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ON CONVERGENCE OF ORTHONORMAL EXPANSIONS FOR EXPONENTIAL WEIGHTS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Let $I = (-d, d)$ be a real interval, finite or infinite, and let $W : I \to (0, \infty)$. Assume that W^2 is a weight, so that we may define orthonormal polynomials corresponding to W^2. For $f : I \to \mathbb{R}$, let $s_m[f]$ denote the mth partial sum of the orthonormal expansion of f with respect to these polynomials. We show that if $f'W \in L_\infty(I) \cap L_2(I)$, then $\|(s_m[f] - f) W\|_{L_\infty(I)} \to 0$ as $m \to \infty$. The class of weights considered includes even exponential weights.

Key words. orthonormal polynomials, de la Vallée Poussin means

AMS subject classifications. 65N12, 65F35, 65J20, 65N55

1. Introduction and Results. Let $I = (-d, d)$ be a real interval, finite or infinite. Let $W : I \to (0, \infty)$ be such that all the power moments

$$\int_I |x|^n W^2(x)\, dx, \qquad n \geq 0,$$

are finite. Then we may define orthonormal polynomials

$$p_n\left(W^2, x\right) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,$$

$n \geq 0$, satisfying for every m, n,

$$\int_I p_m\left(W^2, x\right) p_n\left(W^2, x\right) W^2(x)\, dx = \delta_{mn}.$$

For $f : I \to \mathbb{R}$ such that $f(x) x^j W^2(x) \in L_1(I)$, $j \geq 0$, we may form the formal orthonormal expansion

$$f \sim \sum_{j=0}^{\infty} b_j p_j,$$

where

$$b_j := b_j(f) := \int_I f p_j W^2, \qquad j \geq 0. \tag{1.1}$$

The mth partial sum of this expansion is denoted by

$$s_m[f] := \sum_{j=0}^{m-1} b_j p_j, \qquad m \geq 1.$$

Using (1.1), we obtain the integral representation

$$s_m[f](x) = \int_I f(t) K_m(x, t) W^2(t)\, dt, \tag{1.2}$$

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where it is known that the Christoffel-Darboux kernel K_m can be expressed as

$$K_m(x,t) := \sum_{k=0}^{m-1} p_k(x)\, p_k(t) = \frac{\gamma_{m-1}}{\gamma_m} \frac{p_m(x)\, p_{m-1}(t) - p_m(t)\, p_{m-1}(x)}{x-t}. \tag{1.3}$$

We define the dilated de la Vallée Poussin means by

$$v_n[f](x) := \frac{1}{n} \sum_{m=n+1}^{2n} s_m[f](x). \tag{1.4}$$

A result in [7] (see Theorem 9.1.1) asserts that for a class of Freud weights,

$$\lim_{m\to\infty} \|(f - s_m[f])\, W\|_{L_\infty(I)} = 0, \tag{1.5}$$

provided f is absolutely continuous and $f'W \in L_1(I)$.

In this paper, we generalise this result for a class $\mathcal{F}(C^2)$ of even exponential weights. The definition of this class involves the notion of quasi-increasing and quasi-decreasing. We say that $f : (0,d) \to \mathbb{R}$ is *quasi-increasing* if there exists $C > 0$ such that

$$0 < x < y < d \Rightarrow f(x) \le C f(y).$$

In particular, an increasing function is quasi-increasing. Similarly, we may define the notion of a quasi-decreasing function.

DEFINITION 1.1 (The class of weights $\mathcal{F}(C^2)$). Let $W = e^{-Q}$, where $Q : I \to [0,\infty)$satisfies the following properties:
(a) Q is even and continuous, Q' is continuous in $I = (-d,d)$, and $Q(0) = 0$;
(b) Q'' exists in $I\backslash\{0\}$ and $Q'' \ge 0$ in $I\backslash\{0\}$;
(c)

$$\lim_{t\to d-} Q(t) = \infty;$$

(d) The function

$$T(t) = \frac{tQ'(t)}{Q(t)}, \qquad t \in I\backslash\{0\},$$

is quasi-increasing in $(0,d)$, and for some $\Lambda > 1$,

$$T(t) \ge \Lambda > 1, \qquad t \in I\backslash\{0\};$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{|Q'(x)|}{Q(x)}, \qquad x \in I\backslash\{0\}.$$

Then we write $W \in \mathcal{F}(C^2)$. If there exists a compact subinterval J of I and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \ge C_2 \frac{|Q'(x)|}{Q(x)}, \qquad x \in I\backslash J,$$

then we write $W \in \mathcal{F}\left(C^2+\right)$.

Examples of this weight include the following:

Freud Weights. Assume that $Q' > 0$ in $(0, \infty)$ and that for some $C_1, C_2 > 1$,

$$C_1 \leq T(t) \leq C_2, \qquad t \in (0, \infty).$$

Then W is a Freud weight. For example, if $\alpha > 1$, and

$$W(x) = W_\alpha(x) = \exp\left(-|x|^\alpha\right),$$

then $T(t) = \alpha$ for all t.

Erdös Weights. Here $I = (-\infty, \infty)$ and $T(t) \to \infty$ as $t \to \infty$. The archetypal example is

$$W(x) = \exp\left(\exp_k(0) - \exp_k\left(|x|^\alpha\right)\right) \tag{1.6}$$

where $\alpha > 1$, $k \geq 1$, and

$$\exp_k = \underbrace{\exp\left(\exp\left(\cdots \exp\left(\right) \cdots\right)\right)}_{k \text{ times}}$$

denotes the kth iterated exponential. We also set $\exp_0(x) = x$.

Exponential Weights on $(-1, 1)$. Here $I = (-1, 1)$ and $T(t) \to \infty$ as $t \to 1-$. The archetypal examples are

$$W(x) = \exp\left(1 - \left(1 - x^2\right)^{-\alpha}\right)$$

and

$$W(x) = W^{k,\alpha}(x) = \exp\left(\exp_k(1) - \exp_k\left(1 - x^2\right)^{-\alpha}\right), \qquad x \in (-1, 1) \tag{1.7}$$

where $k \geq 1$, $\alpha > 0$.

In analysis of exponential weights, the Mhaskar-Rakhmanov-Saff number a_n, plays a crucial role. It is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1 - t^2}}\, dt. \tag{1.8}$$

One of its properties is the Mhaskar-Saff identity

$$\|PW\|_{L_\infty(I)} = \|PW\|_{L_\infty(-a_n, a_n)},$$

valid for all polynomials P of degree $\leq n$. We shall need a number of auxiliary quantities. We set

$$\eta_n = \left(nT(a_n)\right)^{-2/3}, \qquad n \geq 1, \tag{1.9}$$

and define the functions

$$\phi_n(x) = \begin{cases} \dfrac{a_n\left|1 - \frac{|x|}{a_{2n}}\right|}{n\sqrt{\left|1 - \frac{|x|}{a_n}\right| + \eta_n}}, & |x| \leq a_n \\ \phi_n(a_n), & |x| > a_n \end{cases} \tag{1.10}$$

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and

$$\Psi_n = \max\left\{\left(\frac{n}{a_n}\phi_n\right)^{1/2}, \left(\frac{n}{a_n}\phi_n\right)^{2/3}\right\}. \tag{1.11}$$

THEOREM 1.2. *Let $W \in \mathcal{F}(C^2)$. Let $f : I \to \mathbb{R}$ be absolutely continuous, let $f'W \in L_\infty(I) \cap L_2(I)$and assume that for each $\varepsilon > 0$,*

$$a_n = O(n^\varepsilon) \quad \text{and} \quad T(a_n) = O(n^\varepsilon) \tag{1.12}$$

Then

$$\lim_{n\to\infty} \|W(f - s_n[f])\|_{L_\infty(I)} = 0. \tag{1.13}$$

Note that the assumption (1.12) is satisfied by the Erdös weights in (1.6) and the exponential weights on $(-1, 1)$ in (1.7). A key ingredient of Theorem 1.2 is a Favard type inequality. For $1 \le p \le \infty$, let

$$E_{n,p}[f]_W := \inf_{\deg(P)\le n} \|(f - P)W\|_{L_p(I)}.$$

This is the error in approximation of f by polynomials of degree $\le n$ in a weighted L_p norm.

THEOREM 1.3. *Let $W \in \mathcal{F}(C^2)$ and $1 \le p \le \infty$. Let $f : I \to \mathbb{R}$ be absolutely continuous, with $f'W \in L_p(I)$. Then*

$$\begin{aligned} E_{2n,p}[f]_W &\le \|W(f - v_n[f])\|_{L_p(I)} \\ &\le C\|f'W\|_{L_p(I)} \frac{a_n}{n} T(a_n)^{\frac{2}{3}-\frac{1}{3p}} \left[1 + \left(\frac{n}{T(a_n)^2}\right)^{4/9}\right]^{1-1/p}. \end{aligned} \tag{1.14}$$

This paper is organised as follows: in Section 2, we record some of the properties of the de la Vallée Poussin means and recall the Nikolskii-type inequality in [3]. In Section 3, we prove Theorems 1.2 and 1.3.

We close this section with more notation. Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, f and polynomials P of degree $\le n$. The same symbol does not necessarily denote the same constant in different occurrences. We denote the set of all polynomials of degree $\le n$ by $\mathcal{P}_n$. If (c_n) and (d_n) are sequences of real numbers, we write $c_n \sim d_n$ if there exist $C_1, C_2 > 0$ such that

$$C_1 \le c_n/d_n \le C_2, \qquad n \ge 1.$$

Similar notation is used for functions and sequences of functions.

2. Technical Estimates. For simplicity, we assume that $W \in \mathcal{F}(C^2)$, although the results hold more generally. The following proposition lists some of the properties of the linear operators v_n.

PROPOSITION 2.1. *Let $n \ge 1$, $1 \le p \le \infty$ and p' be determined by $\frac{1}{p} + \frac{1}{p'} = 1$.*
(a) For P of degree $\le n$,

$$v_n[P] = P. \tag{2.1}$$

(b) If $fW \in L_p(I)$ and $gW \in L_{p'}(I)$, then

$$\int_I v_n[g]\, fW^2 = \int_I v_n[f]\, gW^2. \tag{2.2}$$

Proof. See Proposition 3.4.1 in [7, p. 71]. □

Next, we record a Nikolskii-type inequality:

LEMMA 2.2. *Let $0 < q < p < \infty$. Then there exists $C > 0$ such that for $n \geq 1$ and $P \in \mathcal{P}_n$,*

$$\|PW\|_{L_p(I)} \leq C \left[\frac{n\sqrt{T(a_n)}}{a_n} \right]^{\frac{1}{q}-\frac{1}{p}} \|PW\|_{L_q(I)}. \tag{2.3}$$

Proof. See [3, Theorem 10.3, p. 295]. □

Next, we present an estimate for the error in weighted L_1 approximation by weighted polynomials. This involves the characteristic function χ_x of the interval $(-\infty, x)$:

$$\chi_x(t) = \chi_{(-\infty,x)}(t).$$

LEMMA 2.3. *There exist $C_2 > 0$ and $0 < C_1 < 1$ such that for $n \geq 1$, and $x \in I$,*

$$E_{n,1}[\chi_x]_W \leq C_3 \frac{a_n}{n} W(x). \tag{2.4}$$

Proof. This follows using classical results on Markov-Stieltjes inequalities. Let $x \in (x_{k+1,n}, x_{kn}]$, where $x_{k+1,n}$ and x_{kn} are successive zeros of the nth orthonormal polynomial $p_n(x)$ for the weight W. By Corollary 1.2.6 in [7, p. 17], there exist, for the given x, polynomials R and P of degree $\leq 2n$ such that

$$R \leq \chi_x \leq P \text{ in } I$$

and

$$\int_I [P - R]\, W \leq \lambda_{k+1,n} + \lambda_{k,n},$$

where $\lambda_{k,n}$ is the Christoffel number corresponding to x_{kn}, or equivalently, if $\lambda_n(W, x)$ denotes the nth Christoffel function for W

$$\lambda_{k,n} = \lambda_n(W, x_{kn}).$$

Using the bounds for Christoffel functions in [3, Corollary 1.14, p. 20], and using (12.20) in [3, p. 329], we deduce that

$$\lambda_{k+1,n} W^{-1}(x_{k+1,n}) + \lambda_{k,n} W^{-1}(x_{kn}) \leq C\varphi_n(x)$$

provided $x \in [-a_{2n}, a_{2n}]$. (Here one also uses the relationship between Mhaskar-Rakhmanov-Saff numbers for W and W^2.) Now if in addition $|x| \leq a_{n/2}$, then uniformly in x, n, k,

$$W(x_{kn}) \sim W(x_{k+1,n}).$$

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Indeed, for some ξ between $x_{kn}, x_{k+1,n}$, at least if $0 \notin [x_{k+1,n}, x_{kn}]$,

$$\begin{aligned} |Q(x_{kn}) - Q(x_{k+1,n})| &= |Q'(\xi)|(x_{kn} - x_{k+1,n}) \\ &\leq C|Q'(x_{kn})|\phi_n(x_{kn}) \\ &\leq C\frac{a_n}{n}|Q'(x_{kn})|\sqrt{1-\frac{|x_{kn}|}{a_n}} \leq C, \end{aligned}$$

see [3, (3.41), p. 77] and [3, (1.110), p. 23]. We deduce that

$$E_{2n,1}[\chi_x]_W \leq C\varphi_n(x) W(x), \qquad |x| \leq a_{n/2}.$$

But for this range of x,

$$\varphi_n(x) \sim \frac{a_n}{n}\sqrt{1-\frac{|x|}{a_n}} \leq C\frac{a_n}{n},$$

so

$$E_{2n,1}[\chi_x]_W \leq C\frac{a_n}{n} W(x), \qquad |x| \leq a_{n/2}.$$

Then

$$E_{n,1}[\chi_x]_W \leq C\frac{a_n}{n} W(x), \qquad |x| \leq a_{n/4}$$

For $x > a_{n/4}$, we use the estimate

$$\begin{aligned} E_{n,1}[\chi_x]_W &\leq \int_I |\chi_x - 1| W = \int_x^d W \\ &\leq \frac{1}{Q'(x)}\int_x^d WQ' = \frac{W(x)}{Q'(x)}, \end{aligned}$$

as Q' is increasing. The case $x < -a_{n/4}$ is similar. Finally, from the convexity of Q, for $|x| \geq a_{n/4}$,

$$|Q'(x)| \geq Q'\left(a_{n/4}\right)$$

and by (3.40) in [3, p. 77], and as $C_1 < 1$,

$$\left|Q'\left(a_{n/4}\right)\right| \sim \frac{n}{a_n}\sqrt{T(a_n)} \geq C\frac{n}{a_n}. \qquad \square$$

LEMMA 2.4. *Let $Wh \in L_1(I)$ and*

$$K(h,t) := W^{-2}(t)\int_t^d W^2(u)h(u)\,du, \qquad t \in I. \tag{2.5}$$

Let $n \geq 1, 1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.
(a) Let $Wh \in L_p(I)$ and

$$\int_I W^2 h = 0. \tag{2.6}$$

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Then for some C independent of h,

$$\|WK'(h)\|_{L_p(I)} \leq C \|Wh\|_{L_p(I)} . \tag{2.7}$$

(b) Moreover, if g is absolutely continuous, and $g'W \in L_p(I)$, then

$$\int ghW^2 = \int g'K(h,\cdot)W^2. \tag{2.8}$$

(c) If $Wh \in L_\infty(I)$, $n \geq 1$is an integer, and

$$\int W^2hP = 0, \qquad P \in \mathcal{P}_n, \tag{2.9}$$

then with C_1 as in the previous lemma,

$$\|WK(h,\cdot)\|_{L_\infty(I)} \leq C\frac{a_n}{n} \|Wh\|_{L_\infty(I)} . \tag{2.10}$$

Proof. This is very similar to that in [7, Lemma 4.1.4, p. 84 ff.].
(a) This is actually proved in a more general setting in [2, Lemma 2.2].
(b) This follows by an integration by parts.
(c) Now if P is a polynomial of degree $\leq n$,

$$\begin{aligned}
\left|W^2(t)K(h,t)\right| &= \left|\int_t^d W^2(x)h(x)\,dx\right| \\
&= \left|\int_{-d}^t W^2(x)h(x)\,dx\right| \\
&= \left|\int_I \chi_t(x)W^2(x)h(x)\,dx\right| \\
&= \left|\int_I [\chi_t(x) - P(x)]W^2(x)h(x)\,dx\right| \\
&\leq \|Wh\|_{L_\infty(I)} \int_I |\chi_t(x) - P(x)|\,W(x)\,dx.
\end{aligned}$$

As P is any such polynomial, we obtain

$$\left|W^2(t)K(h,t)\right| \leq \|Wh\|_{L_\infty(I)} E_{n,1}[\chi_t]_W .$$

Now apply the previous lemma, giving

$$\left|W^2(t)K(h,t)\right| \leq \|Wh\|_{L_\infty(I)} CW(t)\frac{a_n}{n}. \qquad \square$$

LEMMA 2.5. *For $1 \leq p \leq \infty$, there exists C independent of n and f such that*

$$\left\|v_n[f]W\Psi_n^{1-\frac{1}{p}}\right\|_{L_p(I)} \leq C\left\|fW\Psi_n^{-\frac{1}{p}}\right\|_{L_p(I)} . \tag{2.11}$$

Proof. This follows directly from Theorem 1.2 in [5, p. 390]. $\square$

We let

$$A_n := \|\max\{\Psi_n, 1\}\|_{L_\infty(I)} \left\|\Psi_n^{-1}\right\|_{L_\infty(I)} . \tag{2.12}$$

LEMMA 2.6. *(a) For $n \geq 1$ and $x \in I$,*

$$C_1 T\left(a_n\right)^{-1/2} \leq \frac{n}{a_n}\phi_n\left(x\right) \leq C_2\left[1+\left(\frac{n}{T\left(a_n\right)^2}\right)^{1/3}\right]. \tag{2.13}$$

(b) For $n \geq 1$ and $x \in I$,

$$C_1 T\left(a_n\right)^{-1/3} \leq \Psi_n\left(x\right) \leq C_2\left[1+\left(\frac{n}{T\left(a_n\right)^2}\right)^{2/9}\right]. \tag{2.14}$$

(c)

$$A_n \leq C T\left(a_n\right)^{1/3}\left[1+\left(\frac{n}{T\left(a_n\right)^2}\right)^{2/9}\right]. \tag{2.15}$$

Proof. (a) For $|x| \leq a_n$,

$$\frac{n}{a_n}\phi_n\left(x\right) = \frac{\left|1-\frac{|x|}{a_{2n}}\right|}{\sqrt{\left|1-\frac{|x|}{a_n}\right|+\eta_n}} \leq \frac{\left|1-\frac{|x|}{a_n}\right|+\frac{|x|}{a_n}\left(1-\frac{a_n}{a_{2n}}\right)}{\sqrt{\left|1-\frac{|x|}{a_n}\right|+\eta_n}}$$

$$\leq \left|1-\frac{|x|}{a_n}\right|^{1/2}+\frac{1-\frac{a_n}{a_{2n}}}{\sqrt{\eta_n}} \leq 1+\frac{C}{T\left(a_n\right)\sqrt{\eta_n}} = 1+\frac{Cn^{1/3}}{T\left(a_n\right)^{2/3}},$$

by definition of η_n. In the third last line we used the estimate [3, (3.50), p. 81],

$$1-\frac{a_n}{a_{2n}} \sim \frac{1}{T\left(a_n\right)}.$$

For the lower bound, we see that if $|x| \leq a_{n/2}$, then

$$\frac{n}{a_n}\phi_n\left(x\right) \geq \frac{\left|1-\frac{a_{n/2}}{a_{2n}}\right|}{\sqrt{\left|1-\frac{a_{n/2}}{a_n}\right|+\eta_n}} \sim \sqrt{1-\frac{a_{n/2}}{a_n}} \sim \frac{1}{\sqrt{T\left(a_n\right)}}.$$

If $a_{n/2} \leq |x| \leq a_n$, then

$$\frac{n}{a_n}\phi_n\left(x\right) \geq \frac{\left|1-\frac{a_n}{a_{2n}}\right|}{\sqrt{\left|1-\frac{a_{n/2}}{a_n}\right|+\eta_n}} \sim \frac{1}{\sqrt{T\left(a_n\right)}},$$

as $\eta_n \ll \frac{1}{T(a_n)}$.
(b) This follows easily from (a) and the definition

$$\Psi_n = \max\left\{\left(\frac{n}{a_n}\phi_n\right)^{1/2}, \left(\frac{n}{a_n}\phi_n\right)^{2/3}\right\}.$$

(c) This follows from (b). $\square$

3. Proof of the Theorems. In this section, we prove Theorem 1.3, but first we need two lemmas. We set

$$\Gamma_{n,p} = \Psi_n^{-1/p} \max\{1, \Psi_n\}. \tag{3.1}$$

LEMMA 3.1. *Let $1 \le p \le \infty$ and $fW \in L_p(I)$. Then for $n \ge 1$,*

$$\left\| W\left(f - v_n[f]\right) \Psi_n^{1-\frac{1}{p}} \right\|_{L_p(I)} \le C E_{n,p}[f]_{W\Gamma_{n,p}}. \tag{3.2}$$

Here C is independent of n and f.

Proof. Let P^* be the polynomial of degree $\le n$ of best approximation to f in the weighted norm L_p norm with weight $W \max \Gamma_{n,p}$. Lemma 2.5 gives

$$\begin{aligned}
\left\| W\left(f - v_n[f]\right) \Psi_n^{1-\frac{1}{p}} \right\|_{L_p(I)} &\le \left\| W\left(f - P^*\right) \Psi_n^{1-\frac{1}{p}} \right\|_{L_p(I)} \\
&\quad + \left\| W\left(P^* - v_n[f]\right) \Psi_n^{1-\frac{1}{p}} \right\|_{L_p(I)} \\
&= \left\| W\left(f - P^*\right) \Psi_n^{1-\frac{1}{p}} \right\|_{L_p(I)} \\
&\quad + \left\| W v_n\left[P^* - f\right] \Psi_n^{1-\frac{1}{p}} \right\|_{L_p(I)} \\
&\le \left\| W\left(f - P^*\right) \Psi_n^{1-\frac{1}{p}} \right\|_{L_p(I)} \\
&\quad + C \left\| \left(P^* - f\right) W \Psi_n^{-\frac{1}{p}} \right\|_{L_p(I)} \\
&\le (C+1) \left\| W\left(f - P^*\right) \Gamma_{n,p} \right\|_{L_p(I)}.
\end{aligned}$$

Our choice of P_n^* gives the result. $\square$

LEMMA 3.2. *Let $n \ge 1$. Let g be absolutely continuous and $g'W \in L_1(I)$. Then*

(a)

$$E_{n,1}[g]_W \le C \frac{a_n}{n} \left\| g'W \right\|_{L_1(I)}. \tag{3.3}$$

(b)

$$\left\| W\left(g - v_n[g]\right) \right\|_{L_1(I)} \le C \left\| g'W \right\|_{L_1(I)} \frac{a_n}{n} T(a_n)^{1/3}. \tag{3.4}$$

Proof. (a) If h is a function such that $hW \in L_\infty(I)$ and h satisfies the orthogonality condition (2.9), then also (2.6) is satisfied, and

$$\begin{aligned}
\left| \int g h W^2 \right| = \left| \int_I g' K(h) W^2 \right| &\le \left\| g'W \right\|_{L_1(I)} \left\| K(h) W \right\|_{L_\infty(I)} \\
&\le \left\| g'W \right\|_{L_1(I)} C \frac{a_n}{n} \left\| hW \right\|_{L_\infty(I)}.
\end{aligned}$$

Taking the sup over all such h gives the result.
(b) Here Lemma 3.1 and (a) give

$$\begin{aligned}\|W(g-v_n[g])\|_{L_1(I)} &\le CE_{n,1}[g]_{W\Gamma_{n,1}}\\ &\le C\frac{a_n}{n}\|g'W\|_{L_1(I)}\left\|\max\left\{1,\Psi_n^{-1}\right\}\right\|_{L_\infty(I)}.\end{aligned}$$

Now apply the lower bound for Ψ_n in Lemma 2.6. ☐

LEMMA 3.3. *Let $n\ge 1$. Let g be absolutely continuous and $g'W\in L_\infty(I)$.*
(a) Then for $n\ge 1$,

$$E_{2n,\infty}[g]_W\le\|g'W\|_{L_\infty(I)}C\frac{a_n}{n}T(a_n)^{1/3}\left[1+\left(\frac{n}{T(a_n)^2}\right)^{2/9}\right]. \tag{3.5}$$

(b)

$$\|W(g-v_n[g])\|_{L_\infty(I)}\le\|g'W\|_{L_\infty(I)}C\frac{a_n}{n}T(a_n)^{2/3}\left[1+\left(\frac{n}{T(a_n)^2}\right)^{4/9}\right]. \tag{3.6}$$

Proof. (a) This follows that in [7, pp. 88–89]. We may assume that $g(0)=0$. Let

$$G(x)=\int_0^x\left[g'(t)-v_n[g'](t)\right]dt.$$

Choose a constant a such that

$$\|W(G-a)\|_{L_p(I)}=E_{0,p}[G]_W.$$

Then

$$V_n(x):=a+\int_0^x v_n[g]'$$

satisfies

$$(W(g-V_n))(x)=W(x)\left[\int_0^x(g-V_n)'-a\right]=W(G-a)(x)$$

so

$$\begin{aligned}E_{2n,\infty}[g]_W\le\|W(g-V_n)\|_{L_\infty(I)}&=\|W(G-a)\|_{L_\infty(I)}\\ =E_{0,\infty}[G]_W&=\left|\int_I GhW^2\right|\end{aligned}$$

where $\|hW\|_{L_1(I)}=1$ and $\int_I hW^2=0$, and we have used duality. Using (2.8), we continue this as

$$\begin{aligned}&=\left|\int_I G'(t)K(h,t)W^2(t)\,dt\right|\\ &=\left|\int_I(g'-v_n[g'])(t)K(h,t)W^2(t)\right|\\ &=\left|\int_I(g'-v_n[g'])(t)(K(h,t)-P(t))W^2(t)\right|,\end{aligned}$$

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for any polynomial P of degree $\leq n$, by orthogonality of $g' - v_n[g']$ to polynomials of degree $\leq n$. We continue this using Hölder's inequality, and by taking the inf over P, as

$$\begin{aligned}
&\leq \|(g' - v_n[g'])\, W\Psi_n\|_{L_\infty(I)}\, E_{n,1}[K(h)]_W \left\|\Psi_n^{-1}\right\|_{L_\infty(I)} \\
&\leq E_{n,\infty}[g']_{W\Gamma_{n,\infty}}\, E_{n,1}[K(h)]_W \left\|\Psi_n^{-1}\right\|_{L_\infty(I)} \\
&\leq E_{n,\infty}[g']_W\, C\frac{a_n}{n} \|K'(h)\, W\|_{L_1(I)} \|\Gamma_{n,\infty}\|_{L_\infty(I)} \left\|\Psi_n^{-1}\right\|_{L_\infty(I)} \\
&\leq E_{n,\infty}[g']_W\, C\frac{a_n}{n} \|\Gamma_{n,\infty}\|_{L_\infty(I)} \left\|\Psi_n^{-1}\right\|_{L_\infty(I)},
\end{aligned}$$

by Lemma 3.1, Lemma 3.2(a) and (2.7). Using our estimates from Lemma 2.6 gives the result.
(b) By Lemma 3.1,

$$\begin{aligned}
\|W(g - v_n[g])\|_{L_\infty(I)} &\leq \|W(g - v_n[g])\,\Psi_n\|_{L_\infty(I)} \left\|\Psi_n^{-1}\right\|_{L_\infty(I)} \\
&\leq CE_{n,\infty}[g]_{W\Gamma_{n,\infty}} \left\|\Psi_n^{-1}\right\|_{L_\infty(I)} \\
&\leq CE_{n,\infty}[g]_W \|\max\{\Psi_n, 1\}\|_{L_\infty(I)} \left\|\Psi_n^{-1}\right\|_{L_\infty(I)}.
\end{aligned}$$

Using (a), (2.12), (2.15), and the fact that $T(a_n) \sim T\left(a_{n/2}\right)$, we continue this as

$$\leq C\,\|g'W\|_{L_\infty(I)} \frac{a_n}{n} T(a_n)^{1/2}\, T(a_n)^{2/3} \left[1 + \left(\frac{n}{T(a_n)^2}\right)^{4/9}\right]. \qquad \square$$

Proof of the Favard Inequality Theorem 1.3. Let us summarize what we have proven in the lemmas above: for $p = 1$ and $p = \infty$,

$$\|W(g - v_n[g])\|_{L_p(I)} \leq \|g'W\|_{L_p(I)}\, C\frac{a_n}{n}\alpha_n^{1-1/p}\beta_n^{1/p},$$

where

$$\begin{aligned}
\alpha_n &= T(a_n)^{2/3}\left[1 + \left(\frac{n}{T(a_n)^2}\right)^{4/9}\right]; \\
\beta_n &= T(a_n)^{1/3}.
\end{aligned}$$

We apply the Riesz-Thorin interpolation theorem [1, Theorem 2.2, p. 196] to the operator

$$\phi(x) \to W(\psi - v_n[\psi]),$$

where

$$\psi(x) = \int_0^x W^{-1}\phi.$$

After a substitution, we obtain for all $1 \leq p \leq \infty$,

$$\begin{aligned}
E_{2n,p}[f]_W &\leq \|W(f - v_n[f])\|_{L_p(I)} \\
&\leq C\,\|f'W\|_{L_p(I)} \frac{a_n}{n}\alpha_n^{1-1/p}\beta_n^{1/p} \\
&\leq C\,\|f'W\|_{L_p(I)} \frac{a_n}{n} T(a_n)^{\frac{2}{3}-\frac{1}{3p}} \left[1 + \left(\frac{n}{T(a_n)^2}\right)^{4/9}\right]^{1-1/p}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.2. Let $n \geq 2$ and m be the largest integer $\leq n/2$. Now

$$\begin{aligned}
(3.7) \qquad \|W(f - s_n[f])\|_{L_\infty(I)} &\leq \|W(f - v_m[f])\|_{L_\infty(I)} \\
&\quad + \|W(v_m[f] - s_n[f])\|_{L_\infty(I)} \\
&\leq \|W(f - v_m[f])\|_{L_\infty(I)} \\
&\quad + \left[\frac{n\sqrt{T(a_n)}}{a_n}\right]^{\frac{1}{2}} \|W(v_m[f] - s_n[f])\|_{L_2(I)},
\end{aligned}$$

by the Nikolskii inequality Lemma 2.2. Since s_n is the best polynomial approximant in the L_2 norm, we see that

$$\begin{aligned}
\|W(v_m[f] - s_n[f])\|_{L_2(I)} &\leq \|W(f - s_n[f])\|_{L_2(I)} + \|W(v_m[f] - f)\|_{L_2(I)} \\
&\leq 2\|W(v_m[f] - f)\|_{L_2(I)} \\
&\leq \|f'W\|_{L_2(I)} \frac{a_n}{n} T(a_n)^{\frac{1}{2}} \left[1 + \left(\frac{n}{T(a_n)^2}\right)^{4/9}\right]^{1/2},
\end{aligned}$$

By Theorem 1.3. If as we assume,

$$a_n = O(n^{\varepsilon}) \quad \text{and} \quad T(a_n) = O(n^{\varepsilon})$$

for each $\varepsilon > 0$, then we have

$$\|W(v_m[f] - s_n[f])\|_{L_2(I)} = O\left(n^{-7/9+\varepsilon}\right),$$

for each $\varepsilon > 0$. Also Theorem 1.2 gives

$$\begin{aligned}
\|W(f - v_m[f])\|_{L_\infty(I)} &\leq C\|f'W\|_{L_p(I)} \frac{a_n}{n} T(a_n)^{\frac{2}{3}} \left[1 + \left(\frac{n}{T(a_n)^2}\right)^{4/9}\right] \\
&= O\left(n^{-5/9+\varepsilon}\right).
\end{aligned}$$

Then substituting in (3.7),

$$\|W(f - s_n[f])\|_{L_\infty(I)} = O\left(n^{-1/18+\varepsilon}\right),$$

giving the asserted result. □

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