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NEW QUADRATURE RULES FOR BERNSTEIN MEASURES ON THE INTERVAL $[-1,1]$*

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Abstract. In the present paper, we obtain quadrature rules for Bernstein measures on $[-1,1]$, having a fixed number of nodes and weights such that they exactly integrate functions in the linear space of polynomials with real coefficients.

Key words. quadrature rules, orthogonal polynomials, measures on the real line, Bernstein measures, Chebyshev polynomials

AMS subject classifications. 33C47, 42C05

1. Introduction. In this paper, we present quadrature rules for Bernstein measures on $[-1,1]$ such that they exactly integrate functions in the linear space of polynomials with real coefficients. These types of rules have a fixed number of nodes and quadrature weights for each Bernstein measure. Since we have proved in the case of the unit circle the existence of a quadrature rule for Bernstein-Szegő measures with similar properties from the point of view of the exactness (see [2]), the use of the transformations between measures supported on the interval $[-1,1]$ and symmetric measures supported on the unit circle $\mathbb{T}=\{z:|z|=1\}$, the so-called Szegő transformation (see [1, 8, 15, 16]), allows us to obtain the results. This approach of relating quadrature rules on the unit circle and the interval $[-1,1]$ has been used successfully to investigate Gauss-Szegő quadrature rules (see [3]).

Indeed, for each Bernstein measure, we give the nodes and the weights. In order to apply our method in the computation of the integral of any polynomial with respect to the Bernstein measure, we only need to know the coefficients of the expansion of the polynomial in the corresponding Chebyshev basis. The well-known Clenshaw-Curtis quadrature rule (see [5]) uses this type of expansion in terms of the Chebyshev polynomials of the first kind but it exactly integrates polynomials only up to a certain degree depending on the number of nodes. Our method has unlimited exactness on the space of polynomials.

The paper has the following structure. In Section 2, we introduce the Bernstein measures on $[-1,1]$ and we recall the Szegő transformations of measures supported on a bounded interval and the unit circle. In Section 3, we deduce a quadrature rule for Bernstein-Szegő measures on the unit circle which is exact in the linear space Π of polynomials with complex coefficients. In Section 4, we prove the main results of the paper concerning quadrature rules for the Bernstein measures supported on the interval $[-1,1]$. In Section 5, we give some numerical examples, and a technical result about the changes of basis is presented.

2. Bernstein measures on the interval $[-1,1]$ and Bernstein-Szegő measures on the unit circle $\mathbb{T}$. The aim of this paper is to obtain quadrature rules for the Bernstein measures corresponding to rational modifications of the Jacobi measures $d\mu_{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}dx$

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for the parameters $\alpha = \pm\frac{1}{2}$ and $\beta = \pm\frac{1}{2}$, i.e., the so-called Chebyshev measures of the first kind $\alpha = \beta = -\frac{1}{2}$, of the second kind $\alpha = \beta = \frac{1}{2}$, of the third kind $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, and of the fourth kind $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ (see [13, 16]).

If $q_k(x) = \sum_{r=0}^{k} a_r x^r$ is a positive polynomial on $[-1,1]$ with real coefficients, we consider the following rational modifications of the above measures, i.e., the Bernstein measures:

$$
\begin{aligned}
d\mu_1(x) &= \frac{dx}{\pi\sqrt{1-x^2}\,q_k(x)}, \qquad & d\mu_2(x) &= \frac{2\sqrt{1-x^2}\,dx}{\pi q_k(x)}, \\
d\mu_3(x) &= \frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}\frac{dx}{q_k(x)}, \qquad & d\mu_4(x) &= \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\frac{dx}{q_k(x)},
\end{aligned}
\tag{2.1}
$$

which, of course, are positive Borel measures on $[-1,1]$.

Taking into account the Féjer-Riesz representation (see [11, 16]), we know that there exists an algebraic polynomial $A_k(z) = \sum_{r=0}^{k} m_r z^r$, with $m_r \in \mathbb{R}$ for $r = 0,\dots,k$, $m_0 > 0$, without zeros in $\overline{\mathbb{D}} = \{z : |z| \le 1\}$ such that $q_k(\cos\theta) = |A_k(e^{i\theta})|^2$.

Next we show that, applying suitable transformations, the measures $d\mu_i$, $i = 1,\dots,4$, become the Bernstein-Szegő measure on $[-\pi,\pi]$, $d\nu(\theta) = \frac{d\theta}{2\pi|A_k(e^{i\theta})|^2}$. In order to obtain this result, we recall the connection between measures supported on $[-\pi,\pi]$ and $[-1,1]$, respectively, through the following four transformations (see [1, 8, 15, 16]).

2.1. Szegő transformations of measures from the interval $[-1,1]$ to the unit circle. Let μ be a nontrivial probability Borel measure on $[-1,1]$, absolutely continuous with respect to the Lebesgue normalized measure, with weight function $\omega(x)$, i.e., $d\mu(x) = \omega(x)dx$. We will transform this measure in the following four different ways.

First transformation. Since $x = \cos\theta$ transforms the interval $[-\pi,\pi]$ into $[-1,1]$, then we can define a measure ν_1 on $[-\pi,\pi]$ such that $d\nu_1(\theta) = \frac{1}{2}\omega(\cos\theta)|\sin\theta|d\theta$ and $\nu_1([-\pi,\pi]) = 1$. It induces another measure on $\mathbb{T}$ that we also denote by ν_1 for the sake of simplicity. Notice that when μ is the Chebyshev measure of the first kind, then ν_1 is the Lebesgue normalized measure.

If $P(x) = \sum_{l=0}^{n} a_l T_l(x)$, with $a_l \in \mathbb{R}$, $l = 0,\dots,n$, where $\{T_n(x)\}_{n\in\mathbb{N}}$ is the sequence of Chebyshev polynomials of the first kind and $Q(z) = \sum_{l=0}^{n} a_l z^l$, then it is easy to relate the inner products induced by both measures μ and ν_1 as follows:

$$\big\langle P(x), T_l(x)\big\rangle_\mu = \frac{1}{2}\big\langle Q(z) + Q\big(z^{-1}\big), z^l\big\rangle_{\nu_1}.$$

Second transformation. The substitution $x = \cos\theta$ and the multiplication by $\frac{1}{4\sin^2\theta}$ defines a finite positive Borel measure ν_2 on $[-\pi,\pi]$ by $d\nu_2(\theta) = \frac{\omega(\cos\theta)}{4|\sin\theta|}d\theta$ if $\int_{-1}^{1}\frac{\omega(x)}{1-x^2}dx < +\infty$. The measure ν_2 induces in a natural way another measure on $\mathbb{T}$ that we also denote by ν_2. Notice that when μ is the Chebyshev measure of the second kind, then ν_2 is the Lebesgue normalized measure.

If $P(x) = \sum_{l=0}^{n} b_l U_l(x)$, with $b_l \in \mathbb{R}$, $l = 0,\dots,n$, where $\{U_n(x)\}_{n\in\mathbb{N}}$ is the sequence of Chebyshev polynomials of the second kind and $Q(z) = \sum_{l=0}^{n} b_l z^l$, then it is easy to obtain the following relation between the inner products corresponding to both measures:

$$\big\langle P(x), U_l(x)\big\rangle_\mu = \big\langle Q(z) - Q\big(z^{-1}\big), z^{l+1}\big\rangle_{\nu_2}.$$

Third transformation. The substitution $x = \cos\theta$ and the multiplication by $\frac{1}{4\cos^2\frac{\theta}{2}}$ defines a finite positive Borel measure ν_3 on $[-\pi,\pi]$ by $d\nu_3(\theta) = \frac{1}{2}\omega(\cos\theta)|\tan\frac{\theta}{2}|d\theta$ if $\int_{-1}^{1}\frac{\omega(x)}{1+x}dx < +\infty$. The measure ν_3 induces a measure on $\mathbb{T}$ that we also denote by ν_3. Notice that when μ is the Chebyshev measure of the third kind, then ν_3 is the Lebesgue normalized measure.

If $P(x) = \sum_{l=0}^{n} c_l W_l(x)$, with $c_l \in \mathbb{R}$, $l = 0,\dots,n$, where $\{W_n(x)\}_{n\in\mathbb{N}}$ is the sequence of Chebyshev polynomials of the third kind and $Q(z) = \sum_{l=0}^{n} c_l z^l$, then the inner products corresponding to both measures are related as follows:

$$\left\langle P(x), W_l(x)\right\rangle_\mu = \left\langle z^{\frac{1}{2}}Q(z) + z^{-\frac{1}{2}}Q\left(z^{-1}\right), z^{l+\frac{1}{2}}\right\rangle_{\nu_3}.$$

Fourth transformation. The substitution $x=\cos\theta$ and the multiplication by $\frac{1}{4\sin^2\frac{\theta}{2}}$ allows us to define a finite positive Borel measure ν_4 on $[-\pi,\pi]$ by $d\nu_4(\theta)=\frac{1}{2}\omega(\cos\theta)|\cot\frac{\theta}{2}|d\theta$ if $\int_{-1}^{1}\frac{\omega(x)}{1-x}dx < +\infty$. The measure ν_4 induces a measure on $\mathbb{T}$ that we also denote by ν_4. Notice that when μ is the Chebyshev measure of the fourth kind, then ν_4 is the Lebesgue normalized measure.

If $P(x) = \sum_{l=0}^{n} d_l V_l(x)$, with $d_l \in \mathbb{R}$, $l = 0,\dots,n$, where $\{V_n(x)\}_{n\in\mathbb{N}}$ is the sequence of Chebyshev polynomials of the fourth kind and $Q(z) = \sum_{l=0}^{n} d_l z^l$, then we can relate the inner products corresponding to both measures as follows:

$$\left\langle P(x), V_l(x)\right\rangle_\mu = \left\langle z^{\frac{1}{2}}Q(z) - z^{-\frac{1}{2}}Q\left(z^{-1}\right), z^{l+\frac{1}{2}}\right\rangle_{\nu_4}.$$

Now, if we consider the Bernstein measures $d\mu_1$, $d\mu_2$, $d\mu_3$, and $d\mu_4$ defined in (2.1) and we transform each $d\mu_i$ by the ith transformation, $i = 1,\dots,4$, then it is easy to prove that we obtain the same Bernstein-Szegő measure $d\nu(\theta) = \frac{d\theta}{2\pi|A_k(e^{i\theta})|^2}$.

3. Quadrature rules for Bernstein-Szegő measures. Since our aim in this work is to obtain quadrature rules for the Bernstein measures $d\mu_i$, $i = 1,\dots,4$, defined in (2.1), we are going to recall a recent result about quadrature rules for Bernstein-Szegő measures on the unit circle $\mathbb{T}$.

Let $A_k(z)$ be an algebraic polynomial of degree k with zeros outside $\overline{\mathbb{D}}$ and $A_k(0) > 0$ and let us consider the Bernstein-Szegő measure $d\nu$ defined on $[-\pi,\pi]$ by

$$d\nu(\theta) = \frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2}. \tag{3.1}$$

If we denote by $\{\Phi_n(z)\}_{n\in\mathbb{N}}$ the monic orthogonal polynomial sequence with respect to the measure $d\nu$ (MOPS(ν)), then $\Phi_n(z) = z^{n-k}\frac{A_k^*(z)}{A_k(0)}$, for $n \geq k$ (see [16]).

In [2] we have obtained the following quadrature rule for Bernstein-Szegő measures.

THEOREM 3.1. *Let $d\nu$ be a Bernstein-Szegő measure like (3.1) and let $\{\Phi_n(z)\}_{n\in\mathbb{N}}$ be the corresponding* MOPS(ν). *Assume that $\Phi_n(z) = z^{n-k}\Pi_{i=1}^{s}(z-z_i)^{\nu_i}$ for $n \geq k$ with $z_i \neq z_j$ if $i \neq j$ and $\sum_{i=1}^{s}\nu_i = k$. Then there exists a quadrature rule with nodes $\{z_1,\dots,z_s\}$ which uses the values of the function and its derivatives in these nodes and such that it exactly integrates functions in the linear space Π of polynomials with complex coefficients, i.e., for every $P \in \Pi$ we get*

$$\int_{\mathbb{T}} P(z)d\mu(z) = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\lambda_{i,j}P^{(j)}(z_i). \tag{3.2}$$

Moreover, $\lambda_{i,\nu_i-1} \neq 0$ for $i = 1,\dots,s$.

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Proof. See [2]. □

Notice that the above result can be completed with the following proposition where the explicit computation of the weights $\lambda_{i,j}$ in the quadrature rule (3.2) is given.

PROPOSITION 3.2. *Let us assume that $A_k^*(z) = c_k \Pi_{i=1}^s (z - z_i)^{\nu_i}$, with $\sum_{i=1}^s \nu_i = k$, and let $A_{i,j}$ be the numerators of the following partial fraction decomposition:*

$$\frac{(-1)^k z^{k-1}}{\left|c_k\right|^2 \Pi_{i=1}^s \overline{z_i}^{\nu_i} \Pi_{i=1}^s (z-z_i)^{\nu_i} \Pi_{i=1}^s \left(z - \frac{1}{\overline{z_i}}\right)^{\nu_i}} = \sum_{i=1}^s \sum_{j=1}^{\nu_i} \frac{A_{i,j}}{(z-z_i)^j} + \sum_{i=1}^s \sum_{j=1}^{\nu_i} \frac{B_{i,j}}{\left(z - \frac{1}{\overline{z_i}}\right)^j}.$$

Then the weights $\lambda_{i,j}$ of the quadrature rule (3.2) satisfy

$$\lambda_{i,j} = \frac{A_{i,j+1}}{j!}.$$

Proof. Since $|A_k(z)| = |A_k^*(z)|$ for $z \in \mathbb{T}$, thus we can write

$$\begin{aligned}
\frac{1}{z\left|A_k(z)\right|^2} = \frac{1}{zA_k^*(z)\overline{A_k^*}\left(\frac{1}{z}\right)} &= \frac{1}{zc_k \Pi_{i=1}^s \left(z-z_i\right)^{\nu_i} \overline{c_k} \Pi_{i=1}^s \left(\frac{1}{z} - \overline{z_i}\right)^{\nu_i}} \\
&= \frac{z^{k-1}}{\left|c_k\right|^2 \Pi_{i=1}^s \left(z-z_i\right)^{\nu_i} \Pi_{i=1}^s \left(1 - z\overline{z_i}\right)^{\nu_i}} \\
&= \frac{(-1)^k z^{k-1}}{\left|c_k\right|^2 \Pi_{i=1}^s \overline{z_i}^{\nu_i} \Pi_{i=1}^s \left(z-z_i\right)^{\nu_i} \Pi_{i=1}^s \left(z - \frac{1}{\overline{z_i}}\right)^{\nu_i}} \\
&= \sum_{j=1}^{\nu_1} \frac{A_{1,j}}{\left(z-z_1\right)^j} + \cdots + \sum_{j=1}^{\nu_s} \frac{A_{s,j}}{\left(z-z_s\right)^j} \\
&\quad + \sum_{j=1}^{\nu_1} \frac{B_{1,j}}{\left(z - \frac{1}{\overline{z_1}}\right)^j} + \cdots + \sum_{j=1}^{\nu_s} \frac{B_{s,j}}{\left(z - \frac{1}{\overline{z_s}}\right)^j}.
\end{aligned}$$

Taking into account the previous decomposition and the Cauchy's theorem, we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} P\left(e^{i\theta}\right) \frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} \\
&\quad = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(z)}{z} \frac{dz}{\left|A_k(z)\right|^2} \\
&\quad = \frac{1}{2\pi i} \int_{\mathbb{T}} P(z) \left(\sum_{i=1}^s \sum_{j=1}^{\nu_i} \frac{A_{i,j}}{\left(z-z_i\right)^j} \right) dz + \frac{1}{2\pi i} \int_{\mathbb{T}} P(z) \left(\sum_{i=1}^s \sum_{j=1}^{\nu_i} \frac{B_{i,j}}{\left(z - \frac{1}{\overline{z_i}}\right)^j} \right) dz \\
&\quad = \sum_{j=1}^{\nu_1} A_{1,j} \frac{P^{(j-1)}(z_1)}{(j-1)!} + \cdots + \sum_{j=1}^{\nu_s} A_{s,j} \frac{P^{(j-1)}\left(z_s\right)}{(j-1)!} = \sum_{i=1}^s \sum_{j=0}^{\nu_i - 1} A_{i,j+1} \frac{P^{(j)}\left(z_i\right)}{j!}.
\end{aligned}$$

Thus using (3.2) we have $\lambda_{i,j} = \frac{A_{i,j+1}}{j!}$ for $i = 1, \ldots, s;\ j = 0, \ldots, \nu_i - 1$. □

In the next section, we obtain quadrature rules for Bernstein measures on $[-1, 1]$ applying the previous results and the transformations of measures introduced in Section 2.

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4. Quadrature rules for Bernstein measures in $[-1,1]$. Let us consider the Bernstein measures given in (2.1) and the Bernstein-Szegő measure (3.1) obtained using the above transformations. Our aim is to prove that for each of these Bernstein measures there exists a quadrature rule with a fixed number of nodes and weights, which is exact in the linear space $\mathbb{P}$ of polynomials with real coefficients.

THEOREM 4.1. *Let $d\mu_1(x) = \frac{dx}{\pi\sqrt{1-x^2}q_k(x)}$. There exists a quadrature rule using the nodes $\{z_i\}_{i=1}^{s}$ and the weights $\{\lambda_{i,j}\}_{i=1,\dots,s;\ j=0,\dots,\nu_i-1}$, with $\sum_{i=1}^{s}\nu_i = k$, such that it exactly integrates polynomials, i.e.,*

$$\forall P \in \mathbb{P}, \quad P(x) = \sum_{l=0}^{M} b_l T_l(x), \text{ with } b_l \in \mathbb{R},$$

we get

$$\int_{-1}^{1} P(x)d\mu_1(x) = 2\Re\left(\sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\lambda_{i,j}\left(\sum_{l=0}^{M}\frac{b_l z^l}{2}\right)^{(j)}_{|z=z_i}\right).$$

Proof. Using the first transformation, the measure $d\mu_1$ becomes the measure $d\nu$ given by $d\nu(\theta) = \frac{d\theta}{2\pi|A_k(e^{i\theta})|^2}$.

If $P(x) = \sum_{l=0}^{M} b_l T_l(x)$ with $b_l \in \mathbb{R}$, $l = 0,\dots,M$, and $Q(z) = \sum_{l=0}^{M}\frac{b_l}{2}z^l$, then $\langle P(x),1\rangle_{\mu_1} = \langle Q(z)+Q(z^{-1}),1\rangle_{\nu}$. Therefore, we can write

$$\int_{-1}^{1} P(x)\frac{dx}{\pi\sqrt{1-x^2}q_k(x)} = \int_{-\pi}^{\pi}\left(\sum_{l=0}^{M}\frac{b_l}{2}z^l\right)\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} + \int_{-\pi}^{\pi}\left(\sum_{l=0}^{M}\frac{b_l}{2}\overline{z}^l\right)\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2}.$$

for $z = e^{i\theta}$.

In order to compute these integrals, we apply Theorem 4.1. Indeed if $z_1,\dots,z_s$ are the zeros of $A_k^*(z)$, which are located in $\mathbb{D}$, and $\nu_1,\dots,\nu_s$ are their multiplicities with $\sum_{i=1}^{s}\nu_i = k$, then there exist weights $\{\lambda_{i,j}\}_{i=1,\dots,s;\ j=0,\dots,\nu_i-1}$ such that for every $R \in \Pi$ we get

$$\int_{-\pi}^{\pi} R(e^{i\theta})\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\lambda_{i,j}R^{(j)}(z_i)$$

and

$$\int_{-\pi}^{\pi} \overline{R(e^{i\theta})}\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\overline{\lambda_{i,j}R^{(j)}(z_i)}.$$

Therefore, if $Q(z) = \sum_{l=0}^{M}\frac{b_l}{2}z^l$ with $b_l \in \mathbb{R}$, then

$$\int_{-\pi}^{\pi}\sum_{l=0}^{M}\left(\frac{b_l\left(z^l+\overline{z}^l\right)}{2}\right)\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} = 2\Re\left(\sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\lambda_{i,j}\left(\sum_{l=0}^{M}\frac{b_l z^l}{2}\right)^{(j)}_{|z=z_i}\right),$$

and, as a consequence, the statement holds. □

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REMARK 4.2. When $A_k^*(z)$ has simple zeros $z_1, \dots, z_k$ in $\mathbb{D}$, the weights are $\lambda_1, \dots, \lambda_k$ and the quadrature rule means that if $P(x) = \sum_{l=0}^{M} b_l T_l(x)$, with $b_l \in \mathbb{R}$, then

$$\int_{-1}^{1} P(x) \frac{dx}{\pi\sqrt{1-x^2}q_k(x)} = 2\Re\left(\sum_{i=1}^{k} \lambda_i \sum_{l=0}^{M} \frac{b_l z_i^l}{2}\right).$$

THEOREM 4.3. *Let* $d\mu_2(x) = \frac{2\sqrt{1-x^2}dx}{\pi q_k(x)}$. *There exists a quadrature rule using the nodes* $\{z_i\}_{i=1}^{s}$ *and the weights* $\{\lambda_{i,j}\}_{i=1,\dots,s;\ j=0,\dots,\nu_i-1}$, *with* $\sum_{i=1}^{s} \nu_i = k$, *such that it exactly integrates polynomials, i.e.,*

$$\forall P \in \mathbb{P}, \quad P(x) = \sum_{l=0}^{M} d_l U_l(x),\ \textit{with}\ d_l \in \mathbb{R},$$

we get

$$\int_{-1}^{1} P(x) d\mu_2(x) = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\left(\lambda_{i,j}\sum_{l=0}^{M}\left(d_l z^l\right)^{(j)}(z_i) - \overline{\lambda_{i,j}}\sum_{l=0}^{M}\overline{\left(d_l z^{l+2}\right)^{(j)}(z_i)}\right).$$

Proof. We write $\int_{-1}^{1} P(x)d\mu_2(x) = \langle P(x), 1\rangle_{\mu_2}, \forall P \in \mathbb{P}$. Using the results about the second transformation given in Section 2, the measure $d\mu_2$ becomes $d\nu(\theta) = \frac{d\theta}{2\pi|A_k(e^{i\theta})|^2}$. If $P(x) = \sum_{l=0}^{M} d_l U_l(x)$ and $Q(z) = \sum_{l=0}^{M} d_l z^{l+1}$, then

$$\left\langle P(x), 1\right\rangle_{\mu_2} = \left\langle Q(z) - Q(\overline{z}), z\right\rangle_{\nu} = \left\langle \overline{z}Q(z) - \overline{z}Q(\overline{z}), 1\right\rangle_{\nu}.$$

Since for $z \in \mathbb{T}$ we have $\overline{z}Q(z) = \sum_{l=0}^{M} d_l z^l$ and $\overline{z}Q(\overline{z}) = \sum_{l=0}^{M} d_l \overline{z}^{l+2}$, then

$$\begin{aligned}&\int_{-1}^{1} P(x)d\mu_2(x)\\ &\quad= \int_{-\pi}^{\pi}\left(\sum_{l=0}^{M} d_l z^l\right)\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} - \int_{-\pi}^{\pi}\left(\sum_{l=0}^{M} d_l \overline{z}^{l+2}\right)\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2}, \quad z = e^{i\theta}.\end{aligned}$$

Proceeding like in the proof of Theorem 4.3, we can apply Theorem 4.1 and get

$$\int_{-\pi}^{\pi}\left(\sum_{l=0}^{M} d_l z^l\right)\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\lambda_{i,j}\left(\sum_{l=0}^{M} d_l z^l\right)^{(j)}(z_i)$$

as well as

$$\int_{-\pi}^{\pi}\left(\sum_{l=0}^{M} \overline{d_l}\overline{z}^{l+2}\right)\frac{d\theta}{2\pi\left|A_k\left(e^{i\theta}\right)\right|^2} = \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\overline{\lambda_{i,j}}\overline{\left(\sum_{l=0}^{M} d_l z^{l+2}\right)^{(j)}(z_i)}.$$

Therefore, we have

$$\begin{aligned}\int_{-1}^{1} P(x)d\mu_2(x) &= \int_{-1}^{1}\left(\sum_{l=0}^{M} d_l U_l(x)\right)d\mu_2(x))\\ &= \sum_{i=1}^{s}\sum_{j=0}^{\nu_i-1}\left[\lambda_{i,j}\sum_{l=0}^{M}\left(d_l z^l\right)^{(j)}(z_i) - \overline{\lambda_{i,j}}\sum_{l=0}^{M}\overline{\left(d_l z^{l+2}\right)^{(j)}(z_i)}\right]. \quad \square\end{aligned}$$

REMARK 4.4. If the zeros $z_1, \dots, z_k$, of $A_k^*(z)$ are simple, then the weights are $\lambda_1, \dots, \lambda_k$ and the quadrature rule becomes

$$\int_{-1}^{1} \left(\sum_{l=0}^{M} d_l U_l(x) \right) d\mu_2(x) = \sum_{i=1}^{k} \left(\lambda_i \sum_{l=0}^{M} d_l z_i^l - \overline{\lambda_i} \sum_{l=0}^{M} d_l \overline{z_i}^{l+2} \right).$$

THEOREM 4.5. *Let us consider the positive measure* $d\mu_3(x) = \frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}\frac{dx}{q_k(x)}$ *supported on* $[-1,1]$. *Then there exists a quadrature rule with nodes* $\{z_i\}_{i=1}^{s}$ *and weights* $\{\lambda_{i,j}\}_{i=1,\dots,s;\ j=0,\dots,\nu_i-1}$, *with* $\sum_{i=1}^{s} \nu_i = k$, *such that it exactly integrates polynomials, i.e.,*

$$\forall P \in \mathbb{P}, \quad P(x) = \sum_{l=0}^{M} e_l V_l(x), \ \textit{with } e_l \in \mathbb{R},$$

we get

$$\int_{-1}^{1} P(x) d\mu_3(x) = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \left(\lambda_{i,j} \sum_{l=0}^{M} \left(e_l z^l\right)^{(j)}(z_i) - \overline{\lambda_{i,j}} \sum_{l=0}^{M} \overline{\left(e_l z^{l+1}\right)^{(j)}(z_i)} \right).$$

Proof. Using the third transformation, $d\mu_3$ becomes $d\nu(\theta) = \frac{d\theta}{2\pi|A_k(e^{i\theta})|^2}$. Then if $P(x) = \sum_{l=0}^{M} e_l V_l(x)$ and $Q(z) = \sum_{l=0}^{M} e_l z^l$, with $e_l \in \mathbb{R}$, we get

$$\left\langle P(x), 1 \right\rangle_{\mu_3} = \left\langle z^{\frac{1}{2}} Q(z) - \overline{z}^{\frac{1}{2}} Q(\overline{z}), z^{\frac{1}{2}} \right\rangle_{\nu} = \left\langle Q(z) - \overline{z} Q(\overline{z}), 1 \right\rangle_{\nu}.$$

Taking into account that, for $z \in \mathbb{T}$, $\overline{z}Q(\overline{z}) = \sum_{l=0}^{M} e_l \overline{z}^{l+1}$, then

$$\int_{-1}^{1} P(x) d\mu_3(x) = \int_{-\pi}^{\pi} \left(\sum_{l=0}^{M} e_l z^l - \sum_{l=0}^{M} e_l \overline{z}^{l+1} \right) \frac{d\theta}{2\pi \left| A_k\left(e^{i\theta}\right) \right|^2}.$$

Proceeding like in the proof of Theorem 4.3, we can apply Theorem 4.1 and deduce

$$\int_{-\pi}^{\pi} \left(\sum_{l=0}^{M} e_l z^l \right) \frac{d\theta}{2\pi \left| A_k\left(e^{i\theta}\right) \right|^2} = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \lambda_{i,j} \sum_{l=0}^{M} \left(e_l z^l\right)^{(j)}(z_i)$$

as well as

$$\int_{-\pi}^{\pi} \left(\sum_{l=0}^{M} \overline{e_l z}^{l+1} \right) \frac{d\theta}{2\pi \left| A_k\left(e^{i\theta}\right) \right|^2} = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \overline{\lambda_{i,j}} \sum_{l=0}^{M} \overline{\left(e_l z^{l+1}\right)^{(j)}(z_i)}.$$

Hence,

$$\int_{-1}^{1} P(x) d\mu_3(x) = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \left(\lambda_{i,j} \sum_{l=0}^{M} \left(e_l z^l\right)^{(j)}(z_i) - \overline{\lambda_{i,j}} \sum_{l=0}^{M} \overline{\left(e_l z^{l+1}\right)^{(j)}(z_i)} \right). \quad \square$$

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REMARK 4.6. If the zeros $z_1, \dots, z_k$ of $A_k^*(z)$ are simple, then the weights are $\lambda_1, \dots, \lambda_k$ and the quadrature rule is

$$\int_{-1}^{1} \left(\sum_{l=0}^{M} e_l V_l(x) \right) d\mu_3(x) = \sum_{i=1}^{k} \left(\lambda_i \sum_{l=0}^{M} e_l z_i^l - \overline{\lambda_i} \sum_{l=0}^{M} e_l \overline{z_i}^{l+1} \right).$$

THEOREM 4.7. *Let us consider the positive measure* $d\mu_4(x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\frac{dx}{q_k(x)}$ *supported on* $[-1,1]$*. Then there exists a quadrature rule with nodes* $\{z_i\}_{i=1}^{s}$ *and weights* $\{\lambda_{i,j}\}_{i=1,\dots,s;\; j=0,\dots,\nu_i-1}$*, with* $\sum_{i=1}^{s} \nu_i = k$*, such that it exactly integrates polynomials, i.e.,*

$$\forall P \in \mathbb{P}, \quad P(x) = \sum_{l=0}^{M} f_l W_l(x), \text{ with } f_l \in \mathbb{R},$$

we get

$$\int_{-1}^{1} P(x) d\mu_4(x) = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \left(\lambda_{i,j} \sum_{l=0}^{M} \left(f_l z^l\right)^{(j)}(z_i) + \overline{\lambda_{i,j}} \sum_{l=0}^{M} \overline{\left(f_l z^{l+1}\right)^{(j)}(z_i)} \right).$$

Proof. Using the fourth transformation, $d\mu_4$ becomes $d\nu(\theta) = \frac{d\theta}{2\pi |A_k(e^{i\theta})|^2}$. Then if $P(x) = \sum_{l=0}^{M} f_l W_l(x)$ and $Q(z) = \sum_{l=0}^{M} f_l z^l$, with $f_l \in \mathbb{R}$, we get

$$\langle P(x), 1 \rangle_{\mu_4} = \langle z^{\frac{1}{2}} Q(z) + \overline{z}^{\frac{1}{2}} Q(\overline{z}), z^{\frac{1}{2}} \rangle_{\nu} = \langle Q(z) + \overline{z} Q(\overline{z}), 1 \rangle_{\nu}.$$

Therefore, taking into account that $\overline{z} Q(\overline{z}) = \sum_{l=0}^{M} f_l \overline{z}^{l+1}$ for $z \in \mathbb{T}$, then

$$\int_{-1}^{1} P(x) d\mu_4(x) = \int_{-\pi}^{\pi} \left(\sum_{l=0}^{M} f_l z^l + \sum_{l=0}^{M} f_l \overline{z}^{l+1} \right) \frac{d\theta}{2\pi \left| A_k\left(e^{i\theta}\right) \right|^2}.$$

Proceeding like in the proof of Theorem 4.3, from Theorem 4.1 we get

$$\int_{-\pi}^{\pi} \left(\sum_{l=0}^{M} f_l z^l \right) \frac{d\theta}{2\pi \left| A_k\left(e^{i\theta}\right) \right|^2} = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \lambda_{i,j} \sum_{l=0}^{M} \left(f_l z^l\right)^{(j)}(z_i)$$

and

$$\int_{-\pi}^{\pi} \left(\sum_{l=0}^{M} \overline{f_l} \overline{z}^{l+1} \right) \frac{d\theta}{2\pi \left| A_k\left(e^{i\theta}\right) \right|^2} = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \overline{\lambda_{i,j}} \sum_{l=0}^{M} \overline{\left(f_l z^{l+1}\right)^{(j)}(z_i)}.$$

Hence,

$$\int_{-1}^{1} P(x) d\mu_4(x) = \sum_{i=1}^{s} \sum_{j=0}^{\nu_i-1} \left(\lambda_{i,j} \sum_{l=0}^{M} \left(f_l z^l\right)^{(j)}(z_i) + \overline{\lambda_{i,j}} \sum_{l=0}^{M} \overline{\left(f_l z^{l+1}\right)^{(j)}(z_i)} \right). \quad \square$$

REMARK 4.8. If the zeros $z_1, \dots, z_k$ of $A_k^*(z)$ are simple, then the weights are $\lambda_1, \dots, \lambda_k$ and the quadrature rule is

$$\int_{-1}^{1} \left(\sum_{l=0}^{M} f_l W_l(x) \right) d\mu_4(x) = \sum_{i=1}^{k} \left(\lambda_i \sum_{l=0}^{M} f_l z_i^l + \overline{\lambda_i} \sum_{l=0}^{M} f_l \overline{z_i}^{l+1} \right).$$

REMARK 4.9. In order to compute the nodes of the quadrature rules, we propose to use the next property. If $(x-a)^\nu$ is a factor of the polynomial $q_k(x)$ of arbitrary degree k, then the Joukowski transformation of a with modulus less than one ($a \pm \sqrt{a^2-1}$ with $|a \pm \sqrt{a^2-1}| < 1$) is a node of the quadrature rule with multiplicity ν. Therefore, if all the factors are linear with real zeros, then the nodes are in $[-1,1]$. Otherwise if the factors are quadratic with complex zeros, we have a pair of complex conjugated nodes. Notice that this fact does not play a relevant role in the method. With this method for determining nodes and the previous one given in Proposition 5.1 for determining the weights, we prepare the method in order to be applied without errors in the calculus.

5. Numerical examples. Let us consider the polynomial $q_1(x) = 5 + 4x$, which is positive on $[-1,1]$ as well as the Bernstein measures $d\mu_i$ for $i = 1, \dots, 4$, given in (2.1) with $q_1(x) = 5 + 4x$. Notice that $q_1(\cos\theta) = |A_1(e^{i\theta})|^2$, with $A_1(z) = z + 2$; see Remark 4.9 for the details. In order to see how our method works and using the theorems of the previous section, for the polynomial $P(x) = 4x^2 - x - 1$, we compute the integrals $\int_{-1}^{1} P(x) d\mu_i(x)$ for $i = 1, \dots, 4$ as follows. First we write $P(x)$ in terms of the four Chebyshev polynomial bases

$$P(x) = 2T_2(x) - T_1(x) + T_0(x), \qquad P(x) = U_2(x) - \tfrac{1}{2}U_1(x),$$
$$P(x) = V_2(x) - \tfrac{3}{2}V_1(x) + \tfrac{3}{2}V_0(x), \quad P(x) = W_2(x) + \tfrac{1}{2}W_1(x) + \tfrac{1}{2}W_0(x).$$

Now we apply the result concerning the Bernstein-Szegő measure in order to get the nodes and the weights. Since $A_1^*(z) = 2z + 1$, then the node is $z_1 = -\frac{1}{2}$ and we compute λ_1 as follows:

$$\lambda_1 = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi|z+2|^2} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\frac{1}{2(z+2)}}{z + \frac{1}{2}} = \frac{1}{3}.$$

Therefore,

$$\int_{-1}^{1} P(x) d\mu_1(x) = \int_{-1}^{1} \left(2T_2(x) - T_1(x) + T_0(x)\right) \frac{dx}{\pi\sqrt{1-x^2}(5+4x)}$$
$$= 2\Re\left(\frac{1}{3} \frac{2z_1^2 - z_1 + 1}{2} \right) = \frac{2}{3},$$

$$\int_{-1}^{1} P(x) d\mu_2(x) = \int_{-1}^{1} \left(U_2(x) - \frac{1}{2}U_1(x) \right) \frac{2\sqrt{1-x^2}dx}{\pi(5+4x)}$$
$$= \frac{1}{3}\left(z_1^2 - \frac{1}{2}z_1 \right) - \frac{1}{3}\left(z_1^4 - \frac{1}{2}z_1^3 \right) = \frac{1}{8},$$

$$\int_{-1}^{1} P(x) d\mu_3(x) = \int_{-1}^{1} \left(V_2(x) - \frac{3}{2}V_1(x) + \frac{3}{2}V_0(x) \right) \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} \frac{dx}{(5+4x)}$$
$$= \frac{1}{3}\left(z_1^2 - \frac{3}{2}z_1 + \frac{3}{2} \right) - \frac{1}{3}\left(z_1^3 - \frac{3}{2}z_1^2 + \frac{3}{2}z_1 \right) = \frac{5}{4},$$

and

$$\int_{-1}^{1} P(x)d\mu_4(x) = \int_{-1}^{1} \left(W_2(x) + \frac{1}{2}W_1(x) + \frac{1}{2}W_0(x)\right)\frac{1}{\pi}\sqrt{\frac{1+x}{1-x}}\frac{dx}{(5+4x)}$$
$$= \frac{1}{3}\left(z_1^2 + \frac{1}{2}z_1 + \frac{1}{2}\right) + \frac{1}{3}\left(z_1^3 + \frac{1}{2}z_1^2 + \frac{1}{2}z_1\right) = \frac{1}{12}.$$

In the previous computations, a key step was the explicit expression of the polynomials in the appropriate basis. In order to simplify these computations, next we present a result about the change of basis.

PROPOSITION 5.1. *If* $P(x) = \sum_{r=0}^{2n+1} a_r x^r$, *then* $P(x) = \sum_{r=0}^{2n+1} b_r T_r(x)$, *with* $b_r = \sum_{j=0}^{2n+1} m_{r,j} a_j$, *where we denote by* $m_{r,j}$ *the entry corresponding to the* r*-row and* j*-column of the matrix of change of basis in* $\mathbb{P}$ *between the canonical basis and the basis of the first kind Chebyshev polynomials. These entries are*

$$\left(m_{0,j}\right)_{j=0,\ldots,2n+1} = \left(1, 0, \frac{1}{2}, 0, \frac{3}{8}, 0, \frac{5}{16}, 0, \frac{35}{128}, 0, \frac{63}{128}, 0, \ldots, 0, \frac{(2n-2)\cdots n}{2\,2^{2n-3}(n-1)!}, 0, \frac{1}{2^{2n}}\binom{2n}{n}\right);$$

for $r = 1, \ldots, n$,

$$\left(m_{2r,j}\right)_{j=0,\ldots,2n+1} = \Bigg(\underbrace{0,\ldots,0}_{2r}, \frac{1}{2^{2r-1}}, 0, \frac{2r+2}{2^{2r+1}}, 0, \frac{(2r+4)(2r+3)}{2!2^{2r+3}}, 0, \frac{(2r+6)(2r+5)(2r+4)}{3!2^{2r+5}};$$
$$0, \ldots, \frac{1}{2^{2n-1}}\binom{2n}{2n-r}\Bigg),$$

and for $r = 0, \ldots, n$,

$$\left(m_{2r+1,j}\right)_{j=0,\ldots,2n+1} = \Bigg(\underbrace{0,\ldots,0}_{2r+1}, \frac{1}{2^{2r}}, 0, \frac{2r+3}{2^{2r+2}}, 0, \frac{(2r+5)(2r+4)}{2!2^{2r+4}}, 0, \frac{(2r+7)(2r+6)(2r+5)}{3!2^{2r+6}},$$
$$0, \ldots, \frac{1}{2^{2n}}\binom{2n+1}{n-r}, 0\Bigg).$$

Notice that if the degree of the polynomial $P(x)$ *is even, then some analogous formulas can be obtained.*

The other changes of bases can be done using the previous ones. Taking into account the expressions of the different Chebyshev polynomials in terms of the Chebyshev polynomials of *the first kind, if* $P(x) = \sum_{r=0}^{n} b_r T_r(x)$, *then*

$$P(x) = \frac{1}{2}\left((2b_0 - b_2)U_0(x) + \sum_{l=1}^{n-2} (b_l - b_{l+2})U_l(x) + b_{n-1}U_{n-1}(x) + b_n U_n(x) \right),$$

$$P(x) = \frac{1}{2}\left((2b_0 + b_1)W_0(x) + \sum_{l=1}^{n-1} (b_l + b_{l+1})W_l(x) + b_n W_n(x) \right),$$

$$P(x) = \frac{1}{2}\left((2b_0 - b_1)V_0(x) + \sum_{l=1}^{n-1} (b_l - b_{l+1})V_l(x) + b_n V_n(x) \right).$$

REMARK 5.2. Taking into account these changes of bases, we can calculate the number of operations (sums and products) that we need for applying our method for the computation of the integral of a polynomial of degree n with respect to the polynomial modifications of the Bernstein measures $d\mu_i$ $(i = 1, 2, 3, 4)$. The number of sums is $\frac{(n+2)(n+1)}{2}$, the number of products is $\frac{(n+1)n}{2}$, and a prefixed number of evaluations of a polynomial of degree n exist. Taking into account the high number of zeros of the matrix of change of basis, a more efficient algorithm could be developed.

If we want to apply the quadrature rule to a function which is not a polynomial, using, e.g., the measure $d\mu_1$, then either the expansion of the function or the expansion of an approximation of the function in the basis $\{T_n(x)\}_{n\in\mathbb{N}}$ is needed.

Let us consider, e.g., an analytic function f on $[-1, 1]$. In order to compute $\int_{-1}^{1} f(x)d\mu_1(x)$, we approximate f by $\sum_{k=0}^{n-1} a_k T_k(x)$ and we compute exactly the integral $\int_{-1}^{1}(\sum_{k=0}^{n-1} a_k T_k(x))d\mu_1(x)$. The idea is to approximate $\int_{-1}^{1} f(x)d\mu_1(x)$ by $\int_{-1}^{1}(\sum_{k=0}^{n-1} a_k \times T_k(x))d\mu_1(x)$, and next to evaluate the error; see [6, 7].

A well-known method based on the FFT (fast Fourier transform) can be used to estimate the expansion in the basis $\{T_n(x)\}_{n\in\mathbb{N}}$ of the interpolatory polynomial of $f(x)$ defined in $[-1, 1]$. It can be applied when the evaluations of the function are determined in fixed nodes.

We will choose $m = 2^p$ and the m zeros $\{x_i\}_{i=1,\dots,m}$ of the polynomial $T_m(x)$ as nodes. Let us consider $\{\theta_i\} \subset [0, \pi]$ such that $\cos\theta_i = x_i$, and $\{\Theta_i\} \subset [-\pi, 0]$ such that $\cos\Theta_i = x_i$, as well as the FFT associated with these 2^{p+1} nodes, i.e., we consider $F(z) = \sum_{k=0}^{2m-1} b_k z^{k-m}$ such that $F(\theta_i) = F(\Theta_i) = f(x_i)$. Then $t(x) = \frac{F(z)+F(\overline{z})}{2} = \sum_{k=0}^{m} a_k T_k(x)$ with $x = \frac{z+\frac{1}{z}}{2}$ satisfies $t(x_i) = f(x_i)$, $\forall i = 1, \dots, m$. From the symmetry of the nodes, the problem has a solution using only functions of the form $\cos(k\theta)$ with $O(m \log_2 m)$ operations and using a well-known algorithm. The solution is the expansion of the minimax interpolation polynomial in terms of the basis $\{T_n(x)\}_{n\in\mathbb{N}}$.

Next we present three numerical experiments based on the ideas developed above. We consider the measure $d\mu_1$ used in Section 5 with $q_1(x) = 5+4x$, i.e., $d\mu_1(x) = \frac{dx}{\pi\sqrt{1-x^2}(5+4x)}$ and we take 2^5 nodes, i.e., a few number of nodes in order to have a low computational cost. The FFT algorithm is well developed in all the standard calculus programs. Although we can use any of them, we apply the FFT algorithm using the Numerical Math "Trig Fit" of Mathematica. The discretization has an effect on the accuracy of the quadrature which is the same as we have when we use a polynomial interpolatory quadrature for a function, i.e., we have an error given by error of the interpolatory process.

For the three analytic functions, we obtain the function $F(z)$ which gives the solution of the complex interpolatory problem and we present a table with the approximation value of the integral using our method, the approximate value of the integral obtained using the command NIntegrate of Mathematica and the difference between both approximations.

5.1. Function $\arctan x$**.**

$$\begin{aligned}F(z) &= -1.82699\,10^{-17} + 0.414214\,z + 2.12513\,10^{-17}\,z^2 - 0.0236893\,z^3\\ &+ 2.05094\,10^{-18}\,z^4 + 0.00243866\,z^5 - 2.79839\,10^{-18}\,z^6 - 0.000298863\,z^7\\ &- 3.93659\,10^{-18}\,z^8 + 0.000039882\,z^9 - 3.72317\,10^{-18}\,z^{10} - 5.59898\,10^{-6}\,z^{11}\\ &+ 3.33897\,10^{-18}\,z^{12} + 8.15588\,10^{-7}\,z^{13} + 7.34596\,10^{-18}\,z^{14} - 1.39154\,10^{-7}\,z^{15}\end{aligned}$$

Function	Approx.	Approx. NIntegrate	Bound error
$\arctan x$	-0.136146	-0.136146	-2.75111×10^{-10}

5.2. Function $e^{x^8+2x^3+3}$**.**

$$\begin{aligned}F(z) &= 4.15411 + 6.22378\,z + 4.68205\,z^2 + 3.86116\,z^3 + 2.81959\,z^4\\ &+ 1.86711\,z^5 + 1.33197\,z^6 + 0.880642\,z^7 + 0.562949\,z^8 + 0.358526\,z^9\\ &+0.224258\,z^{10} + 0.135074\,z^{11} + 0.0813767\,z^{12} + 0.0469612\,z^{13} +0.0254564\,z^{14}\\ &+ 0.0111311\,z^{15}\end{aligned}$$

Function	Approx.	Approx. NIntegrate	Bound error
$e^{x^8+2x^3+3}$	1.24238	1.24238	-9.44322×10^{-8}

5.3. Function e^{5x+3}**.**

$$\begin{aligned}F(z) &= 273.564 + 488.794\,z + 351.61\,z^2 + 207.507\,z^3 + 102.602\,z^4\\ &+ 43.3441\,z^5 + 15.9135\,z^6 + 5.15172\,z^7 + 1.48867\,z^8 + 0.387967\,z^9\\ &+ 0.0919927\,z^{10} + 0.019996\,z^{11} + 0.00401035\,z^{12} + 0.000746306\,z^{13}\\ &+ 0.000129448\,z^{14} + 0.0000205855\,z^{15}\end{aligned}$$

Function	Approx.	Approx. NIntegrate	Bound error
$e^{x^8+2x^3+3}$	64.2682	64.2682	1.21036×10^{-6}

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