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A REMARK ON UNIQUENESS OF BEST RATIONAL APPROXIMANTS OF DEGREE 1 IN L^2 OF THE CIRCLE*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. We derive a criterion for uniqueness of a critical point in H^2 rational approximation of degree 1. Although narrowly restricted in scope because it deals with degree 1 only, this criterion is interesting because it addresses a large class of functions. The method elaborates on the topological approach in [15] and [12].

Key words. rational approximation, uniqueness, Hardy spaces, critical points

AMS subject classifications. 31A25, 30E10, 30E25, 35J05

1. Introduction. Rational approximation to holomorphic functions on compact subsets of their domain of holomorphy is a most classical aspect of function theory. From the very possibility of approximation asserted by Runge's theorem (at least in the finitely connected case), the emphasis has gradually moved towards determining optimal error rates as the degree (hereafter denoted by n) growths large, and constructive means to achieve them. Interpolation theory and logarithmic potential theory have been cornerstones of this development, resulting in a fairly general treatment of asymptotic n-th root error estimates in the sup norm, namely a sharp upper bound on its $\limsup$ [33] and an upper bound on its $\liminf$ [28, 30] (formerly Gonchar's conjecture) whose sharpness was later established in [23] using results from [32]. Besides, strong asymptotic error estimates in the *sup* norm have been derived for more specific functions. Let us mention in particular the case of the exponential function [16] and that of Markov functions [22]; strong asymptotic error estimates for Markov functions were also obtained in H^2-norm on the disk [14], see [6] for a generalization to meromorphic approximation in the H^p norm. In the cases just mentioned, asymptotically optimal sequences of rational approximants can be constructed as Padé approximants or multipoint Padé approximants whose interpolation points, when arranged into a triangular scheme, converge in distribution to some appropriate equilibrium measure arising from a potential-theoretic minimum energy problem.

In contrast, the actual computation of a best or near-best approximant of given degree on a given compact set is still much of an open problem. On the disk, for the uniform norm, a generalization of Remez-like algorithms was proposed in [25, 24], which is however subject to combinatorial choices on the number of points where the error is maximal, to the occurrence of local minima, and for which issues of convergence are apparently still not settled. Suboptimal rational approximants in the uniform norm may be obtained from AAK-theory, but their quality depends on the smallness of the sum of the higher singular values of the Hankel operator having as a symbol the function to be approximated [21], and this sum need neither be small nor even efficiently computable. Finally for L^p-norms, where the criterion is differentiable, methods from optimization are flawed by local minima.

For the case of the H^2-norm on the disk, which is protypical of a smooth criterion, a topological approach was taken in [15] to find conditions under which there is no local minimum except the global one. This property makes for constructive algorithms, because it ensures the convergence of a numerical search. In [15], it was applied to Markov functions: for such functions, it was shown in essence that uniqueness of a critical point holds if the Green ca-

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pacity of the support of the defining measure is less than an absolute constant. But it is in [12] that the connection of the method with interpolation theory was realized and exploited, to the effect that normality and regular error decay of certain multipoint Padé interpolants imply the desired uniqueness property. In particular, it was proved in [12] that uniqueness of a critical point in H^2-rational approximation to the exponential function holds in all degree sufficiently large. The same result was established for Markov functions whose defining measure satisfies the Szegö condition [13], see [4] for a more extensive discussion. Contrary to [15], however, these last two results are not fully constructive in that no lower bound was given on the degree that ensures uniqueness.

The modest objective of the present paper is to give a criterion for uniqueness of a critical point in rational H^2 approximation *of degree 1*. Although narrowly restricted in scope, the criterion is interesting in that it is easily checked and addresses a large class of functions. It is to be hoped that suitable generalizations of the estimates below will allow one to handle higher degree as well.

The method is still that of [15] and [12], except that the estimation of the second derivative proceeds differently. The organization of the paper is as follows. In the next section we set up the notations, and in section 3 we state the rational approximation problem under study. The critical point theory that we need is recalled in section 4, while section 5 is devoted to the proof of our uniqueness criterion in degree 1.

2. Notations and preliminaries. We let $\mathbb{T}$ be the unit circle and $\mathbb{D}$ the open unit disk in the complex plane. We further denote by $L^2 = L^2(\mathbb{T})$ the familiar Lebesgue space with respect to normalized arclength measure on $\mathbb{T}$. The Hardy space H^2 of the unit disk is the closed subspace of L^2 consisting of those functions whose Fourier coefficients of strictly negative index do vanish. By definition H^2 is thus a Hilbert subspace of L^2.

It is a classical fact [18, 26, 20] that members of H^2 are in one-to-one correspondence with nontangential limits of those functions g holomorphic in $\mathbb{D}$ whose L^2 means remain uniformly bounded over all circles centered at 0 of radius less than 1 :

$$\sup_{0<r<1} \int_0^{2\pi} |g(re^{i\theta})|^2 \, d\theta < \infty. \tag{2.1}$$

This allows one to alternatively regard members of H^2 as holomorphic functions in the variable $z \in \mathbb{D}$; the extension from $\mathbb{T}$ to $\mathbb{D}$ is actually obtained through a Cauchy as well as a Poisson integral. Without further notice, we shall consider Hardy functions either as functions of $e^{i\theta} \in \mathbb{T}$ or as functions of $z \in \mathbb{D}$, whichever is more convenient.

From (2.1) and Parseval's theorem, it follows by easily that

$$g(z) \in H^2 \quad \text{iff} \quad g(z) = \sum_{j=0}^{\infty} a_j z^j, \quad \text{with} \quad \sum_{j=0}^{\infty} |a_j|^2 < \infty.$$

Next, we introduce the Hardy space $\overline{H}_0^2$ of the complement of the disk, consisting of L^2 functions whose Fourier coefficients of non-negative index do vanish; these are precisely the complex conjugates of H^2-functions with zero mean, and they can in turn be viewed as nontangential limits of those functions holomorphic in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ that vanish at infinity and have uniformly bounded L^2 means over all circles centered at 0 of radius bigger than 1. Note that a function belongs to $\overline{H}_0^2$ if, and only if it is of the form $e^{-i\theta}\overline{g(e^{i\theta})}$ for some $g \in H^2$. In other words, the map $h \to h^\sigma$, where

$$h^\sigma(z) := \frac{1}{z}\overline{h(\frac{1}{\bar{z}})}, \tag{2.2}$$

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is a bijection from H^2 onto $\overline{H}_0^2$. Clearly, this correspondence is involutive and isometric. In particular, it holds that

$$f(z) \in \overline{H}_0^2 \quad \text{iff} \quad f(z) = \sum_{k=1}^{\infty} b_k z^{-k}, \quad \text{with} \quad \sum_{k=1}^{\infty} |b_k|^2 < \infty,$$

and by Parseval's theorem again we have the orthogonal decomposition:

$$L^2 = H^2 \oplus \overline{H}_0^2. \tag{2.3}$$

Note in passing that the scalar product in L^2 can be rewritten as a line integral:

$$< f, g >= \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})}\, d\theta = \frac{1}{2i\pi} \int_{\mathbb{T}} f(\xi) g^{\sigma}(\xi)\, d\xi. \tag{2.4}$$

Restricting to functions with real Fourier coefficients, one defines analogously $L^2_{\mathbb{R}}$, $H^2_{\mathbb{R}}$, and $\overline{H}^2_{0,\mathbb{R}}$; these are *real* Hilbert spaces. All the previous considerations apply, in particular it is still true that $h \to h^{\sigma}$ is an involutive isometry from $H^2_{\mathbb{R}}$ onto $\overline{H}^2_{0,\mathbb{R}}$.

We will denote by $\mathcal{P}_n$ the space of algebraic polynomials with complex coefficients of degree at most n, and by $\mathcal{M}_n$ the set of *monic* polynomials of exact degree n whose roots lie in $\mathbb{D}$. Taking the coefficients –except the leading one which is 1– as coordinates, $\mathcal{M}_n$ becomes an open subset of $\mathbb{C}^n$. Its closure $\overline{\mathcal{M}}_n$ consists of monic polynomials whose roots lie in $\overline{\mathbb{D}}$; its boundary $\partial\mathcal{M}_n$ is the set of monic polynomials whose roots have modulus at most 1 and at least one of them has modulus 1. For $p \in \mathcal{P}_n$, we denote by $\widetilde{p}$ the *reciprocal* polynomial of p:

$$\widetilde{p}(z) \triangleq z^n \overline{p(1/\bar{z})}, \quad p \in \mathcal{P}_n, \tag{2.5}$$

whose roots are reflected from those of p across the unit circle and whose modulus on $\mathbb{T}$ is the same as the modulus of p. We offer a word of warning about this notation: if $n' > n$ and $p \in \mathcal{P}_n$ is considered as a member of $\mathcal{P}_{n'}$ whose leading coefficients do vanish, the two definitions of $\widetilde{p}$ are inconsistent. For that reason, we always specify the value of n under consideration, as was done in equation (2.5).

The symbols $\mathcal{P}_{n,\mathbb{R}}$, $\mathcal{M}_{n,\mathbb{R}}$, $\overline{\mathcal{M}}_{n,\mathbb{R}}$ and $\partial\mathcal{M}_{n,\mathbb{R}}$ refer to the preceding notions for polynomials with *real* coefficients. This time, of course, $\mathcal{M}_{n,\mathbb{R}}$, $\overline{\mathcal{M}}_{n,\mathbb{R}}$, and $\partial\mathcal{M}_{n,\mathbb{R}}$ are regarded as subsets of $\mathbb{R}^n$ rather than $\mathbb{C}^n$

We further let $\mathcal{R}_{m,n}$ be the set of rational functions of type (m, n), *i.e.* that can be written in the form p/q where $p \in \mathcal{P}_m$ and $q \in \mathcal{P}_n \setminus \{0\}$; we write $\mathcal{R}_{m,n,\mathbb{R}}$ for rational functions with real coefficients, that is if we can choose $p \in \mathcal{P}_{m,\mathbb{R}}$ and $q \in \mathcal{P}_{n,\mathbb{R}}$. By definition, the *degree* of a rational function is $\max\{m, n\}$ where m, n are such that the function belongs to $\mathcal{R}_{m,n}$ but not to $\mathcal{R}_{m',n'}$ whenever $m' < m$ or $n' < n$.

Note that a rational function belongs to H^2 if, and only if its poles lie in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and to $H^2_{\mathbb{R}}$ if, in addition, it has real coefficients. A rational function belongs to $\overline{H}_0^2$ (resp. $\overline{H}^2_{0,\mathbb{R}}$) if, and only if it lies in $\mathcal{R}_{n-1,n}$ (resp. $\mathcal{R}_{n-1,n,\mathbb{R}}$) for some n and its poles lie in $\mathbb{D}$.

3. L^2 rational approximation. We consider the following rational approximation problem:

$\mathbf{P}(f, n)$: *Given $f \in \overline{H}_0^2$ and some positive integer n, minimize*

$$\left\| f - \frac{p}{q} \right\|_{L^2} \tag{3.1}$$

as p/q ranges over $\mathcal{R}_{n-1,n}$.

The version with real coefficients is

$\mathbf{P}_{\mathbb{R}}(f,n)$: *Given $f \in \overline{H}^2_{0,\mathbb{R}}$ and some positive integer n, minimize*

$$\left\| f - \frac{p}{q} \right\|_{L^2} \tag{3.2}$$

as p/q ranges over $\mathcal{R}_{n-1,n,\mathbb{R}}$.

On applying (2.2), problems $\mathbf{P}(f,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$ are immediately seen to be respectively equivalent to

$\mathbf{P}'(g,n)$: *Given $g \in H^2$ and some positive integer n, minimize*

$$\left\| g - \frac{p}{q} \right\|_{L^2}$$

as p/q ranges over $\mathcal{R}_{n-1,n}$

and

$\mathbf{P}'_{\mathbb{R}}(g,n)$: *Given $g \in H^2_{\mathbb{R}}$ and some positive integer n, minimize*

$$\left\| g - \frac{p}{q} \right\|_{L^2}$$

as p/q ranges over $\mathcal{R}_{n-1,n,\mathbb{R}}$.

Although $\mathbf{P}'(g,n)$ and $\mathbf{P}'_{\mathbb{R}}(g,n)$ may look more natural than $\mathbf{P}(f,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$, we rather deal with the latter which are slightly easier to handle. Indeed, by partial fraction expansion, it is easily checked from (2.3) that a solution to $\mathbf{P}(f,n)$ (resp. $\mathbf{P}_{\mathbb{R}}(f,n)$) must lie in $\mathcal{R}_{n-1,n} \cap \overline{H}^2_0$ (resp. $\mathcal{R}_{n-1,n,\mathbb{R}} \cap \overline{H}^2_{0,\mathbb{R}}$), therefore its poles (which are the most important quantities) remain in a bounded set, namely $\mathbb{D}$. We also mention that, surprisingly perhaps, $\mathbf{P}(f,n)$ does not supersede $\mathbf{P}_{\mathbb{R}}(f,n)$ in that a best approximant from $\mathcal{R}_{n-1,n}$ to $f \in \overline{H}^2_{0,\mathbb{R}}$ may fail to belong to $\mathcal{R}_{n-1,n,\mathbb{R}}$. However, the two problems can be approached in a parallel manner so we often dispense with explanations on both cases, leaving it to the reader to transpose the arguments from one case to the other.

The first issue to be addressed is that of existence. The following proposition can be gathered from [19, 27, 33, 31, 17, 8, 7] but we provide a proof for the ease of the reader.

PROPOSITION 3.1. *Problems $\mathbf{P}(f,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$ have a solution; moreover, any solution has exact degree n unless $f \in \mathcal{R}_{n-2,n-1}$.*

Proof. We restrict ourselves to $\mathbf{P}(f,n)$, the case of $\mathbf{P}_{\mathbb{R}}(f,n)$ being argued the same way. Let $p^{(k)}/q^{(k)}$ be a minimizing sequence for $\mathbf{P}(f,n)$. For fixed $q \in \mathcal{M}_n$, the minimum in (3.1) (resp. (3.2)) is attained when p is the orthogonal projection of f onto the linear subspace $\mathcal{P}_{n-1}/q$ of rational functions in $\overline{H}^2_0$ with denominator q. Therefore, we may as well assume for each k that $p^{(k)}/q^{(k)}$ is the orthogonal projection of f onto $\mathcal{P}_{n-1}/q^{(k)}$, and in particular that

$$\left\| \frac{p^{(k)}}{q^{(k)}} \right\|_{L^2} \leq \|f\|_{L^2}.$$

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It follows that $p^{(k)}/q^{(k)}$ is bounded in $\overline{H}^2_0$ and therefore it has a weak limit point, say, $h \in \overline{H}^2_0$. By the weak-compactness of balls we get

$$\|h\|_{L^2} \leq \liminf_{k\to\infty} \left\| \frac{p^{(k)}}{q^{(k)}} \right\|_{L^2},$$

so that h will be a solution to Problem $\mathbf{P}(f,n)$ as soon as we have shown that $h \in \mathcal{R}_{n-1,n}$. This in turn follows from the fact that $\mathcal{R}_{n-1,n}$ is weakly closed because, by a theorem of Kronecker [29, th.3.11], the membership to $\mathcal{R}_{n-1,n}$ can be characterized solely in terms of (infinitely many) algebraic relations between the Fourier coefficients upon writing that the Hankel matrix has rank at most n.

Next, assume that $h \in \mathcal{R}_{n-2,n-1}$. Then, it holds for each $a \in \mathbb{D}$ and $b \in \mathbb{C}$ that $h + b/(z-a) \in \mathcal{R}_{n-1,n}$ and therefore, by definition of h, that

$$\|f-h\|^2_{L^2} \leq \|f-h-b/(z-a)\|^2_{L^2}. \tag{3.3}$$

Expanding (3.3) by bilinearity we obtain

$$0 \leq |b|^2 \|1/(z-a)\|^2_{L^2} - 2\mathbf{Re}\bar{b} < 1/(z-a), f-h >,$$

and taking $|b|$ very small we see that this is possible only if

$$0 =< 1/(z-a), f-h >= (f-h)^\sigma(a),$$

where the second equality comes from (2.4) and the residue formula. Now, the H^2-function $(f-h)^\sigma$ is identically zero since $a \in \mathbb{D}$ was arbitrary, and therefore $f = h \in \mathcal{R}_{n-2,n-1}$ as announced. $\square$

REMARK 1. *Let us agree that a local minimum in Problem $\mathbf{P}(f,n)$ (resp. $\mathbf{P}_{\mathbb{R}}(f,n)$) is a member of $\mathcal{R}_{n-1,n}$ (resp. $\mathcal{R}_{n-1,n,\mathbb{R}}$) that minimizes $p/q \mapsto \|f-p/q\|$ over a neighborhood of itself in $\overline{H}^2_0 \cap \mathcal{R}_{n-1,n}$ (resp. $\overline{H}^2_{0,\mathbb{R}} \cap \mathcal{R}_{n-1,n,\mathbb{R}}$). Then, because it is enough to deal with arbitrary small $|b|$, the argument in the second part of the above proof generalizes to* local minima *in Problems $\mathbf{P}(f,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$, showing that these have exact degree n unless $f \in \mathcal{R}_{n-2,n-1}$ (in which case there is no other local minimum than the global one which is just f itself, see [10]).*

A solution to Problem $\mathbf{P}(f,n)$ or $\mathbf{P}_{\mathbb{R}}(f,n)$ needs not be unique: for example, any non-rational even function in $\overline{H}^2_0$ (resp. $\overline{H}^2_{0,\mathbb{R}}$) has at least two best approximants from $\mathcal{R}_{n-1,n}$ (resp. $\mathcal{R}_{n-1,n,\mathbb{R}}$) when n is odd [31, 8, 7]. In the case of $\mathbf{P}(f,n)$, an extreme example of a function with infinitely many best approximants of given order can even be obtained [17]. All the above examples exploit some symmetry of the function f. In another vein, relying on topological methods, one can adapt [16, ch. 5, thm 1.6] to the present situation and show that any $(n+1)$-dimensional subspace of $\overline{H}^2_0$ (resp. $\overline{H}^2_{0,\mathbb{R}}$) containing no non-zero member of $\mathcal{R}_{n-1,n}$ (resp. $\mathcal{R}_{n-1,n,\mathbb{R}}$) must contain a function with at least two best approximants.

Nevertheless, by a general theorem of Stechkin on Banach space approximation from approximately compact sets [16], the solution to $\mathbf{P}(f,n)$ (resp. $\mathbf{P}_{\mathbb{R}}(f,n)$) is unique for f in a dense subset of $\overline{H}^2_0$ (resp. $\overline{H}^2_{0,\mathbb{R}}$). Reference [9] improves this in the case of $\mathbf{P}_{\mathbb{R}}(f,n)$ to the effect that the dense subset in question contains an *open* set; the argument there would carry over to $\mathbf{P}(f,n)$ as well. Even if the solution to $\mathbf{P}(f,n)$ or $\mathbf{P}_{\mathbb{R}}(f,n)$) is unique, though, there may be several local minima that impede a numerical search for a solution. This is why it is of particular importance to set up conditions on f that ensure uniqueness of a local minimum. However, from first principles in differential topology, local minima turn out to be difficult to analyse independently from other *critical points*, namely saddles and local maxima. In the next section, we gather some facts from the critical point theory of problems $\mathbf{P}(f,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$.

4. Critical points. For fixed $q \in \mathcal{M}_n$ (resp. $\mathcal{M}_{n,\mathbb{R}}$), as noticed already in the proof of Proposition 3.1, the minimum in (3.1) (resp. (3.2)) is attained when p is the orthogonal projection of f onto the subspace $\mathcal{P}_{n-1}/q$ of rational functions in $\overline{H}_0^2$ (resp. $\overline{H}_{0,\mathbb{R}}^2$) with denominator q. Write $L_n^f(q)/q$ for this projection, where $L_n^f(q) \in \mathcal{P}_{n-1}$ is a function of q parametrized by f and n. Here, there is no need to adopt a special notation like $L_{n,\mathbb{R}}^f(q)$ when $f \in \overline{H}_{0,\mathbb{R}}^2$ and $q \in \mathcal{M}_{n,\mathbb{R}}$, because then the projection of f onto $\mathcal{P}_{n-1}/q$ in $\overline{H}_0^2$ and the projection of f onto $\mathcal{P}_{n-1,\mathbb{R}}/q$ in $\overline{H}_{0,\mathbb{R}}^2$ coincide.

The next proposition formulates in terms of division a characterization of $L_n^f(q)$ which in essence goes back to [33].

PROPOSITION 4.1. *For $f \in \overline{H}_0^2$ and $q \in \mathcal{M}_n$, let $r \in \mathcal{P}_{n-1}$ be the remainder of the division of $f^\sigma \widetilde{q}$ by q:*

$$f^\sigma \widetilde{q} = vq + r, \quad v \in H^2,\ r \in \mathcal{P}_{n-1}. \tag{4.1}$$

Then it holds that $L_n^f(q) = \widetilde{r}$. Moreover, we can write

$$\widetilde{L}_n^f(q)(z) = \frac{1}{2i\pi} \int_T \frac{f^\sigma(\xi)\widetilde{q}(\xi)}{q(\xi)} \frac{q(\xi) - q(z)}{\xi - z} d\xi. \tag{4.2}$$

Proof. Applying (2.2) to (4.1), we see that the latter is equivalent to the relation

$$f - \widetilde{r}/q = v^\sigma \widetilde{q}/q. \tag{4.3}$$

Hence if we pick $p \in \mathcal{P}_{n-1}$, we get

$$< f - \widetilde{r}/q, p/q >=< v^\sigma \widetilde{q}/q, p/q >,$$

and since multiplication by $q/\widetilde{q}$ is an isometry of L^2 –because $q/\widetilde{q}$ has modulus 1 on $\mathbb{T}$– we obtain

$$< f - \widetilde{r}/q, p/q >=< v^\sigma, p/\widetilde{q} > .$$

Now, q has its roots in $\mathbb{D}$ so that $\widetilde{q}$ has its roots outside $\overline{\mathbb{D}}$, which entails that $p/\widetilde{q} \in H^2$ while $v^\sigma \in \overline{H}_0^2$. As (2.3) is an orthogonal sum, we therefore deduce that

$$< f - \widetilde{r}/q, p/q >= 0$$

meaning that $L_n^f(q) = \widetilde{r}$. The representation (4.2) now follows from the Hermite formula for the remainder of polynomial division [33]. ∎

By definition of $L_n^f(q)$, the minimization in (3.1) (resp. (3.2)) can be replaced by the minimization over $\mathcal{M}_n$ (resp. $\mathcal{M}_{n,\mathbb{R}}$) of the function:

$$\Psi_n^f(q) \triangleq \|f - L_n^f(q)/q\|_{L^2}^2 \tag{4.4}$$

which depends on q only. The notation Ψ_n^f will refer to the function (4.4) defined on $\mathcal{M}_n$, whereas if $f \in \overline{H}_{0,\mathbb{R}}^2$ and $q \in \mathcal{M}_{n,\mathbb{R}}$ we shall write $\Psi_{n,\mathbb{R}}^f$ to mean the *restriction* of Ψ_n^f to $\mathcal{M}_{n,\mathbb{R}}$.

Differentiating under the integral sign, one deduces from (2.4) and (4.2) that Ψ_n^f (resp. $\Psi_{n,\mathbb{R}}^f$) is a smooth function of q. By definition, a *critical point* is any q where the gradient of Ψ_n^f (resp. $\Psi_{n,\mathbb{R}}^f$) vanishes (recall q is coordinatized by its coefficients except the leading one).

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Denominators of solutions to $\mathbf{P}(f,n)$ (resp. $\mathbf{P}_{\mathbb{R}}(f,n)$) are critical points of Ψ_n^f (resp. $\Psi_{n,\mathbb{R}}^f$) but there may be more, *e.g.* local minima, saddles and maxima.

It is worth characterizing critical points in terms of divisibility, as is done in the next proposition, which appears in [12] for the case of $\mathbf{P}_{\mathbb{R}}(f,n)$.

PROPOSITION 4.2. *Let $f \in \overline{H}_0^2$ (resp. $\overline{H}_{0,\mathbb{R}}^2$) and $q \in \mathcal{M}_n$ (resp. $\mathcal{M}_{n,\mathbb{R}}$). In view of Proposition 4.1, write the division of $f^\sigma \widetilde{q}$ by q in the form*

$$f^\sigma \widetilde{q} = vq + \widetilde{L}_n^f, \quad v \in H^2, \; L_n^f \in \mathcal{P}_{n-1}. \tag{4.5}$$

Then q is a critical point of Ψ_n^f (resp. $\Psi_{n,\mathbb{R}}^f$) if, and only if q divides $L_n^f v$ in H^2 (resp. in $H^2_{\mathbb{R}}$).

Proof. Write

$$q(z) = z^n + q_{n-1}z^{n-1} + \ldots + q_0$$

and put $q_k = x_k + iy_k$ for the real and imaginary parts of the coefficients. Introduce the differential operator

$$\partial_{q_k} = \frac{1}{2}\left(\partial/\partial x_k - i\partial/\partial y_k\right).$$

Because Ψ_n^f is real-valued, to say that q is critical for Ψ_n^f amounts to say that

$$\partial_{q_k}\Psi_n^f(q) = 0, \quad 0 \le k \le n-1.$$

On differentiating under the integral sign, this equality in turn yields:

$$\begin{aligned} &2\mathbf{Re} < \partial_{q_k} L_n^f(q)/q - L_n^f(q)z^k/q^2, f - L_n^f(q)/q > \\ &\qquad = -2\mathbf{Re} < L_n^f(q)z^k/q^2, f - L_n^f(q)/q >= 0, \quad 0 \le k \le n-1, \end{aligned}$$

where the first equality comes from the characteristic property of the orthogonal projection and the fact that $\partial_{q_k} L_n^f(q) \in \mathcal{M}_{n-1}$. Substituting (4.3) where $\widetilde{r} = L_n^f(q)$ (compare Proposition 4.1) we obtain

$$\mathbf{Re} < L_n^f(q)z^k/q^2, v^\sigma \widetilde{q}/q >= 0, \quad 0 \le k \le n-1,$$

and multiplying throughout by $q/\widetilde{q}$ while forming arbitrary linear combinations of these equations gives us

$$\mathbf{Re} < L_n^f(q)p/(q\widetilde{q}), v^\sigma >= 0, \quad \text{for all } p \in \mathcal{P}_{n-1}. \tag{4.6}$$

Let d be the monic $g.c.d.$ of q and $L_n^f(q)$, and $\xi_1, \ldots \xi_\kappa$ be the roots of q/d with multiplicities, say, $m_1, \ldots, m_\kappa$. When p range over the family of polynomials

$$(i)^\varepsilon (z-\xi_k)^{\ell_k} \Pi_{j \ne k}(z-\xi_j)^{m_j}, \quad \varepsilon = 0,1, \quad 1 \le k \le \kappa, \quad 0 \le \ell_k \le m_k - 1, \tag{4.7}$$

we see on computing (4.6) *via* the residue formula using (2.4) –where $g = v^\sigma$–that it is equivalent to

$$v^{(\ell_k)}(\xi_k) = 0, \quad 1 \le k \le d, \quad 0 \le \ell_k \le m_k - 1,$$

where the superscript (ℓ_k) indicates the ℓ_k-th derivative. In other words, q is critical if, and only if q/d divides v, that is to say if, and only if q divides $L_n^f v$ as desired. The argument for

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$\Psi^f_{n,\mathbb{R}}$ is similar but slightly simpler, since the q_k are real hence the introduction of ∂_{q_k} is not necessary and the real part does not occur in (4.6) so that the factor $(i)^\varepsilon$ can be omitted from (4.7). $\square$

Proposition 4.2 allows one to connect Problems $\mathbf{P}(f,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$ with interpolation theory. Indeed, suppose that $q \in \mathcal{M}_n$ is such that q and $L^f_n(q)$ have g.c.d. $d \in \mathcal{M}_k$ for some $k \in \{0,\dots,n\}$ (if $k=0$ then $d=1$ and if $k=n$ then $d=q$ while $L^f_n(q)=0$). Then, if we write for simplicity $q = dq_1$ and $L^f_n(q) = dp_1$, we get from (4.3) and Proposition 4.1 that

$$f - L^f_n(q)/q = f - p_1/q_1 = v^\sigma \widetilde{q}/q. \tag{4.8}$$

Now, if q is a critical point of Ψ^f_n (resp. $\Psi^f_{n,\mathbb{R}}$), it follows from Proposition 4.2 that the right-hand side of (4.8) is divisible by $\overline{q_1(1/\overline{z})q(1/\overline{z})}$ in $\overline{H}^2_0$ (resp. $\overline{H}^2_{0,\mathbb{R}}$) so that $L^f_n(q)/q$, which is of type $(n-k-1, n-k)$, interpolates f in at least $2n-k$ points of $\overline{\mathbb{C}}$ counting multiplicities, notwithstanding one structural interpolation condition at ∞ where all the functions involved do vanish. Since $2n-k \geq 2n-2k$, it follows that $L^f_n(q)/q$ is a multipoint Padé approximant to f [2]. Of course, the difference with classical interpolation theory is that the interpolation points are not known in advance but rather depend on the interpolant (they must include the reflections of its poles across $\mathbb{T}$, each of them with multiplicity twice the multiplicity of that pole). This implicit determination of the interpolation points accounts for the nonlinearity of the problem and capsulizes its difficulty. In the particular case of best approximants, we obtain the following corollary that appears already in [19, 27, 17] (the first of these references deals with simple poles only).

COROLLARY 4.3. *Let $f \in \overline{H}^2_0$ (resp. $\overline{H}^2_{0,\mathbb{R}}$). If p/q is a solution to $\mathbf{P}(f,n)$ (resp. $\mathbf{P}_{\mathbb{R}}(f,n)$), then p/q interpolates f with order two at the reflections of its poles across the unit circle.*

Proof. If $f \in \mathcal{R}_{n-1,n}$ (resp. $f \in \mathcal{R}_{n-1,n,\mathbb{R}}$), then $p/q = f$ and there is nothing to prove. Otherwise, we know from Proposition (3.1) that a solution to $\mathbf{P}(f,n)$ (resp. $\mathbf{P}_{\mathbb{R}}(f,n)$) has exact degree n, so the result follows from the previous discussion where $k=0$. $\square$

Note that, in view of the remark after Proposition 3.1, the corollary is still valid for *local minima* in Problems $\mathbf{P}(f,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$, and more generally for any *irreducible* critical point, that is any critical point such that q and $L^f_n(q)$ are coprime. This was importantly used in [12] and [14, 13], where the interpolation properties of the critical points team up with somewhat *ad hoc* bootstrap propositions on the behaviour of the poles in order to obtain asymptotic error estimates for critical points in $\mathbf{P}'_{\mathbb{R}}(e^z,n)$ and $\mathbf{P}_{\mathbb{R}}(f,n)$ respectively, with f a Markov function. In [12] and [13], these estimates were used to prove asymptotic uniqueness of a critical point in the cases just mentioned, using Theorem 4.6 below, see the remark after that theorem.

In order to study the critical points of Ψ^f_n or $\Psi^f_{n,\mathbb{R}}$ using classical tools from differential topology (centering around the notion of topological degree), we need to compactify the domain of definition of these functions. This requires additional assumptions on f. The one below could be weakened but is convenient to work with and already general enough for many applications:

HYPOTHESIS (H): the function f is analytic in $|z| > 1-\varepsilon$ for some $\varepsilon > 0$.

LEMMA 4.4. *Let $f \in \overline{H}^2_0$ (resp. $\overline{H}^2_{0,\mathbb{R}}$) satisfy hypothesis* **(H)**. *Then Ψ^f_n (resp. $\Psi^f_{n,\mathbb{R}}$) extends smoothly to a neighborhood of $\overline{\mathcal{M}}_n$ (resp. $\overline{\mathcal{M}}_{n,\mathbb{R}}$) in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$).*

Proof. By Pythagora's theorem and the characteristic property of the orthogonal projection, we can write

$$\Psi^f_n(q) = \|f\|^2_{L^2} - \|L^f_n(q)/q\|^2_{L^2} = \|f\|^2_{L^2} - < L^f_n(q)/q, f > \tag{4.9}$$

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so that it is enough to smoothly extend $< L_n^f(q)/q, f >$. But using (2.4) we have that

$$< L_n^f(q)/q, f >= \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{L_n^f}{q}(\xi)\, f^\sigma(\xi)\, d\xi,$$

and since f^σ is holomorphic for $|z| < 1+\varepsilon$ by hypothesis **(H)** we may, by Cauchy's theorem, deform $\mathbb{T}$ into a slightly larger circle, say, of radius $1+\varepsilon/2$, without changing the value of the integral. The new expression is nonsingular as long as the zeros of q remain of modulus strictly less than $1+\varepsilon/2$, and this provides us with the desired extension of Ψ_n^f. The argument applies to $\Psi_{n,\mathbb{R}}^f$ as well. $\square$

When f satisfies hypothesis **(H)**, Lemma 4.4 allows us to define critical points of Ψ_n^f (resp. $\Psi_{n,\mathbb{R}}^f$) in $\overline{\mathcal{M}}_n$ (resp. $\overline{\mathcal{M}}_{n,\mathbb{R}}$) and not just in $\mathcal{M}_n$ (resp. $\mathcal{M}_{n,\mathbb{R}}$). In particular, it will make sense to talk about critical points lying on $\partial\mathcal{M}_n$ (resp. $\partial\mathcal{M}_{n,\mathbb{R}}$). For these, the characterization in Proposition 4.2 no longer holds, essentially because zeros of q with modulus 1 must cancel automatically with a zero of $L_n^f(q)$ (since the L^2-norm of $L_n^f(q)/q$ remains finite). Characterizing critical points lying on $\partial\mathcal{M}_n$ or $\partial\mathcal{M}_{n,\mathbb{R}}$ is a technical exercise that we can safely dispense with here. We simply state the result for completeness and refer the interested reader to [12] for an argument in the case of $\Psi_{n,\mathbb{R}}^f$ which is easily carried over to Ψ_n^f.

PROPOSITION 4.5. *Let $f \in \overline{H}_0^2$ (resp. $\overline{H}_{0,\mathbb{R}}^2$) and $q \in \partial\mathcal{M}_n$ (resp. $\partial\mathcal{M}_{n,\mathbb{R}}$). In view of Proposition 4.1, write the division of $f^\sigma\widetilde{q}$ by q in the form*

$$f^\sigma\widetilde{q} = vq + \widetilde{L}_n^f, \quad v \in H^2,\ L_n^f \in \mathcal{P}_{n-1}. \tag{4.10}$$

Decompose q into $q = q_1 u$ where $q_1 \in \mathcal{M}_{n-k}$ (resp. $q_1 \in \mathcal{M}_{n-k,\mathbb{R}}$) and u is a monic polynomial of degree $k \in \{1,\dots,n\}$, each root of which has modulus 1 (if $k = n$ then $q_1 = 1$). Let $\zeta_1,\dots,\zeta_\ell \in \mathbb{T}$ be the roots of u, and ν_j be the multiplicity of ζ_j so that $\sum_j \nu_j = k$. Form the polynomial

$$Q(z) = \Pi_{j=1}^{\ell}(z-\zeta_j)^{[(1+\nu_j)/2]},$$

where the brackets in $[(1+\nu_j)/2]$ indicate the integer part. Then, q is a critical point of Ψ_n^f (resp. $\Psi_{n,\mathbb{R}}^f$) if, and only if qQ divides $L_n^f v$ in H^2 (resp. in $H_{\mathbb{R}}^2$).

A critical point is said to be *nondegenerate* if the second derivative at that point is a nondegenerate quadratic form. In this case, the number of negative eigenvalues of the second derivative is called the *Morse index* of the critical point and it is invariant by smooth changes of coordinates. The theorem below lies a little too deep in differential topology for us to prove it here. It was established in [3] for the case of $\Psi_{n,\mathbb{R}}^f$ (see also [10]) and substantially outlined in ([4]) for the case of Ψ_n^f.

THEOREM 4.6. (The Index Theorem) *Let $f \in \overline{H}_0^2$ (resp. $\overline{H}_{0,\mathbb{R}}^2$) satisfy hypothesis* **(H)**, *and assume that Ψ_n^f (resp. $\Psi_{n,\mathbb{R}}^f$) has only nondegenerate critical points $\overline{\mathcal{M}}_n$ (resp. $\overline{\mathcal{M}}_{n,\mathbb{R}}$) none of which lies on $\partial\mathcal{M}_n$ (resp. $\partial\mathcal{M}_{n,\mathbb{R}}$). Let $\mathcal{C}$ be the collection of these critical points and $\varepsilon(q)$ designate the Morse index of $q \in \mathcal{C}$. Then $\mathcal{C}$ is finite and*

$$\sum_{q\in\mathcal{C}}(-1)^{\varepsilon(q)} = 1.$$

REMARK 2. *The nondegeneracy of all critical points is a generic (i.e. open and dense) property with respect to f in various function spaces, see [3] for a discussion of $\Psi_{n,\mathbb{R}}^f$ on*

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trigonometric polynomials of sufficiently large degree or on functions analytic in $|z| > 1 - \varepsilon$ endowed with the topology of uniform convergence on compact sets. One can in fact prove that critical points on $\partial\mathcal{M}_n$ (resp. $\partial\mathcal{M}_{n,\mathbb{R}}$) are necessarily degenerate, so that the hypotheses we made in the above theorem are somewhat redundant, but this is of no importance to us.

The index theorem provides us with a criterion for uniqueness of a critical point: if no critical point lies on $\partial\mathcal{M}_n$ (resp. $\partial\mathcal{M}_{n,\mathbb{R}}$) and if each of them is nondegenerate with even Morse index –or, equivalently, if the second derivative at each of them has strictly positive determinant – then there can be only one. This is what we use in the forthcoming section.

5. A uniqueness theorem in degree 1.

This section, which constitutes the truly original contribution of the paper, is devoted to proving the following theorem.

THEOREM 5.1. *Let $f \in \overline{H}^2_0$ (resp. $\overline{H}^2_{0,\mathbb{R}}$) satisfy hypothesis* **(H)**. *Assume that f^σ has no zero on $\overline{\mathbb{D}}$ and that $|(f^\sigma)'/f^\sigma| \leq 1$ on $\mathbb{T}$, where the superscript "prime" indicates the derivative. Then Ψ_1^f (resp. $\Psi_{1,\mathbb{R}}^f$) has a unique critical point in $\mathcal{M}_1$ (resp. $\mathcal{M}_{1,\mathbb{R}}$). In particular, $\mathbf{P}(f,1)$ (resp. $\mathbf{P}_{\mathbb{R}}(f,1)$) has a unique solution which is also the unique local minimizer of (3.1) (resp. (3.2)) in degree 1.*

Proof. Let us consider Ψ_1^f and put for simplicity $g = f^\sigma$. For $q \in \mathcal{M}_1$, write $q(z) = z - a$ with $a \in \mathbb{D}$, $a = x + iy$; then $\widetilde{q}(z) = 1 - \overline{a}z$, and from Proposition 4.1 we deduce that

$$L_1^f = \overline{g(a)}(1 - |a|^2) \tag{5.1}$$

(note that L_1^f is indeed a complex number since it is a polynomial of degree 0). Subsequently, as it is immediate from (2.4) and the residue formula that

$$\left\| \frac{1}{z-a} \right\|_{L^2}^2 = < \frac{1}{z-a}, \frac{1}{z-a} > = (1 - |a|^2)^{-1},$$

we see from (5.1) and the first equality in (4.9) that

$$\Psi_1^f(q) = \|f\|_{L^2}^2 - |g(a)|^2(1 - |a|^2). \tag{5.2}$$

As predicted by Lemma 4.4, this formula extends smoothly to $q \in \overline{\mathcal{M}_1}$ (that is: to $a \in \overline{\mathbb{D}}$) if f satisfies hypothesis **(H)**.

Consider the differential operators

$$\partial_a = \frac{1}{2}\left(\partial/\partial x - i\partial/\partial y\right), \quad \partial_{\overline{a}} = \frac{1}{2}\left(\partial/\partial x + i\partial/\partial y\right),$$

having the property that $\partial_a\,\overline{a} = \partial_{\overline{a}}\,a = 0$, and observe from (5.2), since $\partial_a \overline{g(a)} = 0$ by the holomorphy of g, that

$$\partial_a \Psi_1^f = -\overline{g(a)}\Big(g'(a)(1 - |a|^2) - g(a)\overline{a}\Big). \tag{5.3}$$

Because Ψ_1^f is real-valued, a is critical if, and only if $\partial_a \Psi_1^f = 0$, because the real and imaginary parts of $\partial_a \Psi_1^f$ are respectively half and minus half of the components of the gradient of Ψ_1^f in the coordinates x, y. Thus, as g has no zero on $\overline{\mathbb{D}}$ by assumption, we deduce from (5.3) that a is critical if, and only if

$$g'(a)(1 - |a|^2) - g(a)\overline{a} = 0. \tag{5.4}$$

In particular a cannot be critical if $|a|=1$, that is to say no critical point lies on $\partial\mathcal{M}_1$. Next, it is clear that

$$\begin{aligned} 2\frac{\partial}{\partial x}\partial_a &= \frac{\partial^2}{\partial x^2} - i\frac{\partial^2}{\partial x\partial y},\\ 2\frac{\partial}{\partial y}\partial_a &= \frac{\partial^2}{\partial x\partial y} - i\frac{\partial^2}{\partial y^2},\end{aligned}$$

hence the determinant of the second derivative of the *real-valued* function Ψ_1^f at q – computed in the coordinates x, y– is given by

$$\frac{\partial^2\Psi_1^f}{\partial x^2}\frac{\partial^2\Psi_1^f}{\partial y^2} - \left(\frac{\partial^2\Psi_1^f}{\partial x\partial y}\right)^2 = 4\mathbf{Im}\left(\frac{\partial}{\partial x}\partial_a\Psi_1^f\ \overline{\frac{\partial}{\partial y}\partial_a\Psi_1^f}\right). \tag{5.5}$$

Now, assume that a is critical so that (5.4) holds, or equivalently:

$$\frac{g'(a)}{g(a)} - \frac{\overline{a}}{(1-|a|^2)} = 0. \tag{5.6}$$

Let us put for simplicity $F(a) = g'(a)/g(a)$. In view of (5.3) and (5.6), we compute

$$\begin{aligned} \frac{\partial}{\partial x}\partial_a\Psi_1^f &= (\partial_a + \partial_{\overline{a}})\ \partial_a\,\Psi_1^f\\ &= -\big(\partial_a + \partial_{\overline{a}}\big)\ \left(|g(a)|^2(1-|a|^2)\left(F(a) - \frac{\overline{a}}{(1-|a|^2)}\right)\right)\\ &= -|g(a)|^2(1-|a|^2)\left(F'(a) - \frac{\overline{a}^2}{(1-|a|^2)^2} - \frac{1}{(1-|a|^2)^2}\right)\\ &= -|g(a)|^2(1-|a|^2)\left(F'(a) - F^2(a) - \frac{1}{(1-|a|^2)^2}\right)\end{aligned} \tag{5.7}$$

where we have used in the next-to-last equality that $\partial\overline{a}\,F(a) = 0$ by holomorphy. Similarly, we get that

$$\begin{aligned} \frac{\partial}{\partial y}\partial_a\Psi_1^f &= i(\partial_a - \partial_{\overline{a}})\ \partial_a\,\Psi_1^f\\ &= -i\big(\partial_a - \partial_{\overline{a}}\big)\ \left(|g(a)|^2(1-|a|^2)\left(F(a) - \frac{\overline{a}}{(1-|a|^2)}\right)\right)\\ &= -i|g(a)|^2(1-|a|^2)\left(F'(a) - F^2(a) + \frac{1}{(1-|a|^2)^2}\right).\end{aligned} \tag{5.8}$$

Therefore, from (5.7), (5.8), and (5.5), we see that the determinant of the second derivative of Ψ_1^f at the critical point $q(z) = z - a$ is equal to

$$4|g(a)|^4(1-|a|^2)^2\left(\frac{1}{(1-|a|^2)^4} - |F'(a) - F^2(a)|^2\right)$$

whose strict positivity is equivalent to

$$\frac{1}{(1-|a|^2)^2} > |F'(a) - F^2(a)|. \tag{5.9}$$

By assumption $|F| \le 1$ on $\mathbb{T}$ hence also on $\mathbb{D}$ by the maximum principle. Hence by the Schwarz-Pick lemma (see *e.g.* [1]) it holds that

$$|F'(a)| \le \frac{1-|F(a)|^2}{1-|a|^2}.$$

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Consequently

$$|F'(a) - F^2(a)| \leq \frac{1-|F(a)|^2}{1-|a|^2} + |F(a)|^2 = \frac{1-|aF(a)|^2}{1-|a|^2},$$

so that clearly

$$(1-|a|^2)^2|F'(a) - F^2(a)| \leq (1-|aF(a)|^2)(1-|a|^2) < 1$$

which implies (5.9). The result now follows from the index theorem. The case of $\Psi^f_{1,\mathbb{R}}$ is similar but simpler, since then $a \in [-1,1]$ hence the computation becomes 1-dimensional. $\square$

In [12], it was shown that $\mathbf{P'}_{\mathbb{R}}(e^z, n)$ has a unique critical point for n large enough (no estimate on how large is available). To estimate the index of a critical point, [12] makes use of error estimates in interpolation to e^z at a conjugate-symmetric set of points that were previously given in [11]. The case of $\mathbf{P'}(e^z, n)$ has apparently not been studied so far, although the same result could be obtained by the same technique using instead of [11] the results of [34] that relax the assumption of conjugate-symmetry. From Theorem 5.1, we deduce one more (modest) piece of information:

COROLLARY 5.2. *Problems $\mathbf{P'}(e^z, 1)$ and $\mathbf{P'}_{\mathbb{R}}(e^z, 1)$ have a unique critical point, hence a unique local minimum which is also, of necessity, the solution to the problem.*

Proof. It is equivalent to prove uniqueness of a critical point in Problems $\mathbf{P}(e^{1/z}/z, 1)$ and $\mathbf{P}_{\mathbb{R}}(e^{1/z}/z, 1)$. But if $f(z) = e^{1/z}/z$ then $f^\sigma(z) = e^z$, and since the exponential is its own derivative and has no zeros we can apply the theorem. $\square$

6. Conclusion. In this paper, we have shown that uniqueness in L^2-rational approximation of order 1 on the unit circle, to a function holomorphic in the complement of a compact subset of the open unit disk, will hold if only the reflected function has no zeros in the closed disk and has a logarithmic derivative which is bounded by 1 in modulus there. Although extremely limited in scope since it only deals with degree 1, this criterion is interesting because it can be checked explicitly from the function and qualifies an *open* subset of the class of holomorphic functions in $|z| > r$ (for some fixed r such that $0 < r < 1$) endowed with the topology of uniform convergence on compact sets. These features make the present criterion unique. It is to be hoped that suitable refinements of the estimates of the present paper will enable one to address the problem in higher degree.

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