phi_j(0, t) = \pm y_0, \quad j = a, b, \end{aligned}$$

where $\pm y_0 \in (0, \pm D)$ is a constant that adopts the $+$ sign if $j = a$ and the $-$ sign if $j = b$.

In [4], the nonlinear convection term in the Navier-Stokes equations (3.8) was semi-discretized using the characteristics method; see [1, 3, 6]. With this, the position of the fluid particles is a function of $t$ and the convection term is the total derivative (or the material derivative) which is approximated by

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{d\mathbf{u}}{dt} \approx \frac{\mathbf{u}^{m+1}_m - \mathbf{u}_m\left(\mathbf{X}^m\right)}{\delta t},$$

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where $\delta t > 0$ is the time step, $\mathbf{u}_m^{m+1}$ is an approximation of the fluid's velocity at time $t^{m+1}$ defined on the approximate domain at time $t^m$, and $\mathbf{X}^m = \mathbf{X}(t^m)$ is the characteristic at time $t^m$. Then, the Navier-Stokes equations in (3.8) are transformed into the known momentum equations, and the generalized Stokes system is obtained for incompressible flow (for simplicity, the dependency on each time $t^m$ was suppressed):

$$\begin{aligned} \alpha\rho\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{g} + \rho\alpha\mathbf{w} \quad & \text{in each } \Omega^i,\ i = 1, 2, \\ \nabla\cdot\mathbf{u} = 0 \quad & \text{in } \Omega, \end{aligned} \tag{3.15}$$

where $\alpha$ represents $1/\delta t$ and $\mathbf{w}$ represents $\mathbf{u}_m\left(\mathbf{X}^m\right)$ (known at a previous time $t^{m-1}$). The boundary conditions are given by (3.10) and the problem posed in these terms is known as a *generalized Stokes problem* (cf. [18, 20]), which can be solved for each time step given the surface that describes the interface. The first condition for the interfaces (3.11) remains the same, and since the position of each interface is now known, the second condition simplifies, and (3.11) is transformed into

$$\left[\mathbf{u}^i\right]_{\Gamma_j} = \mathbf{0},\quad [\sigma]_{\Gamma_j}\cdot\mathbf{n}_j^1 = -K_j\mathbf{n}_j^1 \quad \text{for } i = 1, 2 \text{ and } j = a, b, \tag{3.16}$$

where $K_j$, used to represent $\kappa_j/R_j$ for each $j = a, b$, is now a known function. Finally, (3.15), (3.10), and (3.16) are the expressions that define the generalized Stokes problem with non-standard boundary conditions. Let us consider a variational formulation equivalent to the generalized Stokes problem with non-standard boundary conditions (3.15), (3.10) and (3.16), studied in [4]: given the functions $\mathbf{w}$, $p_{out}$, $\mu$, $\rho$, $K_j$, $j = a, b$, and the constant $\alpha$, find $\mathbf{u} \in X + \overline{\mathbf{U}}$ and $p \in M$ to solve

$$\begin{aligned} \forall\mathbf{v}\in X,\quad & a\left(\mathbf{u},\mathbf{v}\right) + b\left(\mathbf{v},p\right) = \ell\left(\mathbf{v}\right), \\ \forall q \in M,\quad & b\left(\mathbf{u},q\right) = 0, \end{aligned} \tag{3.17}$$

where the function $\overline{\mathbf{U}}$ is an extension to all $\Omega$ of the inlet velocity $\mathbf{U}$,

$$X = \left\{\mathbf{v}\in\left[H^1(\Omega)\right]^2;\ \ \mathbf{v}|_{\Gamma_{in}} = \mathbf{0},\ \mathbf{v}|_{\Gamma_{0_j}} = \mathbf{0} \text{ for } j = a, b\right\}, \tag{3.18}$$

$$M = \left\{q : \Omega \to \mathbb{R};\ \ q \in L^2(\Omega)\right\}, \tag{3.19}$$

$a : X \times X \to \mathbb{R}$ is a *bilinear form* defined as

$$a(\mathbf{u},\mathbf{v}) = \alpha\int_\Omega \rho\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} + \frac{1}{2}\int_\Omega \mu A_1(\mathbf{u}) : A_1(\mathbf{v})\,d\mathbf{x},$$

$b : X \times M \to \mathbb{R}$ is a *bilinear form* defined by

$$b(\mathbf{v},p) = -\int_\Omega p\nabla\cdot\mathbf{v}\,d\mathbf{x}$$

and, finally, $\ell : X \to \mathbb{R}$ is a *linear form* defined by

$$\ell(\mathbf{v}) = \int_\Omega \rho\left(\mathbf{g} + \alpha\mathbf{w}\right)\cdot\mathbf{v}\,d\mathbf{x} - \int_{\Gamma_{out}} p_{out}\mathbf{n}\cdot\mathbf{v}\,ds - \int_{\Gamma_a} K_a\mathbf{n}_a^1\cdot\mathbf{v}\,ds - \int_{\Gamma_b} K_b\mathbf{n}_b^1\cdot\mathbf{v}\,ds.$$

The pressure has no indeterminate constant because the outflow condition in (3.10) and the transmission condition in (3.11) involve the stress tensor.

In [4], it is demonstrated that the bilinear form $a(\cdot,\cdot)$ is $X$-elliptic, the linear form $\ell(\cdot)$ is a continuous linear functional, and the bilinear form $b(\cdot,\cdot)$ satisfies an inf-sup condition over $X \times M$. Therefore, the variational formulation (3.17) is a well-posed problem.

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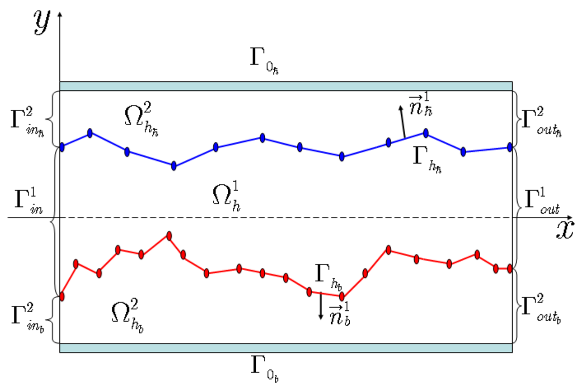

FIGURE 4.1. *Domain $\Omega_h$ and boundary $\partial\Omega_h$*

**4. Spatial discretization.** Taking into account that $K_j$, $j = a, b$, is related to the surface tension, from this section onwards we shall assume that each $\Gamma_j$ is a curve of class $\mathcal{C}^1 \cap H^2(\Omega)$ with a horizontal tangent at the inlet point $(0, y_0)$ for $\Phi_a$ and at $(0, -y_0)$ for $\Phi_b$, i.e.,

$$\Phi_a'(0) = 0 \quad \text{y} \quad \Phi_b'(0) = 0.$$

Then we triangulate separately each subdomain $\Omega^i$ with a triangulation $\mathcal{T}_h^i$, so that the global triangulation given by

$$\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2,$$

with $\mathcal{T}_h^2 = \mathcal{T}_{h,a}^2 \cup \mathcal{T}_{h,b}^2$, $h = \max_{T \in \mathcal{T}_h}(h_T)$, is conforming and regular in the sense that there is no overlapping among the elements of triangulation, and there exists a constant $\zeta$ independent of $h$ such that

$$\forall\, T \in \mathcal{T}_h, \quad \frac{h_T}{\rho_T} \leq \zeta,$$

where $h_T$ denotes the diameter of each $T$ and $\rho_T$ is the diameter of the circle inscribed in $T$. Let us also consider that $\mathcal{T}_h$ contains two polygonal lines $\Gamma_{h,a}$ and $\Gamma_{h,b}$ whose nodes are on the interfaces $\Gamma_a$ and $\Gamma_b$, respectively. Then we denote by $\Omega_h^i$ the domain approximating $\Omega^i$, i.e., the region bounded by $\Gamma_{h,a}$, $\Gamma_{h,b}$, $\Gamma_0^i$, $\Gamma_{in}^i$ and $\Gamma_{out}^i$ as illustrated in Figure 4.1.

In order to employ the Crouzeix-Raviart/$P_0$ finite elements, we denote by $\mathcal{E}(T)$ the set of all sides of an element $T \in \mathcal{T}_h$, $\mathcal{E} = \bigcup_{T\in\mathcal{T}_h} \mathcal{E}(T)$ the set of all sides in $\mathcal{T}_h$, $\mathcal{E}_{\partial\Omega_h} = \{e \in \mathcal{E};\ \ e \subset \Omega_h\}$ the set of sides at the boundary, and $\mathcal{E}_0 = \mathcal{E} \setminus \mathcal{E}_{\partial\Omega_h}$ the set of internal sides, including even those corresponding to the two polygonal lines which approximate the interfaces. For a given piecewise continuous function $v$, the jump $[v]$ on a side $e \in \mathcal{E}$ is defined by

$$[v](\mathbf{x}) = v(\mathbf{x})\big|_{T_1} - v(\mathbf{x})\big|_{T_2}, \quad e = (T_1 \cap T_2) \not\subset \partial\Omega_h,\ T_1, T_2 \in \mathcal{T}_h,$$

where $\mathbf{x}$ is a point in $e$. Finally, for every $e \in \mathcal{E}$, its midpoint shall be denoted by $\mathbf{x}_e$.

From everything dealt with above, the nonconforming Crouzeix-Raviart space of degree one [11] is defined by

$$P_h = \left\{ v_h \in L^2(\Omega) : \ \ \forall T \in \mathcal{T}_h,\ v_h\big|_T \in P_1(T);\ \forall e \in \mathcal{E},\ v_h \text{ is continuous on } \mathbf{x}_e \right\},$$

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in which $P_1(T)$ is the space of polynomials of degree less than or equal to one defined on each $T \in \mathcal{T}_h$. The property that $v_h$ is only continuous on $\mathbf{x}_e$ implies that

$$\forall e \in \mathcal{E}_0, \quad \int_e [v_h]\, ds = 0.$$

If $e_i$, $i = 1, 2, 3$, is the $i$th side of $T \in \mathcal{T}_h$ opposite to its $i$th vertex $\mathbf{a}_i$, then for the space $P_h$ the base functions associated to the midpoint $\mathbf{x}_{e_i}$ are given by

$$\varphi_{i,T} = 1 - 2\lambda_i, \quad i = 1, 2, 3,$$

where $\lambda_i$ is the usual barycentric coordinate relative to the vertex $\mathbf{a}_i$. Furthermore, for each $\varphi_{i,T}$, $i = 1, 2, 3$, we have that

$$\int_{e_i} \varphi_{j,T}\, ds = |e_i|\delta_{ij}, \tag{4.1}$$

where $|e_i|$ is the length of the side $e_i$, $i = 1, 2, 3$, and $\delta_{ij}$ is the Kronecker delta function. We introduce now the global Crouzeix-Raviart interpolation operator, $\Pi_h \in \mathcal{L}(H^1(\Omega), P_1)$, defined as

$$\forall v \in H^1(\Omega),\ \forall T \in \mathcal{T}_h,\ \ \Pi_h(v)\big|_T = \Pi_T(v\big|_T),$$

where $\Pi_T \in \mathcal{L}(H^1(T), P_1(T))$ is the local Crouzeix-Raviart operator defined by

$$\int_{e_i} \Pi_T(v)\, ds = \int_{e_i} v\, ds, \quad i = 1, 2, 3.$$

Now, considering (4.1), we have that for $\mathbf{x}_{e_i}$

$$\Pi_T(v)(\mathbf{x}_{e_i}) = \frac{1}{|e_i|}\int_{e_i} v\, ds, \quad i = 1, 2, 3.$$

Hence, $\Pi_T : H^1(T) \to P^1(T)$ can be written as

$$\forall v \in H^1(T),\ \ \Pi_T(v) = \sum_{i=1}^{3} \Pi_T(v)(\mathbf{x}_{e_i})\varphi_{i,T}.$$

Thus, the velocity $\mathbf{u}$ of the problem shall be approximated by a function $\mathbf{u}_h$ belonging to the following space:

$$X_h = \left\{\mathbf{v}_h \in L^2(\Omega)^2;\ \forall T \in \mathcal{T}_h,\ \mathbf{v}_h\big|_T \in P_h^2\right\}. \tag{4.2}$$

Moreover, we define the space of functions $\mathbf{v}_h$ with zero discrete divergence in each $T \in \mathcal{T}_h$ as follows:

$$V_h = \left\{\mathbf{v}_h \in X_h;\ \ \nabla_h \cdot \mathbf{v}_h = 0\right\}, \tag{4.3}$$

in which $\nabla_h \cdot \mathbf{v}_h = \nabla \cdot (\mathbf{v}_h\big|_T)$ for each $T \in \mathcal{T}_h$. Similarly, we approximate the pressure $p$ by a function $p_h$ from the space

$$M_h = \left\{q_h \in L^2(\Omega);\ q_h\big|_T \in P_0(T),\ \forall T \in \mathcal{T}_h\right\},$$

where $P_0(T)$ is the space of zero degree polynomials defined on $T \in \mathcal{T}_h$.

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Now, we must define the lifting of the inlet velocity $\mathbf{U}$, so that the lifting can be appropriately approximated, considering that it is not reasonable to assume that $U$ is globally smooth. Due to physical considerations, the regularity of $U$ cannot be greater than that of the *inflow* velocity of the static problem, specifically, $H^{3/2-\varepsilon}(\Gamma_{in})$. The following proposition demonstrates the existence of a lifting $\overline{\mathbf{U}}$.

PROPOSITION 4.1. *Assume that $\Gamma_j$, $j = a, b$, satisfies the above assumptions, and that $U$ belongs to $H^1(\Gamma_{in})$ and its restriction to $\Gamma^i_{in}$ belongs to $H^2(\Gamma^i_{in})$ for $i = 1, 2$. Then there exists a function $\overline{\mathbf{U}} \in [H^1(\Omega)]^2$ such that $\operatorname{div}(\overline{\mathbf{U}}) = 0$ in $\Omega$, $\overline{\mathbf{U}} = \mathbf{0}$ on $\Gamma_{0_j}$, $j = a, b$, $\overline{\mathbf{U}} = (U, 0)^t$ on $\Gamma_{in}$, its restriction $\overline{\mathbf{U}}^i$ to $\Omega^i$ belongs to $[H^2(\Omega^i)]^2$ for $i = 1, 2$, and*

$$\|\overline{\mathbf{U}}\|_{[H^2(\Omega^i)]^2} \leq C \|U\|_{H^2(\Gamma^i_{in})} \quad \textit{for } i = 1, 2, \tag{4.4}$$

*with a constant $C$ that is independent of $U$.*

*Proof*. The proof shall be presented schematically for $\Omega^1$; it is similar for $\Omega^2$. The construction of $\overline{\mathbf{U}}$ is based on the observation that if $\Phi_a(x) = y_0$ and $\Phi_b(x) = -y_0$ for all $x \in [0, L]$, then the function $\overline{\mathbf{U}}$ satisfies all the requirements in this proposition. Therefore, as a first intermediate step, we propose the lifting

$$\widetilde{\mathbf{U}} = (\widetilde{U}, 0)^t,$$

in which

$$\forall (x, y) \in \Omega, \quad \widetilde{U}(x, y) = U\left(\mathscr{L}(x, y)\right) a(x) + c b(x) \varrho^1(y), \tag{4.5}$$

for which $a \in \mathcal{C}^\infty$ is a truncating function satisfying

$$0 \leq a(x) \leq 1 \text{ in } [0, L], \quad a(x) = 1 \text{ in } [0, L/2], \quad a(x) = 0 \text{ in } [3L/4, L];$$

$b \in \mathcal{C}^\infty$ is given by $b(x) = 1 - a(x)$; $c$ is a constant such that for each connected component $\Omega^i$, $i = 1, 2$, we have

$$\int_{\partial\Omega^i} \widetilde{\mathbf{U}} \cdot \mathbf{n} \, ds = 0;$$

$\mathscr{L}$ is the function

$$\begin{aligned} &\forall x \in [0, L], \ \forall y \in [\Phi_b(x), \Phi_a(x)], \\ &\qquad \mathscr{L}(x, y) = \left(\frac{2y_0}{\Phi_a(x) - \Phi_b(x)}\right) y - y_0 \left(\frac{\Phi_a(x) + \Phi_b(x)}{\Phi_a(x) - \Phi_b(x)}\right); \end{aligned}$$

and, finally, $\varrho^1$ is the restriction to $\Gamma^i_{out}$ of a smooth non-negative function $\varrho$ that vanishes identically in a neighborhood of $\partial\Omega$ except in the neighborhood of $\Gamma_{out}$. Moreover, its trace on $\Gamma_{out}$ has compact support and satisfies

$$\int_{\Gamma_{out}} \varrho \, ds = 1.$$

Since $\partial\Omega^1$ is closed, and considering the definition of $c$, we have

$$\int_{\Omega^1} \operatorname{div}(\widetilde{\mathbf{U}}) \, d\mathbf{x} = 0.$$

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The foregoing considerations imply that $\widetilde{\mathbf{U}}$ satisfies all the hypotheses in the proposition, except that its divergence does not necessarily have to be zero. In effect,

$$\operatorname{div}(\widetilde{\mathbf{U}}) = g(x, y),$$

with

$$g(x, y) = 2y_0 \left( \frac{[\Phi_b(x) - y]\Phi_a'(x) - [\Phi_a(x) - y]\Phi_b'(x)}{(\Phi_a - \Phi_b)^2} \right) a(x) \frac{\partial U}{\partial \mathscr{L}} + U(\mathscr{L}(x, y))a'(x) + c\varrho^1(y)b'(x),$$

where $a'$ and $b'$ are the first derivatives with respect to $x$ of the functions $a$ and $b$, respectively.

It can also be shown that

$$\frac{\widetilde{U}}{\delta} \in L^2(\Omega^1), \tag{4.6}$$

where $\delta(\mathbf{x})$ is the minimum distance of $\mathbf{x} \in \Omega^1$ to the corners of $\partial\Omega^1$. Then, according to Kellogg and Osborn [24], the condition given by (4.6) implies that there exists a function $\mathbf{v} \in [H^2(\Omega^1) \cap H_0^1(\Omega^1)]^2$ such that

$$\operatorname{div}(\mathbf{v}) = g \ \text{ in } \Omega^1 \quad \text{and} \quad \|\mathbf{v}\|_{[H^2(\Omega^1)]^2} \leq C\|g\|_{H^1(\Omega^1)}.$$

This is so even when the domain $\Omega^1$ is not convex, since in this case it is only required that the angles at the two corners of the boundary $\Gamma_{in}^1$ be smaller than $\pi$; this is satisfied due to the hypotheses established on the interfaces. Finally, the proof is concluded by taking

$$\overline{\mathbf{U}} = \widetilde{\mathbf{U}} - \mathbf{v}, \tag{4.7}$$

as the appropriate lifting that satisfies all the requirements of this proposition. □

Since a function that belongs to $H^1(\Omega^i)$ for $i = 1, 2$ also belongs to $H^{1/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$, Proposition 4.1 has the following corollary.

COROLLARY 4.2. *Under the assumptions of Proposition 4.1, the function $\overline{\mathbf{U}}$ defined by (4.7) belongs to $[H^{3/2-\varepsilon}(\Omega)]^2$ for any $\varepsilon > 0$, and there exists a constant $C(\varepsilon)$ independent of $\overline{\mathbf{U}}$ and $U$ such that*

$$\|\overline{\mathbf{U}}\|_{[H^{3/2-\varepsilon}(\Omega)]^2} \leq C(\varepsilon)\|U\|_{H^{3/2-\varepsilon}(\Gamma_{in})} \leq C(\varepsilon) \sum_{i=1}^{2} \|U\|_{H^2(\Gamma_{in}^i)}. \tag{4.8}$$

As $\overline{\mathbf{U}} \in [H^1(\Omega)]^2$, it can be approximated component by component using the global Crouzeix-Raviart interpolator, since it is enough for it to possess a continuous trace in each segment. We define this approximation by

$$\overline{\mathbf{U}}_h = (\Pi_h(U), 0)^t$$

and we have the following result.

PROPOSITION 4.3. *Let $\overline{\mathbf{U}}$ be the function of the Proposition 4.1 and $\overline{\mathbf{U}}_h$ its approximation by means of the global Crouzeix-Raviart interpolator. Then*

$$\forall q_h \in M_h, \ \int_\Omega q_h \, \nabla_h \cdot \overline{\mathbf{U}}_h \, d\mathbf{x} = 0.$$

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*Proof*. For all $q_h \in M_h$ we have

$$\int_\Omega q_h \, \nabla_h \cdot \overline{\mathbf{U}}_h \, d\mathbf{x} = \sum_{T\in\mathcal{T}_h} q_h\big|_T \sum_{i=1}^{3} \int_{e_i} \overline{\mathbf{U}}_h \cdot \mathbf{n}_{e_i} \, ds,$$

where $\mathbf{n}_{e_i}$ is the normal unit vector external to $T$ on its $i$th side $e_i$. By the definition of the Crouzeix-Raviart interpolation operator

$$\int_{e_i} \overline{\mathbf{U}}_h \cdot \mathbf{n}_{e_i} \, ds = \int_{e_i} \overline{\mathbf{U}} \cdot \mathbf{n}_{e_i} \, ds.$$

Thus,

$$\forall q_h \in M_h, \quad \int_\Omega q_h \nabla_h \cdot \overline{\mathbf{U}}_h \, d\mathbf{x} = \int_\Omega q_h \nabla \cdot \overline{\mathbf{U}} \, d\mathbf{x}$$

and the proof concludes with the fact that $\nabla \cdot \overline{\mathbf{U}} = 0$ in $\Omega$. ☐

Another point that must be addressed prior to proposing a discrete variational formulation involves the approximation of the integral term that takes surface tension into account,

$$K_j(\mathbf{v}) = \int_{\Gamma_j} K_j \mathbf{n}_j^1 \cdot \mathbf{v} \, ds, \quad j = a, b.$$

The proposed approximation is motivated by the fact that with the sign convention used for each $R_j$, we have

$$\frac{\mathbf{n}_j^1}{R_j} = -\frac{\mathbf{n}_j}{\overline{R}_j} = \frac{d\mathbf{t}_j}{ds}, \;\; j = a, b,$$

where $\mathbf{t}_j$ is the tangent to $\Gamma_j$ in the direction of increasing $s$, which is the same as the direction of increasing $x$; $\mathbf{n}_j$ is the principal normal vector to $\Gamma_j$, i.e., parallel to $\mathbf{n}_j^1$ and directed toward the *center of curvature* of $\Gamma_j$; and $\overline{R}_j$ is the positive *radius of curvature*, i.e., $\overline{R}_j = R_j$ if the center of the curvature is located inside of $\Omega^1$ and $\overline{R}_j = -R_j$ if the center of curvature is on the other side. Therefore, in the case of test functions $\mathbf{v}_h \in X_h$, we have

$$K_j(\mathbf{v}_h) = \int_{\Gamma_j} \kappa_j \mathbf{v}_h \cdot \frac{d\mathbf{t}_j}{ds} \, ds = \kappa_j \int_{\Gamma_j} \mathbf{v}_h \cdot \frac{d\mathbf{t}_j}{ds} \, ds, \;\; j = a, b. \tag{4.9}$$

Now, let

$$\mathbf{x}_r^j = \left(x_r^j, y_r^j = \Phi_j(x_r^j)\right), \quad 0 \le r \le N, \;\; j = a, b$$

be the points defined in each $\Gamma_j$, with

$$0 = x_0^j < x_1^j < \cdots < x_{(N-1)}^j < x_N^j = L, \;\; j = a, b.$$

Let $\tilde{S}_r^j$ be the arch of $\Gamma_j$ with end points $\mathbf{x}_r^j$ and $\mathbf{x}_{r+1}^j$. The polygon $\Gamma_{h,j}$ is built from straight lines that connect the nodes of $\mathbf{x}_r^j$ to $\mathbf{x}_{r+1}^j$ for every $0 \le r \le N-1$, and we denote by $S_r^j$ the chord $[\mathbf{x}_r^j, \mathbf{x}_{r+1}^j]$. We define the unit tangent $\mathbf{t}^j$ along the chord $S_r^j$ by

$$\mathbf{t}_r^j = \frac{\mathbf{x}_{r+1}^j - \mathbf{x}_r^j}{\left|\mathbf{x}_{r+1}^j - \mathbf{x}_r^j\right|}, \;\; 0 \le r \le N, \;\; j = a, b,$$

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and approximate $K_j(\mathbf{v}_h)$ by the following formula:

$$\forall \mathbf{v}_h \in X_h, \quad K_{h_j}(\mathbf{v}_h) = \kappa_j \sum_{r=1}^{N-1} \mathbf{v}_h(\mathbf{x}^j_{r+1/2}) \cdot (\mathbf{t}^j_r - \mathbf{t}^j_{r-1}), \quad j = a, b, \tag{4.10}$$

where $\mathbf{x}^j_{r+1/2}$ is the midpoint of the segment $S^j_r$ where the function is approximated with the Crouzeix-Raviart element.

Since elements of $P_h$ do not have global weak derivatives, we must use a modification of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in order to state the discretization of the problem. Thus, we define

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} \Big[ \alpha \int_T \rho_h \mathbf{u}_h \cdot \mathbf{v}_h \, d\mathbf{x} \\ + \frac{1}{2} \int_T \mu_h (\nabla_h \mathbf{u}_h + (\nabla_h \mathbf{u}_h)^t) : (\nabla_h \mathbf{v}_h + (\nabla_h \mathbf{v}_h)^t) \, d\mathbf{x} \Big] \\ + J_h(\mathbf{u}_h, \mathbf{v}_h), \end{aligned} \tag{4.11}$$

where for all $\mathbf{v}_h \in X_h$, the term $\nabla_h \mathbf{v}_h = \nabla(\mathbf{v}_h\big|_T)$ is the discrete gradient in $T \in \mathcal{T}_h$, $\rho_h$ and $\mu_h$ are approximations to the density and the viscosity respectively defined by

$$\rho_h\big|_{\Omega^i_h} = \rho^i, \quad \mu_h\big|_{\Omega^i_h} = \mu^i, \quad i = 1, 2, \tag{4.12}$$

and

$$\forall \mathbf{u}_h, \mathbf{v}_h \in X_h, \quad J_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{e\in\mathcal{E}} \frac{1}{|e|} \int_e [\mathbf{u}_h] \cdot [\mathbf{v}_h] \, ds$$

is the factor that controls the *rigid body rotations* which cause a lack of coercivity for the Crouzeix-Raviart approximation when applied to the Stokes formulation written in terms of the strain tensor $A_1(\mathbf{u})$ [8]. Likewise,

$$b_h(\mathbf{u}_h, q_h) = -\sum_{T\in\mathcal{T}_h} \int_T q_h \, \nabla_h \cdot \mathbf{v}_h \, d\mathbf{x}. \tag{4.13}$$

Based on the foregoing, the discrete variational formulation is as follows: find $\mathbf{u}_h \in X_h + \overline{\mathbf{U}}_h$ and $p_h \in M_h$ that satisfy

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = \ell(\mathbf{v}_h), \\ \forall q_h \in M_h, \quad b_h(\mathbf{u}_h, q_h) = 0, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} \ell(\mathbf{v}_h) = \int_\Omega \rho_h \, (\mathbf{g} + \alpha \mathbf{w}) \cdot \mathbf{v}_h \, d\mathbf{x} \\ - \int_{\Gamma_{out}} p_{out} \mathbf{v}_h \cdot \mathbf{n} \, ds - K_{h,a}(\mathbf{v}_h) - K_{h,b}(\mathbf{v}_h). \end{aligned} \tag{4.15}$$

In the following, we will define the norm associated to the space $X_h$ to be

$$\forall \mathbf{v}_h \in X_h, \quad \|\mathbf{v}_h\|_h^2 = \|\mathbf{v}_h\|^2_{[L^2(\Omega)]^2} + \|\nabla \mathbf{v_h}\|^2_{[L^2(\Omega)]^2} + J_h(\mathbf{v}_h, \mathbf{v}_h), \tag{4.16}$$

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where

$$\forall \mathbf{v} \in [L^2(\Omega)]^2, \quad |\!|\!|\nabla \mathbf{v}|\!|\!|^2_{[L^2(\Omega)]^2} = \sum_{T \in \mathcal{T}_h} \|\nabla \mathbf{v}\|^2_{[L^2(T)]^2}, \tag{4.17}$$

is the *broken norm* [19]. The following result on boundedness for $J_h$ is obtained.

PROPOSITION 4.4. *There exist constants $C'$ and $C''$ independent of $h$ such that*

$$\forall \mathbf{v}_h \in X_h, \quad J_h(\mathbf{v}_h, \mathbf{v}_h) \leq C' |\!|\!|\nabla \mathbf{v}_h|\!|\!|^2_{[L^2(\Omega)]^2} \leq C'' \|\mathbf{v}_h\|^2_h. \tag{4.18}$$

*Proof.* The proof is similar to that presented by Girault et al. in [19]. ☐

On the other hand, with Korn's inequality for piecewise $H^1$ vector fields [9],

$$\forall \mathbf{v}_h \in X_h, \quad |\!|\!|\nabla \mathbf{v}_h|\!|\!|^2_{[L^2(\Omega)]^2} \leq C_1 \Big( |\!|\!|A_1(\mathbf{v}_h)|\!|\!|^2_{[L^2(\Omega)]^2} + J_h(\mathbf{v}_h, \mathbf{v}_h) \Big), \tag{4.19}$$

Poincaré's inequality for piecewise $H^1$ vector fields [8],

$$\forall \mathbf{v}_h \in X_h, \quad \|\mathbf{v}_h\|^2_{[L^2(\Omega)]^2} \leq C_2 \Big( |\!|\!|\nabla \mathbf{v}_h|\!|\!|^2_{[L^2(\Omega)]^2} + J_h(\mathbf{v}_h, \mathbf{v}_h) \Big), \tag{4.20}$$

and the above result of boundedness for $J_h$, we have the following equivalence lemma between the norm associated to the space $X_h$ and the broken norm of the gradient of every function $\mathbf{v}_h \in X_h$.

LEMMA 4.5. *There exist positive constants $\beta_1$ and $\beta_2$ such that*

$$\forall \mathbf{v}_h \in X_h, \quad \beta_1 |\!|\!|\nabla \mathbf{v}_h|\!|\!|^2_{[L^2(\Omega)]^2} \leq \|\mathbf{v}_h\|^2_h \leq \beta_2 |\!|\!|\nabla \mathbf{v}_h|\!|\!|^2_{[L^2(\Omega)]^2}. \tag{4.21}$$

Now, using norm $\|\cdot\|_h$ previously defined, we have the following lemma for the global Crouzeix-Raviart operator.

LEMMA 4.6. *There is a constant $C$, independent of $h$, so that*

$$\forall \mathbf{v} \in [H^1(\Omega)]^2, \quad \|\Pi_h(\mathbf{v})\|^2_h \leq C \|\mathbf{v}\|^2_{[H^1(\Omega)]^2}. \tag{4.22}$$

*Proof.* The proof is similar to that presented, in a more general setting, by J.-L. Guermond and A. Ern in [15]. ☐

**4.1. Uniform stability analysis.** The first step in the numerical analysis of the problem (4.14) consists in proving a *uniform discrete inf-sup condition* for the pair of spaces $X_h$, $M_h$. This is the objective of the following proposition.

PROPOSITION 4.7. *Let $\mathcal{T}_h$ be the triangulation previously defined. There exists a constant $\beta^\star > 0$, independent of $h$, such that the following discrete inf-sup condition holds:*

$$\forall\, q_h \in M_h, \quad \sup_{\mathbf{v}_h \in X_h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_h} \geq \beta^\star \|q_h\|_{L^2(\Omega)}. \tag{4.23}$$

*Proof.* By Fortin's Lemma (see [16]), proving the discrete inf-sup condition is equivalent to demonstrating that there exists a restriction operator $\Pi_h \in \mathcal{L}(X, X_h)$ and a constant $C$, independent of $h$, such that

$$\forall \mathbf{v} \in X, \;\; \|\Pi_h(\mathbf{v})\|_X \leq C \|\mathbf{v}\|_X, \tag{4.24}$$

$$\forall q_h \in M_h, \;\; b_h(\Pi_h(\mathbf{v}) - \mathbf{v}, q_h) = 0. \tag{4.25}$$

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It is easy to check that if $\Pi_h$ is taken as the Crouzeix-Raviart operator, then (4.24) and (4.25) are verified and the proposition is established with

$$\beta^\star = \frac{\beta}{C},$$

where $\beta > 0$ is the constant for the continuous inf-sup condition and $C$ is the constant of Lemma 4.6. □

**4.2. Results on the existence and uniqueness of the discrete solution $(\mathbf{u}_h, p_h)$.** Let us write the velocity $\mathbf{u}_h$ as

$$\mathbf{u}_h = \mathbf{u}_{0,h} + \overline{\mathbf{U}}_h,$$

where $\overline{\mathbf{U}}_h$ satisfies Proposition 4.3. Then the problem (4.14) is the following linear system of equations: find $(\mathbf{u}_{0,h}, p_h) \in X_h \times M_h$ that satisfy

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad a_h\left(\mathbf{u}_{0,h}, \mathbf{v}_h\right) + b_h\left(\mathbf{v}_h, p_h\right) = \ell\left(\mathbf{v}_h\right) - a_h\left(\overline{\mathbf{U}}_h, \mathbf{v}_h\right), \\ \forall q_h \in M_h, \quad b_h\left(\mathbf{u}_{0,h}, q_h\right) = 0. \end{aligned} \tag{4.26}$$

To prove the existence and uniqueness of the solution $\mathbf{u}_{0,h} \in X_h$, the following must be proved:

- Bicontinuity of the bilinear form $a_h(\cdot,\cdot)$ over $X_h \times X_h$: there exists a constant $\gamma > 0$ such that

$$\forall \mathbf{u}_h, \mathbf{v}_h \in X_h, \quad |a_h(\mathbf{u}_h, \mathbf{v}_h)| \le \gamma \|\mathbf{u}_h\|_h \, \|\mathbf{v}_h\|_h. \tag{4.27}$$

- Coercitivity of the bilinear form $a_h(\cdot,\cdot)$ over $X_h \times X_h$: there exists a constant $\alpha^* > 0$ such that

$$\forall \mathbf{v}_h \in X_h, \quad a_h(\mathbf{v}_h, \mathbf{v}_h) \ge \alpha^* \|\mathbf{v}_h\|_h^2. \tag{4.28}$$

On the other hand, the existence and uniqueness of $p_h$ holds automatically from Proposition 4.7 once we have the result for $\mathbf{u}_h$. This is established with the following proposition.

PROPOSITION 4.8. *The system of linear equations given by (4.26) has a unique solution $(\mathbf{u}_{0,h}, p_h) \in X_h \times M_h$.*

*Proof*. Bicontinuity (4.27) is obtained immediately by considering the definition of the bilinear form $a_h$, the norm $\|\cdot\|_h$ associated to space $X_h$, and the Cauchy-Schwartz inequality. Indeed, we have

$$\gamma = \max\left\{\alpha \max_{1\le i\le 2}(\rho^i),\ \frac{1}{2}\max_{1\le i\le 2}(\mu^i), 1\right\}.$$

Finally, coercitivity (4.28) is verified with

$$\alpha^* = \frac{\tilde{C}}{C_1\beta_2}, \quad \tilde{C} = \min\left\{\alpha \min_{1\le i\le 2}(\rho^i),\ \frac{1}{2}\min_{1\le i\le 2}(\mu^i), 1\right\},$$

as an immediate consequence of (4.19) and Lemma 4.5. □

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**5. Numerical simulation.** In this section we present a numerical simulation of the generalized Stokes problem at an initial time $t_0$, considering each interface as a horizontal line within the domain. Therefore, the terms $K_{h,j}(\mathbf{v}_h)$, $j = a, b$, are zero for every $\mathbf{v} \in X_j$. The condition at the pipeline outlet is established for a manometric pressure given by $p_{out} = 0$, and $\mathbf{g} = 9.806\mathbf{e}_2$.

The simulation is based on the following parameters: $\rho_1 = 1$, $\rho_2 = 1.1$, $\mu_1 = 0.1$ and $\mu_2 = 0.01$. The inlet velocity $\mathbf{U} = (U(y))\mathbf{e}_1$ is defined by the function $U$ given by

$$\forall y \in \Gamma^2_{in}, \quad U(y) = \left(-\frac{\xi}{2L\mu_2}\right)y^2 + \left(-\frac{\xi}{2L\mu_2}\right)D^2$$

and

$$\forall y \in \Gamma^1_{in}, \quad U(y) = \left(-\frac{\xi}{2L\mu_1}\right)y^2 + \left(-\frac{\xi}{2L\mu_2}\right)\left[\frac{|\Gamma^1_{in}|^2}{4}\left(\frac{\mu_2 - \mu_1}{\mu_1}\right) + D^2\right],$$

where $\xi > 0$ is a hydrodynamic parameter taken from [7] for this test problem.

Figures 5.1 and 5.2 visualize, respectively, the velocity field and the pressure field. The pressure field is shown with greater intensity at the inlet of the pipe (red color) and less intensity at the outlet (blue color), which is consistent with the type of shear stresses that must be maintained to obtain the velocity field as illustrated in Figure 5.1; see [7, 22, 23].

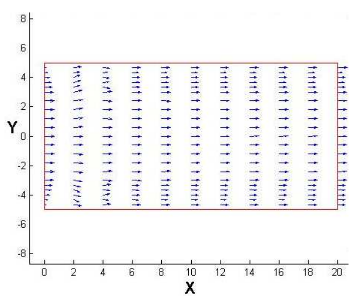


FIGURE 5.1. *Velocity field.*

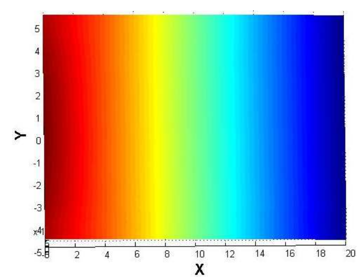


FIGURE 5.2. *Pressure field.*

Following Hughes [21], we determine the constraint ratio for the system equations (4.14). The value of this ratio was $r = 1.999$, very close to 2, which is the optimal value. Therefore, in heuristic terms, the calculations verify the discrete inf-sup condition.

**6. Conclusions.** We have described a discretization of a water-petroleum two-phase flow problem based on a generalized Stokes problem with non-standard boundary conditions [4], using the Crouzeix-Raviart mixed finite element with spaces $P_1$ and $P_0$ for the velocity and pressure, respectively. Under an adequate variational formulation, we demonstrated that the discretization generates a linear system of equations whose solution exists, is unique, and is uniformly stable, since a discrete inf-sup condition was demonstrated over the nonconforming Crouzeix-Raviart finite element spaces $P_1/P_0$. The existence of a lifting function which extends the Dirichlet condition to the entire pipeline was also demonstrated, based on the regularity and properties that make it adequate to conserve an incompressible flow condition and be approximated by means of the Crouzeix-Raviart interpolation operator. Based on this, a numerical simulation was presented to visualize the velocity field and pressure field for an initial time in which the interfaces between the two fluids are straight lines.

The constraint ratio was also calculated, and it was consistent with the theoretical results, since the uniform stability of the solution was thus heuristically verified.

The finite element of Crouzeix-Raviart has been used by numerous authors for the solution of problems where the Navier-Stokes equations arise in the simulation of incompressible flows; see, for example, [28]. Like the mini-element, the Crouzeix-Raviart element is a stable mixed combination. However, since there is no physical reason to assume continuous pressure between the two fluids [22, 23], we consider the Crouzeix-Raviart element in the space discretization of the two-phase flow studied in this work a better tool to approximate the pressure.

In future work, we would like to do a convergence analysis and to compare our numerical results with those generated by the "mini-element." Also, we would like to compute solutions for several time steps and approximations to the interface between both fluids.

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