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PICK FUNCTIONS RELATED TO ENTIRE FUNCTIONS HAVING NEGATIVE ZEROS*

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Abstract. For any sequence $\{a_k\}$ satisfying $0 < a_1 \leq a_2 \leq \dots$ and $|a_k - k| \leq \text{Const}$ we find the Stieltjes representation of the function

$$z \mapsto \frac{\log P(z)}{z \operatorname{Log} z},$$

where P denotes the canonical product of genus 1 having $\{-a_k\}$ as its zero set.

We also find conditions on the zeros (e.g. $a_k \in [k, k+1]$ for $k \geq 1$) in order that the function

$$z \mapsto \frac{-\log P(z) + z \log P(1)}{z \operatorname{Log} z}$$

be a Pick function. We find the corresponding representation in terms of a positive density on the negative axis. We thereby generalize earlier results about the Γ-function. We also show that another related function is a Pick function.

Key words. pick function, canonical product, integral representation

AMS subject classifications. 30E20, 30D15, 30E15, 33B15

1. Introduction.

The n-dimensional volume V_n of the unit ball in $\mathbb{R}^n$ can be expressed as

$$V_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)},$$

where Γ is Euler's gamma function. In [3] the asymptotic behaviour of the $n\log n$'th root of V_n was studied (the limit as n tends to infinity is seen to be $1/\sqrt{e}$, by applying Stirlings formula). This initiated an investigation of monotonicity properties of the function

$$f(x) = \frac{\log\Gamma(x+1)}{x\log x}, \quad x > 0,$$

see [2], [1, Kapitel 2], [7] and [4]. In [5] we proved that a holomorphic extension of f to the cut plane

$$\mathcal{A} = \mathbb{C} \setminus]-\infty, 0],$$

is a Pick function. We also found the corresponding integral representation in terms of a positive density on the negative axis.

The proof consisted in applying a Phragmén-Lindelöf argument to the harmonic function

$$z \mapsto \Im\left(\frac{\log\Gamma(z+1)}{z \operatorname{Log} z}\right)$$

in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \Im z > 0\}$. In order to do so it was necessary to investigate the growth at infinity and the boundary behaviour on the real line of this harmonic function. These investigations depended heavily on the functional equation

$$\Gamma(z+1) = z\Gamma(z)$$

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of the Γ-function.

In this paper we generalize the results about the Γ-function to a class of entire functions having negative zeros.

The reciprocal of the Γ-function is an entire function having negative zeros. Indeed, the Weierstrass factorization of the Γ-function states that

$$\frac{1}{\Gamma(z+1)} = \exp(\gamma z) \prod_{k=1}^{\infty} (1 + z/k) \exp(-z/k),$$

where γ is Euler's constant. If P_0 denotes the infinite product on the right hand side then we have

$$\log \Gamma(z+1) = -\log P_0(z) - \gamma z = -\log P_0(z) + z \log P_0(1).$$

Therefore it seems natural to consider functions of the form

$$\frac{-\log P(z) + z \log P(1)}{z \operatorname{Log} z},$$

where P denotes an infinite product having negative zeros.

We recall that if $\{b_k\}$ is any sequence of complex numbers ($\neq 0$) such that $\sum_k |b_k|^{-2}$ converges then

$$z \mapsto \prod_{k=1}^{\infty} (1 - z/b_k) \exp(z/b_k)$$

defines an entire function. It is commonly denoted the canonical product of genus 1 associated with the sequence $\{b_k\}$, or having zeros at b_k. (The genus is defined as the smallest integer $\kappa \geq 0$ such that $\sum_k |b_k|^{-\kappa-1}$ converges.)

Throughout this paper $\{a_k\}$ denotes a sequence satisfying

$$< a_1 \leq a_2 \leq \dots \tag{1.1}$$

and

$$n(r) \leq \text{Const}\, r, \tag{1.2}$$

for all $r > 0$. Here n is the counting function associated with the sequence $\{a_k\}$:

$$n(r) = \#\{k \,|\, a_k \leq r\}.$$

We shall consider the canonical product P of genus 1 having zeros at $-a_k$, $k \geq 1$. This function is defined because of (1.2). Since all its zeros are negative we may define

$$\log P(z) = \sum_{k=1}^{\infty} \operatorname{Log}(1 + z/a_k) - z/a_k,$$

for $z \in \mathcal{A}$. (Here Log denotes the principal logarithm, defined in terms of the principal argument Arg). This is the unique branch of the logarithm of $P(z)$ that is real on the positive axis. Its imaginary part is given as

$$\arg P(z) = \sum_{k=1}^{\infty} \operatorname{Arg}(1 + (x+iy)/a_k) - y/a_k,$$

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for $z = x + iy \in \mathcal{A}$.

We recall that a *Pick function* is a holomorphic function φ in the upper half-plane $\mathbb{H}$ with $\Im\varphi(z) \geq 0$ for $z \in \mathbb{H}$. Pick functions are extended by reflection to holomorphic functions in $\mathbb{C} \setminus \mathbb{R}$ and they have the following integral representation

$$\varphi(z) = az + b + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t), \tag{1.3}$$

where $a \geq 0$, $b \in \mathbb{R}$ and μ is a non-negative Borel measure on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{t^2+1} < \infty.$$

See e.g. [6]. It is known that

$$a = \lim_{y\to\infty} \varphi(iy)/(iy), \qquad b = \Re\varphi(i), \qquad \mu = \lim_{y\to 0_+} \frac{\Im\varphi(t+iy)dt}{\pi},$$

where the last limit is in the *vague* topology, and finally that φ has a holomorphic extension to $\mathcal{A}$ if and only if $\operatorname{supp}(\mu) \subseteq]-\infty, 0]$.

We describe our main results.

THEOREM 1.1. *If $\{a_k\}$ satisfies (1.1) and $|k - a_k| \leq$ Const for $k \geq 1$, we have*

$$\frac{-\log P(z)}{z \operatorname{Log} z} = 1 + \frac{\log P(1)}{1-z} + \int_{-\infty}^{0} \frac{D(t)}{t-z}\, dt,$$

where D is defined as

$$D(x) = \frac{-\log|P(x)| + k\log|x|}{|x|((\log|x|)^2 + \pi^2)} \tag{1.4}$$

for $x \in [-a_{k+1}, -a_k]$ and $k \geq 1$ and

$$D(x) = \frac{-\log|P(x)|}{|x|((\log|x|)^2 + \pi^2)}$$

for $x \in [-a_1, 0]$.

Theorem 1.1 fournishes the Stieltjes representation of $\log P(z)/(z \operatorname{Log} z)$ on a half-line. The representing real-valued measure has density w.r.t. Lebesgue measure on the negative line and has a point mass at 1.

We could equally well have given the representation of

$$\frac{-\log P(z) + z\log P(1)}{z \operatorname{Log} z}$$

in terms of a density on the negative half-line (there is no support of the measure on the positive half-line, since the function has a removable singularity at 1). In this general setup there is no particular reason for this density to be positive. However, when $P = P_0$ is the canonical product in the Weierstrass factorization of the Γ-function above, the corresponding density *is* positive. This leads to the question of finding conditions on the distribution of the zeros in order that the corresponding density be positive, or, what amounts to the same, that

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the function above is a Pick function. Our second result (Theorem 1.2) gives such a condition and is a generalization of the main result of [5].

THEOREM 1.2. *If $\{a_k\}$ satisfies (1.1) and if $k \le a_k \le k+1$ for $k \ge 1$ then*

$$g(z) \equiv \frac{-\log P(z) + z \log P(1)}{z \operatorname{Log} z}$$

is a Pick function. It has the representation

$$g(z) = 1 + \int_{-\infty}^{0} \frac{d(t)}{t-z}\, dt,$$

where the positive *density d is defined as*

$$d(x) = \frac{-\log|P(x)| + \log P(1)x + k\log|x|}{|x|((\log|x|)^2 + \pi^2)} \tag{1.5}$$

for $x \in [-a_{k+1}, -a_k]$ and $k \ge 1$ and

$$d(x) = \frac{-\log|P(x)| + \log P(1)x}{|x|((\log|x|)^2 + \pi^2)}$$

for $x \in [-a_1, 0]$.

We notice that Theorem 1.2 can be generalized, by shifting the zeros to the left; see Theorem 5.7.

In terms of the counting function associated with the zeros, the positivity of the density d can be expressed more compactly as

$$\log|P(x)| - \log P(1)x \le n(-x)\log|x|,$$

for $x < 0$. This gives an upper bound on $|P(x)|$ on the negative axis. We shall describe the asymptotic behaviour of the maximum of $\log|P(x)|$ as x tends to $-\infty$.

To prove our main results we shall use a Phragmén-Lindelöf argument, and need to find new arguments (avoiding the functional equation of the Γ-function) in order to investigate the growth at infinity and the boundary behaviour on the real line.

2. Growth estimates. Throughout this section P denotes the canonical product of genus 1 having negative zeros $-a_k$, where $\{a_k\}$ satisfies (1.1) and (1.2). We shall estimate the growth of the holomorphic function

$$z \mapsto \frac{\log P(z)}{z \operatorname{Log} z}$$

in the upper half-plane. To do this we need some preliminary results about the growth of the canonical product P in the half-plane.

LEMMA 2.1. *Suppose that $\{a_k\}$ satisfies (1.1) and (1.2). Then*

$$\log|P(z)| \le \mathit{Const}\, |z| \log|z|$$

for all large values of $|z|$.

The proof is relatively straight forward. We have included it here for the readers convenience and also in order to illustrate the use of $P(z)P(-z)$.

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Proof. We get, by partial integration,

$$\begin{aligned}\log P(z) &= \int_0^\infty (\mathrm{Log}(1+z/t) - z/t)\,dn(t)\\ &= -z\int_0^\infty \frac{n(t)}{t}\left(\frac{1}{t}-\frac{1}{t+z}\right)dt,\end{aligned}$$

so that

$$\begin{aligned}\log|P(z)| = -x\int_0^\infty \frac{n(t)}{t}\left(\frac{1}{t}-\frac{x+t}{(x+t)^2+y^2}\right)dt\\ +y^2\int_0^\infty \frac{n(t)}{t}\frac{1}{(x+t)^2+y^2}dt.\end{aligned}$$

If $x \geq 0$ then

$$\frac{x+t}{(x+t)^2+y^2} \leq \frac{1}{x+t} \leq \frac{1}{t},$$

so that the first term in the relation above comes out negative. Hence, by the assumption (1.2),

$$\begin{aligned}\log|P(z)| &\leq |y|\int_0^\infty \frac{n(t)}{t}\frac{|y|}{(x+t)^2+y^2}dt\\ &\leq \mathrm{Const}\,|z|.\end{aligned}$$

We conclude that $|P|$ is of finite exponential growth in the right half-plane. To estimate the growth in the left half-plane we use the identity

$$P(z)P(-z) = f(z) \equiv \prod_{k=1}^\infty \left(1-\frac{z^2}{a_k^2}\right).$$

It is easily seen that f is an entire function of finite exponential type. We thus get

$$\log|P(-z)| \leq \mathrm{Const}\,|z| - \log|P(z)| \leq \mathrm{Const}\,|z| - \log|P(x)|,$$

since also $|P(z)| \geq |P(x)|$. Therefore, a *lower* bound on $|P|$ in the right half-plane will give us an *upper* bound on $|P|$ in the left half-plane; we thus consider

$$-\log P(x) = x\int_0^\infty \frac{n(t)}{t}\left(\frac{1}{t}-\frac{1}{x+t}\right)dt, \tag{2.1}$$

for $x \geq 0$. We find, with $\varepsilon > 0$ smaller than a_1,

$$\begin{aligned}x\int_0^\infty \frac{n(t)}{t}\left(\frac{1}{t}-\frac{1}{x+t}\right)dt &\leq \mathrm{Const}\,x\int_\varepsilon^\infty \left(\frac{1}{t}-\frac{1}{x+t}\right)dt\\ &= \mathrm{Const}\,x\log(1+x/\varepsilon)\\ &\leq \mathrm{Const}\,|z|\log|z|,\end{aligned}$$

for $|z|$ large. The lemma is proved. ☐

We also need an estimate of $\arg P$.

LEMMA 2.2. *Suppose that $\{a_k\}$ satisfies (1.1) and (1.2). Then*

$$|\arg P(z)| \leq \mathit{Const}\,|z|\log|z| + \mathit{Const}$$

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for all $z \in \mathcal{A}$.

Proof. We may assume that $z = x + iy \in \mathbb{H}$. We have

$$\arg P(x+iy) - \arg P(iy) = \int_0^x \frac{\partial}{\partial x} \arg P(t+iy)dt. \tag{2.2}$$

We shall now find estimates of $\arg P(iy)$ and of $\frac{\partial}{\partial x} \arg P(x+iy)$. From these estimates the relation above can be used to find estimates of $\arg P(x+iy)$ in $\mathbb{H}$. A simple computation shows that

$$\frac{\partial}{\partial x} \arg P(x+iy) = -\int_0^\infty \frac{y}{(s+x)^2+y^2} dn(s).$$

We also find that

$$|\arg P(iy)| \leq \text{Const}\, y \log(y+1).$$

This relation is straight forward to verify, e.g. by using partial integration.

Suppose first that $x \geq 0$. Then, by (2.2),

$$\begin{aligned} |\arg P(x+iy)| &\leq |\arg P(iy)| + \int_0^x \int_0^\infty \frac{y}{(s+t)^2+y^2} dn(s)dt \\ &\leq \text{Const}\, |z| \log|z| + \int_0^\infty \int_0^x \frac{y}{(s+t)^2+y^2} dt dn(s) \\ &\leq \text{Const}\, |z| \log|z| + x \int_0^\infty \frac{y}{s^2+y^2} dn(s). \end{aligned}$$

Here, by integration by parts,

$$x \int_0^\infty \frac{y}{s^2+y^2} dn(s) = 2x \int_0^\infty \frac{n(s)}{s} \frac{y}{s^2+y^2} \frac{s^2}{s^2+y^2} ds \leq \text{Const}\, x.$$

Suppose next that $x < 0$. In this situation,

$$|\arg P(x+iy)| \leq |\arg P(iy)| + \int_0^{-x} \int_0^\infty \frac{y}{(s-t)^2+y^2} dn(s)dt,$$

and we turn to estimate the double integral on the right hand side. We write it as

$$\begin{aligned} \int_0^{-x} \int_0^\infty \frac{y}{(s-t)^2+y^2} dn(s)dt &= \int_0^{-x} \int_0^{-x+1} \frac{y}{(s-t)^2+y^2} dn(s)dt \\ &\quad + \int_0^{-x} \int_{-x+1}^\infty \frac{y}{(s-t)^2+y^2} dn(s)dt, \end{aligned}$$

and we estimate these integrals separately. We find

$$\begin{aligned} \int_0^{-x} \int_0^{-x+1} \frac{y}{(s-t)^2+y^2} dn(s)dt &= \int_0^{-x+1} \int_0^{-x} \frac{y}{(s-t)^2+y^2} dt dn(s) \\ &\leq \int_0^{-x+1} \int_{-\infty}^\infty \frac{y}{(s-t)^2+y^2} dt dn(s) \\ &= \pi n(-x+1) \leq \text{Const}\, x. \end{aligned}$$

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The second integral is more delicate to estimate: first of all, if $t \in [0, -x]$, we get by partial integration,

$$\int_{-x+1}^{\infty} \frac{y}{(s-t)^2+y^2} dn(s) = -n(-x+1)\frac{y}{(t+x-1)^2+y^2} + \int_{-x+1}^{\infty} \frac{n(s)}{s} \frac{2(s-t)sy}{((s-t)^2+y^2)^2} ds,$$

so that

$$\int_0^{-x}\int_{-x+1}^{\infty} \frac{y}{(s-t)^2+y^2} dn(s)dt \le \int_0^{-x}\int_{-x+1}^{\infty} \frac{n(s)}{s} \frac{2(s-t)sy}{((s-t)^2+y^2)^2} dsdt.$$

This integral we write again as

$$\int_0^{-x}\int_{-x+1}^{\infty} \frac{n(s)}{s} \frac{s}{s-t} \frac{2(s-t)^2}{(s-t)^2+y^2} \frac{y}{(s-t)^2+y^2} dsdt,$$

and, using that $s \mapsto s/(s-t)$ is decreasing for $s > t$ and the assumed bound on $n(s)$, we see that this integral is bounded by

$$\begin{aligned}
&\text{Const} \int_0^{-x}\int_{-x+1}^{\infty} \frac{-x+1}{-x+1-t} \frac{2y}{(s-t)^2+y^2} dsdt \\
&\le \text{Const} \int_0^{-x} \frac{-x+1}{-x+1-t} \int_{-\infty}^{\infty} \frac{2y}{(s-t)^2+y^2} dsdt \\
&= \text{Const} \int_0^{-x} \frac{-x+1}{-x+1-t} dt \\
&= \text{Const}\,(-x+1)\log(-x+1).
\end{aligned}$$

The lemma follows by combining all these estimates. ◻

PROPOSITION 2.3. *If $\{a_k\}$ satisfies (1.1) and (1.2) there exist a constant C and a sequence $\{r_n\}$ tending to infinity such that*

$$\left|\frac{\log P(z)}{z \operatorname{Log} z}\right| \le C$$

for all $z \in \mathcal{A}$ of absolute value r_n and all n.

Proof. A classical result, going back to Littlewood, see [8], states that for some sequence $r_n \to \infty$,

$$\inf_{|z|=r_n} \log|P(z)| \ge -\text{Const} \sup_{|z|=r_n} \log|P(z)|.$$

Therefore, by Lemma 2.1,

$$|\log|P(z)|| \le \text{Const}\,|z|\log|z|$$

for all z satisfying $|z| = r_n$ and all n. The proposition now follows from Lemma 2.2. ◻

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3. An auxiliary Pick function. In this section we suppose that the sequence $\{a_k\}$ satisfies (1.1) and also

$$|a_k - k| \leq \text{Const.} \tag{3.1}$$

This is equivalent to

$$|n(r) - r| \leq \text{Const.} \tag{3.2}$$

The main result of this section is the following theorem.

THEOREM 3.1. *For any given sequence $\{a_k\}$ satisfying (1.1) and (3.1) there exists a real constant $A \leq \log P(1)$ such that*

$$g_A(z) = \frac{-\log P(z) + Az}{z \operatorname{Log} z}$$

is a Pick function.

REMARK 3.2. *If $A > \log P(1)$ then the function in the theorem above is certainly* not *a Pick function (its imaginary part tends to $-\infty$ as z tends to 1). Thus $A = \log P(1)$ yields the strongest result. In many cases the constant A should be taken* smaller *than $\log P(1)$ in order that g_A be a Pick function. As an example, consider*

$$P(z) = (1+z)\exp(-z) \cdot \prod_{k=5}^{\infty} (1 + z/k)\exp(-z/k),$$

which has zeros at $-1, -5, -6, \ldots$. In this case, computer experiments indicate that there are points $x \in [-5, -1]$ such that $\log|P(x)| - x \log P(1) > \log|x|$. This implies (see the proof of Theorem 3.1 below) that the imaginary part of

$$\frac{-\log P(z) + z \log P(1)}{z \operatorname{Log} z}$$

in the upper half plane has some negative boundary values and hence that the function cannot be a Pick function. However, in §5 we show that if $k \leq a_k \leq k+1$ then we may take A equal to $\log P(1)$.

To prove the theorem we need some lemmas.

LEMMA 3.3. *For the canonical product P of genus 1 associated with a sequence $\{a_k\}$ satisfying (1.1) and (3.1) we have (with $\varepsilon \leq a_1$)*

$$|-\log P(t) - t\log(1 + t/\varepsilon)| \leq \mathit{Const}\, t.$$

for $t > 0$.

Proof. From (2.1) we have (with $\varepsilon \leq a_1$)

$$-\log P(t) = t \int_{\varepsilon}^{\infty} \frac{n(s)}{s} \left(\frac{1}{s} - \frac{1}{s+t} \right) ds,$$

for $t > 0$. Therefore, and by (3.2),

$$\begin{aligned} |-\log P(t) - t\log(1+t/\varepsilon)| &= \left| -\log P(t) - t \int_{\varepsilon}^{\infty} \left(\frac{1}{s} - \frac{1}{s+t} \right) ds \right| \\ &= \left| t \int_{\varepsilon}^{\infty} \left(\frac{n(s)}{s} - 1 \right) \left(\frac{1}{s} - \frac{1}{s+t} \right) ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \text{Const } t \int_\varepsilon^\infty \frac{1}{s}\left(\frac{1}{s}-\frac{1}{s+t}\right) ds \\
&\leq \text{Const } t \int_\varepsilon^\infty \frac{t}{s^2(s+t)} ds \\
&\leq \text{Const } t.
\end{aligned}$$

The lemma is proved. $\square$

LEMMA 3.4. *For any given sequence $\{a_k\}$ satisfying (1.1) and (3.1) there exists a real constant A such that* $\log|P(x)| - Ax \leq 0$ *for all* $x \in [-a_1, 0]$ *and*

$$\log|P(x)| - Ax \leq k \log|x|$$

for all $x \in [-a_{k+1}, -a_k]$ *and all* $k \geq 1$.

Proof. For $x \in [-a_1, 0]$ we have $0 \leq P(x) \leq 1$, so the first assertion of the lemma is evidently true. Given any bounded interval of the form $[-K, 0]$, it is possible to choose A such that the asserted inequalities hold on any interval $[-a_{k+1}, -a_k] \subseteq [-K, 0]$. Hence we may assume that $x \leq -1$. If $k \geq 1$ and $x \in [-a_{k+1}, -a_k]$, we put $t = -x$ and find

$$-\log|P(x)| - Ax = \log|f(t)| - \log P(t) + At.$$

Here, $f(z) = \prod_{k=1}^\infty (1 - z^2/a_k^2)$ is (as used before) an entire function of exponential type. Hence, by Lemma 3.3,

$$\begin{aligned}
-\log|P(x)| - Ax &\leq \text{Const } t + t\log(1 + t/\varepsilon) + At \\
&\leq \text{Const } t + t\log t + At.
\end{aligned}$$

We get, for $t \in [a_k, a_{k+1}]$,

$$t\log t - k\log t \leq |a_{k+1} - k| \log t \leq \text{Const } \log t,$$

so that $-\log|P(x)| - Ax \leq \text{Const } t + At + k\log t$. From this relation we see that it is possible to choose A such that $-\log|P(x)| - Ax \leq k\log|x|$ for all k and all $x \in [-a_{k+1}, -a_k]$. This completes the proof. $\square$

Proof of of Theorem 3.1. We consider the harmonic function

$$V(z) = \Im\left(\frac{-\log P(z) + Az}{z \operatorname{Log} z}\right)$$

in the upper half-plane. Our goal is to show that $V \geq 0$ in $\mathbb{H}$. We claim that

$$\liminf_{z \to t, z \in \mathbb{H}} V(z) \geq 0$$

for all $t \in \mathbb{R}$. Indeed, if $t > 0$ and $t \neq 1$ then $V(z) \to V(t) = 0$ as $z \to t$. for z near 1 we have

$$g(z) = \frac{A - \log P(1)}{z - 1} + \phi(z),$$

where ϕ is holomorphic at $z = 1$. Thus,

$$V(z) = -\frac{(A - \log P(1))y}{|z-1|^2} + \Im\phi(z),$$

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so $\liminf_{z\to 1} V(z) \geq 0$ (and is 0 if $A = \log P(1)$). Since $-\log P(z) + Az$ (which is actually holomorphic in a neighbourhood of the origin) has a zero there,

$$|V(z)| \leq \frac{\text{Const}}{|\operatorname{Log} z|} \longrightarrow 0$$

as $z \to 0$ within $\mathbb{H}$. If $t \in (-a_{k+1}, -a_k)$ for some $k \geq 1$,

$$\begin{aligned} V(z) &\longrightarrow \Im\left(\frac{-\log|P(t)| - ik\pi + At}{t(\log|t| + i\pi)}\right) \\ &= \frac{-\pi}{t((\log|t|)^2 + \pi^2)}(-\log|P(t)| + At + k\log|t|) \end{aligned}$$

(and a similar estimate holds for $k = 0$ and $t \in (-a_1, 0)$). From the lemma above we see that these expressions are non-negative. If $t = -a_k$ for some k then one sees that $V(z) \to \infty$. Our claim is verified.

The function $-V$ is a harmonic function in the upper half-plane, which has, as we have just verified, non-positive boundary values on the real line. It is (by Proposition 2.3) bounded from above by some fixed constant on some semicircles, whose radii tend to infinity. The ordinary maximum principle yields that $-V$ is bounded from above in all of the upper half-plane, and hence, by applying an extended maximum principle, that $-V$ is non-positive in the upper half-plane (see e.g. [9, p. 23]). The theorem is proved. $\square$

4. Integral representation. We shall find an integral representation of functions of the form

$$z \mapsto \frac{-\log P(z)}{z \operatorname{Log} z},$$

where P is the usual canonical product associated with the sequence $\{-a_k\}$ satisfying (1.1) and (3.1). We shall do this by using the Pick functions described in Theorem 3.1. We choose $A \leq \log P(1)$ such that

$$g_A(z) = \frac{-\log P(z) + Az}{z \operatorname{Log} z}$$

is a Pick function. We shall find the integral representation of g_A as expressed in (1.3). We have

LEMMA 4.1. *We have, with $V_A(z) = \Im g_A(z)$,*

$$\frac{1}{\pi} V_A(t + i/n)dt \longrightarrow_n d_A(t)dt + (\log P(1) - A)\varepsilon_1$$

in the vague topology. Here ε_1 is the point mass at 1 and d_A is defined as $d_A(t) = 0$ for $t \geq 0$,

$$d_A(t) = -\frac{-\log|P(t)| + At + k\log|t|}{t((\log|t|)^2 + \pi^2)},$$

for $t \in [-a_{k+1}, -a_k]$ and $k \geq 1$ and

$$d_A(t) = -\frac{-\log|P(t)| + At}{t((\log|t|)^2 + \pi^2)},$$

for $t \in [-a_1, 0]$.

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Proof. A computation shows that

$$|z\operatorname{Log} z|^2 V_A(z) = (x\log|P(z)| + y\arg P(z))\operatorname{Arg} z \\ +(y\log|P(z)| - x\arg P(z))\log|z| \\ -A|z|^2\operatorname{Arg} z. \tag{4.1}$$

Here we notice that $\arg P(z)$ is bounded on compact subsets of the upper half-plane (Lemma 2.2) and that $\log|P(z)|$ involves only logarithmic singularities. If $h \in C_c(\mathbb{R})$ has its support in $[-K, 1/2]$ it thus follows that

$$\int_{-K}^{1/2} h(t)\log|P(t+i/n)|dt \longrightarrow_n \int_{-K}^{1/2} h(t)\log|P(t)|dt,$$

and that

$$\int_{-K}^{1/2} h(t)\arg P(t+i/n)dt \longrightarrow_n \int_{-K}^{1/2} h(t)\arg P(t)dt.$$

By (4.1) we see that

$$\frac{1}{\pi}\int_{-K}^{1/2} h(t)V_A(t+i/n)dt \longrightarrow_n \int_{-K}^{1/2} h(t)d_A(t)dt,$$

where d_A is given as in the statement of the lemma. (We notice that the origin does not represent any difficulty since, as noted before, $V_A(z) \leq \text{Const}\,/|\operatorname{Log} z|$ for z near zero.)

If h has its support in $[1-\delta, 1+\delta]$, then

$$\frac{1}{\pi}\int_{1-\delta}^{1+\delta} h(t)V_A(t+i/n)dt \longrightarrow_n -(A - \log P(1))h(1).$$

This follows from the (already used) fact that

$$g(z) = \frac{(A - \log P(1))}{z-1} + \phi(z),$$

where ϕ is holomorphic near $z = 1$. The lemma is proved. ◻

We have thus found the integral representation

$$g_A(z) = \int_{-\infty}^{0}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)d_A(t)\,dt \\ +(\log P(1) - A)\left(\frac{1}{1-z} - \frac{1}{2}\right) + \alpha_A z + \beta_A,$$

where

$$\alpha_A = \lim_{y\to\infty} g_A(iy)/(iy) = 0,$$

and

$$\beta_A = \Re g_A(i) = \frac{2}{\pi}\log|P(i)|.$$

On the other hand, we *know* that $A/\operatorname{Log} z$ is a Pick function (since $A<0$) and that

$$\frac{-1}{\operatorname{Log} z}=\int_{-\infty}^{0}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)\frac{1}{(\log|t|)^2+\pi^2}\,dt+\frac{1}{1-z}-\frac{1}{2}.$$

This gives us

$$\begin{aligned}\frac{-\log P(z)}{z\operatorname{Log} z}&=g_A(z)-\frac{A}{\operatorname{Log} z}\\ &=\int_{-\infty}^{0}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)D(t)\,dt\\ &\quad+\log P(1)\left(\frac{1}{1-z}-\frac{1}{2}\right)+\frac{2}{\pi}\log|P(i)|,\end{aligned}\tag{4.2}$$

where D is given in (1.4).

Concerning the growth of D we note:

LEMMA 4.2.

$$\int_{-\infty}^{0}\frac{|tD(t)|}{t^2+1}\,dt<\infty.$$

Proof. We shall again bring in the function $f(z)=\prod_{k=1}^{\infty}(1-z^2/a_k^2)$. We have earlier used that f is of exponential type and now we shall also use that

$$\int_{-\infty}^{\infty}\frac{\log^+|f(t)|}{t^2+1}\,dt<\infty.\tag{4.3}$$

This relation follows since f is of polynomial growth on the real line (see e.g. [10]; one may also use estimates similar to those in §5). It is known (see e.g. [9, p. 50]) that the exponential growth together with (4.3) implies

$$\int_{-\infty}^{\infty}\frac{|\log|f(t)||}{t^2+1}\,dt<\infty.\tag{4.4}$$

We find, for $t\in[-a_{k+1},-a_k]$,

$$\begin{aligned}\frac{|t|D(t)}{t^2+1}&=\frac{-\log|P(t)|+k\log|t|}{(t^2+1)((\log|t|)^2+\pi^2)}\\ &=\frac{-\log|f(t)|}{(t^2+1)((\log|t|)^2+\pi^2)}+\frac{\log P(-t)+k\log(-t)}{(t^2+1)((\log|t|)^2+\pi^2)}.\end{aligned}$$

Hence (with $a_0=0$),

$$\begin{aligned}\int_{-\infty}^{0}\frac{|tD(t)|}{t^2+1}\,dt&\le\int_{-\infty}^{0}\frac{|\log|f(t)||}{t^2+1}\frac{1}{(\log|t|)^2+\pi^2}\,dt\\ &\quad+\sum_{k=0}^{\infty}\int_{a_k}^{a_{k+1}}\frac{|\log P(t)+k\log t|}{(t^2+1)((\log t)^2+\pi^2)}\,dt.\end{aligned}$$

The integral involving $|\log|f(t)||$ is, by (4.4), finite. It therefore suffices to estimate the infinite sum in the last line of the relation above. To do this we use Lemma 3.3, from which we get (for large t)

$$|\log P(t)+k\log t|\le\text{Const}\,t,\quad t\in[a_k,a_{k+1}],$$

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where the constant does not depend on k. The infinite sum of integrals is therefore bounded from above by a constant times the finite integral

$$\int_2^\infty \frac{dt}{t(\log t)^2}.$$

The lemma is proved. $\square$

This lemma makes it possible for us to split the integral in (4.2) into a sum of two and we thus get

$$\frac{-\log P(z)}{z \operatorname{Log} z} = \int_{-\infty}^0 \frac{D(t)}{t-z}\,dt + \frac{\log P(1)}{1-z} + C,$$

for some constant C. We may identify C by noting that

$$\lim_{x\to\infty} \int_{-\infty}^0 \frac{D(t)}{t-x}\,dt = \lim_{x\to\infty} \frac{\log P(1)}{1-x} = 0,$$

so that (by Lemma 3.3),

$$C = \lim_{x\to\infty} \frac{-\log P(x)}{x\log x} = 1.$$

We conclude: If the sequence $\{a_k\}$ satisfies (1.1) and (3.1) we have the representation

$$\frac{-\log P(z)}{z \operatorname{Log} z} = \int_{-\infty}^0 \frac{D(t)}{t-z}\,dt + \frac{\log P(1)}{1-z} + 1, \qquad z \in \mathcal{A},$$

of the canonical product P of genus 1 associated with the sequence $\{-a_k\}$. The density D is defined in (1.4).

We have proved Theorem 1.1.

5. Zero distribution and positivity of density. In this section we prove Theorem 1.2: We show that Theorem 3.1 holds with $A = \log P(1)$ provided that the sequence $\{a_k\}$ satisfies

$$a_k \in [k, k+1], \qquad k \geq 1. \tag{5.1}$$

We begin our investigation by finding estimates of the function

$$f(z) = \prod_{k=1}^\infty \left(1 - \frac{z^2}{a_k^2}\right)$$

on the real line. For this we need

LEMMA 5.1. *For any $k \geq 2$,*

$$\left|\frac{\sin(\pi x)}{\pi x}\frac{1}{1-x^2/k^2}\right| \leq \begin{cases} 1 & \text{for } x \in [k-1,k], \\ 1/2 & \text{for } x \in [k,k+1]. \end{cases}$$

Proof. For $k \geq 2$ and $x \in [k, k+1]$,

$$\left|\frac{\sin(\pi x)}{\pi x}\frac{1}{1-x^2/k^2}\right| = \left|\frac{\sin(\pi x - \pi k)}{\pi x - \pi k}\right| \frac{k^2}{x(x+k)} \leq \frac{k^2}{k(k+k)} \leq \frac{1}{2}.$$

For $k \geq 3$ and $x \in [k-1, k]$,

$$\left|\frac{\sin(\pi x)}{\pi x}\frac{1}{1-x^2/k^2}\right| = \left|\frac{\sin(\pi x - \pi k)}{\pi x - \pi k}\right| \frac{k^2}{x(x+k)} \leq \frac{k^2}{(k-1)(2k-1)} \leq 1,$$

since $k \geq 3$. If $k = 2$ we should check that

$$\left|\frac{\sin(\pi x)}{\pi x}\frac{1}{1-x^2/4}\right| \leq 1$$

for $x \in [1, 2]$. This inequality *does* hold, but we do not give a detailed argument here. The lemma follows. ∎

PROPOSITION 5.2. *Suppose that* $\{a_k\}$ *satisfies* (5.1). *For any* $k \geq 2$, $|f(x)| \leq 1$ *for* $x \in [a_{k-1}, k]$ *and* $|f(x)| \leq 1/2$ *for* $x \in [k, a_k]$.

Proof. Suppose that $x \in [a_{k-1}, a_k]$, $k \geq 2$. Since

$$\left|1 - \frac{x^2}{a_l^2}\right| = \frac{x^2}{a_l^2} - 1 \leq \frac{x^2}{l^2} - 1$$

for $l = 1, \ldots, k-1$, and

$$\left|1 - \frac{x^2}{a_l^2}\right| = 1 - \frac{x^2}{a_l^2} \leq 1 - \frac{x^2}{(l+1)^2}$$

for $l \geq k$, we get

$$\begin{aligned}
|f(x)| &= \prod_{l=1}^{k-1}\left|1 - \frac{x^2}{a_l^2}\right| \cdot \prod_{l=k}^{\infty}\left|1 - \frac{x^2}{a_l^2}\right| \\
&\leq \prod_{l=1}^{k-1}\left|1 - \frac{x^2}{l^2}\right| \cdot \prod_{l=k}^{\infty}\left|1 - \frac{x^2}{(l+1)^2}\right| \\
&= \left|\frac{\sin(\pi x)}{\pi x}\frac{1}{1-x^2/k^2}\right|.
\end{aligned}$$

Thus the proposition follows from the lemma above. ∎

PROPOSITION 5.3. *Suppose that* $\{a_k\}$ *satisfies* (5.1). *We have*

$$-\log P(t) + t\log P(1) = \int_0^\infty \frac{n(s)}{s}\frac{1}{(s+t)(s+1)}\,ds \cdot t(t-1),$$

for $t > 0$ *and in particular*

$$-\log P(t) + t\log P(1) \leq \log\Gamma(t+1),$$

for $t \geq 1$.

Proof. A computation, based on (2.1), yields the identity in the lemma. The counting function $n(s)$ is, by the assumption on the a_k's, bounded from above by the counting function n_0 corresponding to the case of $a_k = k$. Thus, when $t \geq 1$,

$$\begin{aligned}
\int_0^\infty \frac{n(s)}{s}\frac{1}{(s+t)(s+1)}\,ds \cdot t(t-1) &\leq \int_0^\infty \frac{n_0(s)}{s}\frac{1}{(s+t)(s+1)}\,ds \cdot t(t-1) \\
&= -\log P_0(t) + t\log P_0(1),
\end{aligned}$$

where P_0 denotes the canonical product with zeros $-1, -2, \ldots$. But then we know that $-\log P_0(t) + t \log P_0(1) = \log \Gamma(t+1)$. The proposition follows. ☐

LEMMA 5.4. *Let* $k \geq 1$. *For* $x \in [-a_{k+1}, -a_k]$ *we have*

$$\log |P(x)| - x \log P(1) \leq k \log |x|.$$

Proof. If $t = -x \in [a_k, a_{k+1}]$,

$$\begin{aligned}\log |P(x)| - x \log P(1) &= \log |f(t)| - \log P(t) + t \log P(1) \\ &\leq \log |f(t)| + \log \Gamma(t+1),\end{aligned}$$

by the lemma above. It is therefore enough to show that

$$\log |f(t)| + \log \Gamma(t+1) \leq k \log t, \qquad a_k \leq t \leq a_{k+1},$$

for $k \geq 1$. Suppose first that $t \in [a_k, k+1] \subseteq [k, k+1]$. Then $\log |f(t)| \leq 0$ by Proposition 5.2, so we should verify that $\log \Gamma(t+1) \leq k \log t$ for $t \in [k, k+1]$ and all $k \geq 1$. This last inequality follows e.g. by induction from the functional equation for the Γ-function:

$$\log \Gamma(t+2) = \log \Gamma(t+1) + \log(t+1) \leq k \log t + \log(t+1) \leq (k+1) \log(t+1).$$

Suppose next that $t \in [k+1, a_{k+1}] \subseteq [k+1, k+2]$. Here, $\log |f(t)| \leq -\log 2$ by Proposition 5.2, so that

$$\log |f(t)| + \log \Gamma(t+1) \leq \log \Gamma(t+1) - \log 2.$$

We now claim that $\log \Gamma(t+1) - \log 2 \leq k \log t$ for $t \in [k+1, k+2]$ and $k \geq 1$. If $k = 1$, the inequality is true ($\log \Gamma(t) \leq \log 2$ for $t \in [2, 3]$). To go from k to $k+1$, we note that

$$\begin{aligned}\log \Gamma(t+2) - \log 2 &\leq \log \Gamma(t+1) + \log(t+1) - \log 2 \\ &\leq k \log t + \log(t+1) \leq (k+1) \log(t+1).\end{aligned}$$

The lemma is proved. ☐

We can now give the proof of Theorem 1.2 (the promised generalization of [5, Theorem 1.2]):

Lemma 5.4 shows that Theorem 3.1 holds with $A = \log P(1)$. Then Lemma 4.1 and Lemma 4.2 yield the desired representation

$$\frac{-\log P(z) + z \log P(1)}{z \operatorname{Log} z} = \int_{-\infty}^{0} \frac{d(t)}{t - z}\, dt + 1,$$

where the density (defined in (1.5)) is non-negative.

By differentiating under the integral sign, we find the following corollary.

COROLLARY 5.5. *We have, for* $n \geq 1$,

$$f^{(n)}(z) = (-1)^{n+1} n! \int_0^{\infty} \frac{d(-s)}{(s+z)^{n+1}}\, ds,$$

where

$$f(z) = \frac{-\log P(z) + z \log P(1)}{z \operatorname{Log} z},$$

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and P is the canonical product associated with a sequence $\{-a_k\}$ satisfying (1.1) and (5.1). In particular, f' is completely monotone on $]0,\infty[$.

REMARK 5.6. *One may describe the asymptotic behaviour of the maximum of $\log|P(x)|$ on the negative line for a canonical product P having negative zeros. As an example we shall study*

$$\sup\{\log|P(x)| - x\log P(1)\,|\, x\in[-a_k,-a_{k-1}]\,\}$$

as k tends to infinity. Here $\{a_k\}$ satisfies (5.1). We have (with $t=-x\in[a_{k-1},a_k]$)

$$\log|P(x)| - x\log P(1) = \log|f(t)| - \log P(t) + t\log P(1).$$

From the proof of Lemma 5.4 we know that this is bounded from above by $\log\Gamma(t+1)$. To find a lower bound on the supremum above, we need a lower bound on $|f|$. Such a lower bound can be found by using the inequalities (for $t\in[a_{k-1},a_k]$):

$$\left|1+\frac{t}{a_l}\right| \geq \left|1+\frac{t}{l+1}\right|,$$

for $l\geq 1$,

$$\left|1-\frac{t}{a_l}\right| \geq \left|1-\frac{t}{l+1}\right|,$$

for $1\leq l\leq k-2$, and

$$\left|1-\frac{t}{a_l}\right| \geq \left|1-\frac{t}{l}\right|,$$

for $l\geq k+1$. One obtains

$$|f(t)| \geq \left|\frac{\sin\pi(t-k)}{\pi(t-k)}\,\frac{k(1-t/a_k)(1-t/a_{k-1})}{1-t^2}\right|.$$

We now put $b_k=(a_k+a_{k-1})/2$ and note that

$$b_k\in[a_{k-1},a_k]\cap[k-1/2,k+1/2].$$

Therefore

$$\begin{aligned}\sup\{|f(t)|\,|\,t\in[a_{k-1},a_k]\,\} &\geq |f(b_k)|\\ &\geq \mathit{Const}\inf_{|s|\leq 1/2}\left|\frac{\sin(\pi s)}{\pi s}\right|\frac{(a_k-a_{k-1})^2}{k^3}.\end{aligned}$$

Furthermore, $-\log P(t)+t\log P(1)\geq\log\Gamma(t+2)-t\log 2$ for $t>1$ (by an argument as in Proposition 5.3) and we thus obtain, for a suitable constant C,

$$\begin{aligned}&\log\Gamma(k+1)-k\log 2-3\log k+2\log(a_k-a_{k-1})+C\\ &\qquad\leq\sup\{\log|P(x)|-x\log P(1)\,|\,x\in[-a_k,-a_{k-1}]\}\\ &\qquad\leq\log\Gamma(k+2).\end{aligned}$$

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Let us end this section by finding ways of weakening the assumption (5.1). As indicated in Remark 3.2, *removal* of some zeros of P may destroy the Pick property of

$$\frac{-\log P(z) + z\log P(1)}{z\,\mathrm{Log}\,z}.$$

It is possible to obtain the following result, dealing with the case of the zeros being shifted to the left.

THEOREM 5.7. *Let $a \geq 1$ and suppose that $a_k \in [a+k-1, a+k]$ for $k = 1, 2, \ldots$. Then*

$$z \mapsto \frac{-\log P(z) + z\log P(1)}{z\,\mathrm{Log}\,z}$$

is a Pick function. Here P denotes the canonical product of genus 1 having zeros at $\{-a_k\}$.

Proof. The proof follows the same lines as the proof of Theorem 1.2 and is based on the inequalities

$$\log|P(x)| - x\log P(1) \leq k\log|x|$$

for $x \in [-a_{k+1}, -a_k]$ and $k \geq 1$. We shall briefly indicate how to verify these inequalities.

First of all, for $t \in [a_k, a_{k+1}] \subseteq [a+k-1, a+k+1]$,

$$\begin{aligned} |f(t)| &= \prod_{l=1}^{\infty}\left|1 - \frac{t^2}{a_l^2}\right| \\ &\leq \frac{\Gamma(a)^2\Gamma(t-a)(t-a)^2}{\Gamma(t+a)}\left|\frac{\sin\pi(t-a)}{\pi(t-a)}\frac{1}{1-(t-a)^2/k^2}\right|. \end{aligned}$$

This estimate can be deduced in the same way as the estimate in Proposition 5.2 and using the identity

$$\prod_{l=0}^{\infty}\left(1 - \frac{t^2}{(a+l)^2}\right) = \frac{\Gamma(a)^2}{\Gamma(t+a)\Gamma(-t+a)}.$$

Secondly, from Proposition 5.3, for $t \geq 1$,

$$-\log P(t) + t\log P(1) \leq -\log P_a(t) + t\log P_a(1),$$

where P_a has its zeros at $-(a+k-1), k \geq 1$. The relation

$$-\log P_a(t) + t\log P_a(1) = \log\Gamma(t+a) - \log\Gamma(a) - t\log a$$

follows either by computation by noting that $1/\Gamma(z+a)$ is an entire function with zeros at $-(a+k-1), k \geq 1$ and of order (at most) 1. Thus, by Hadamards factorization theorem, it is of the form $P_a(z)\exp(A+Bz)$, where one finds $A = \log\Gamma(a)$ and $B = \log a + \log P_a(1)$.

Now, suppose that $t \in [a_k, a_{k+1}]$ ($\subseteq [a+k-1, a+k+1]$). We find

$$\begin{aligned} \log|P(x)| - x\log P(1) &\leq \log\Gamma(t+a) - \log\Gamma(a) - t\log a + \log|f(t)| \\ &\leq \log\Gamma(a) - t\log a + \log\Gamma(t-a+1) \\ &\quad + \log(t-a) + \eta(t), \end{aligned}$$

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where $\eta(t) = 0$ for $a+k-1 \le t \le a+k$ and $\eta(t) = -\log 2$ for $a+k \le t \le a+k+1$ according to Lemma 5.1. If we put $s = t-a$ we see that the desired inqualities are verified if

$$\log\Gamma(a) - (s+a)\log a + \log\Gamma(s+1) + \log s \le k\log(s+a),$$

for $k-1 \le s \le k$ and

$$\log\Gamma(a) - (s+a)\log a + \log\Gamma(s+1) + \log s - \log 2 \le k\log(s+a),$$

for $k \le s \le k+1$. To verify these inequalities we use that $\log\Gamma(a) \le a\log a$ for $a \ge 1$, $s\log a \ge 0$, $\log(s+a) \ge \log s$ and the inequalities for Γ used in the proof of Lemma 5.4. The theorem is proved. □

If we take $a_k = R+k-1, k \ge 1$ for some $R \ge 1$ then we obtain the following corollary.

COROLLARY 5.8. *For any $R \ge 1$, the function*

$$z \mapsto \frac{\log\Gamma(z+R) - \log\Gamma(R) - z\log R}{z\,\mathrm{Log}\,z}$$

is a Pick function.

REMARK 5.9. *It should be noted that e.g. the assumption $|a_k - k| \le 1/2$, for $k \ge 1$ is* not *sufficient to produce a Pick function. Even for arithmetic sequences $a_k = r+k-1$, where $r > 0$ is close to zero the result need not be a Pick function.*

A convergent sum of Pick functions is again a Pick function. As a consequence we mention the following result, the moral being "the more zeros the better".

COROLLARY 5.10. *Let $\{P_k\}$ be a sequence of canonical products of genus 1 and suppose that P_k is associated with $\{-a_l^{(k)}\}$, where $a_l^{(k)} \in [a^{(k)}+l-1, a^{(k)}+l]$, for $l \ge 1$ and some numbers $a^{(k)} \ge 1$. If $\sum_{k=1}^\infty \sum_{l=1}^\infty (a_l^{(k)})^{-2} < \infty$ then*

$$z \mapsto \frac{-\log P(z) + z\log P(1)}{z\,\mathrm{Log}\,z}$$

is a Pick function, where $P(z) = \prod_{k=1}^\infty P_k(z)$ is the canonical product of genus 1 associated with $\{-a_l^{(k)}\}$.

6. A related Pick function. We note the following generalization of [5, Theorem 5.1]. The proof is exactly the same and we shall not repeat it here.

PROPOSITION 6.1. *If $0 < b_1 \le b_2 \le \ldots$ and $\sum_{k=1}^\infty b_k^{-2} < \infty$ then*

$$z \mapsto -\frac{\log P(z)}{z} = \sum_{k=1}^{\infty}\left(\frac{1}{b_k} - \frac{\mathrm{Log}(1+z/b_k)}{z}\right)$$

is a Pick function and it has the integral representation

$$-\frac{\log P(z)}{z} = b + \int_{-\infty}^{0}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\xi(t)\,dt,$$

where

$$b = \sum_{k=1}^{\infty}\left(\frac{1}{b_k} - \arctan\left(\frac{1}{b_k}\right)\right)$$

and ξ is defined as $\xi(t) = 0$ for $t \ge -b_1$ and

$$\xi(t) = \frac{-k}{t} \quad \text{for } t \in \,]-b_{k+1}, -b_k[\,,\ k = 1, 2, \ldots.$$

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