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MORE EXAMPLES ON GENERAL ORDER MULTIVARIATE PADÉ APPROXIMANTS FOR PSEUDO-MULTIVARIATE FUNCTIONS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Although general order multivariate Padé approximants have been introduced some decades ago, very few explicit formulas have been given so far. We show in this paper that, for any given pseudo-multivariate function, we can compute its (M, N) general order multivariate Padé approximant for some given index sets M, N with the usage of Maple or other software. Examples include a multivariate form of the sine function

$$S(x,y) = (x+y)\sum_{i,j=0}^{\infty}(-1)^{i+j}\frac{x^{2i}y^{2j}}{(2(i+j)+1)!},$$

a multivariate form of the logarithm function

$$L(x,y) = \sum_{i+j\geq 1}\frac{x^i y^j}{i+j},$$

a multivariate form of the inverse tangent function

$$T(x,y) = (x+y)\sum_{i,j=0}^{\infty}(-1)^{i+j}\frac{x^{2i}y^{2j}}{2(i+j)+1},$$

and many others.

Key words. multivariate Padé approximant; pseudo-multivariate function

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1. Introduction. Multivariate Padé approximants have been extensively investigated in the past few decades. The existence, uniqueness and non-uniqueness for homogeneous and general order multivariate Padé approximants and some convergence theorems have been established [3], [4], [5]. Despite all these activities, there are very few explicit constructions of multivariate Padé approximants. By using the residue theorem and the functional equation method, several researchers have successfully constructed multivariate Padé approximants to some functions which satisfy functional equations [2], [11], [12], [13]. Unfortunately, not many functions satisfy those functional equations. Besides, because the index sets for the numerator and denominator polynomials can not be chosen freely, most numerators of the approximants look complicated. In Cuyt-Tan-Zhou [7], we explicitly construct multivariate Padé approximants to so-called pseudo-multivariate functions, by using the Padé approximants of particular univariate functions which, in most of the cases, are the univariate projections of the pseudo-multivariate functions obtained by letting all but one variable be zero. Examples given in [7] are the general order multivariate Padé approximants for the multivariate form of the exponential function

$$E(x,y) = \sum_{i,j=0}^{\infty}\frac{x^i y^j}{(i+j)!},$$

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the multivariate form of the $q-$exponential function (see also [1], [12])

$$E_q(x,y) = \sum_{i,j=0}^{\infty} \frac{x^i y^j}{[i+j]_q!}, \quad |q| > 1,$$

the Appell function (see also [6], [9])

$$F_1(a,1,1;c;x,y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}\, x^i y^j}{(c)_{i+j}}, \;\; c > a > 0,$$

and the multivariate form of the partial theta function (see also [12], [8], [1])

$$T_q(x,y) = \sum_{i,j=0}^{\infty} q^{(i+j)(i+j+1)/2} x^i y^j, \quad |q| < 1.$$

Our aim in this paper is to show that, by using Theorem 2.1 in [7] and Maple, we can compute the (M,N) general order multivariate Padé approximant to any given pseudo-multivariate function for M, N, E defined in Theorem 2.1 in [7] (and stated as Theorem 1.3 in this paper).

DEFINITION 1.1. *Let*

$$F(x,y) := \sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j, \quad c_{ij} \in \mathbb{C}$$

be a formal power series, and let M, N, E be index sets in $\mathbb{N}\times\mathbb{N} = \mathbb{N}^2$. The (M,N) general order multivariate Padé approximant to $F(x,y)$ on the set E is a rational function

$$[M/N]_E(x,y) := \frac{P(x,y)}{Q(x,y)}$$

where the polynomials

$$P(x,y) := \sum_{(i,j)\in M} a_{ij} x^i y^j, \;\; a_{ij} \in \mathbb{C},$$

$$Q(x,y) := \sum_{(i,j)\in N} b_{ij} x^i y^j, \;\; b_{ij} \in \mathbb{C},$$

are such that

$$(FQ-P)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij} x^i y^j, \;\; d_{ij} \in \mathbb{C} \tag{1.1}$$

and E satisfying the inclusion property

$$(i,j)\in E,\; 0 \le k \le i,\; 0 \le l \le j \Longrightarrow (k,l) \in E. \tag{1.2}$$

DEFINITION 1.2. *A multivariate function $F(x,y)$ is said to be pseudo-multivariate if the coefficients of its formal power series*

$$F(x,y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j$$

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satisfy

$$c_{ij} = g\left(i+j\right), \quad i,j = 0,1,\cdots,$$

where $g\left(k\right)$ *is a certain function of* k.

Please see [7] for more details on definitions of the (M,N) general order multivariate Padé approximant to $F(x,y)$ on the set E, which is slightly different from the one given in earlier papers, and the pseudo-multivariate functions.

THEOREM 1.3. (Theorem 2.1 in [7]) *Let*

$$F\left(x,y\right) := \sum_{k=0}^{\infty} g\left(k\right) \sum_{i+j=k} x^i y^j$$

be a pseudo-multivariate function. For $m,n \in \mathbb{N}$, *let*

$$\frac{p_{m,n}}{q_{m,n}}\left(z\right) := \frac{\sum_{j=0}^{m} \alpha_j z^j}{\sum_{j=0}^{n} \beta_j z^j}, \quad \beta_0 = 1 \tag{1.3}$$

be the (m,n) *Padé approximant of the function*

$$h\left(z\right) := \sum_{k=0}^{\infty} g\left(k\right) z^k,$$

let $s = \max\{m,n\}$, *and let*

$$N := \{(i,j) : 0 \le i,j \le n\}, \tag{1.4}$$

$$M := \{(i,j) : 0 \le i,j \le s\} \cap \{(i,j) : 0 \le i+j \le m+n\}, \tag{1.5}$$

$$E := \{(i,j) : 0 \le i+j \le m+n,\ i,j \ge 0\} \tag{1.6}$$

be index sets in $\mathbb{N}^2$. *Then the* (M,N) *general order multivariate Padé approximant to* $F(x,y)$ *on the index set* E *is*

$$[M/N]_E(x,y) = \frac{P(x,y)}{Q(x,y)},$$

where

$$Q(x,y) := q_{m,n}\left(x\right) q_{m,n}\left(y\right), \tag{1.7}$$

and

$$\begin{aligned} P(x,y) &:= \frac{x p_{m,n}(x) q_{m,n}(y) - y q_{m,n}(x) p_{m,n}(y)}{x-y} \\ &= \sum_{i=0}^{m} \left(\sum_{j=0}^{\min\{i,n\}} \alpha_i \beta_j \left(x^i y^j + x^{i-1} y^{j+1} + \cdots + x^j y^i\right) \right. \\ &\quad \left. - \sum_{j=\min\{i,n\}+1}^{n} \alpha_i \beta_j \left(x^{i+1} y^{j-1} + x^{i+2} y^{j-2} + \cdots + x^{j-1} y^{i+1}\right) \right). \end{aligned} \tag{1.8}$$

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The proof of this theorem in [7] uses the fact that if $x \neq y$,

$$F(x, y) = \frac{xh(x) - yh(y)}{x - y},$$

and if $x = y$,

$$F(x, x) = \frac{d}{dx}(xh(x)) = \frac{d}{dy}(yh(y)) = F(y, y).$$

REMARK 1.4. *It is easy to see that $P(x, y)/Q(x, y)$ is irreducible, as $p_{m,n}(z)/q_{m,n}(z)$ is irreducible.*

2. More Explicit Constructions. The importance of finding more explicit formulas of multivariate Padé approximants to some given functions is obvious as very few of them are known so far. For any given pseudo-multivariate function, we can compute its (M, N) general order multivariate Padé approximant on the set E for given m, n, where M, N, E, and m, n are defined in Theorem 1.3. For some pseudo-multivariate functions whose one variable projection functions have general explicit formulas of Padé approximants, like the ones given in [7], we can use Theorem 1.3 to write their general order multivariate Padé approximants. For those pseudo-multivariate functions whose one variable projection functions don't have general explicit formulas of Padé approximants, we can use Maple or other software to compute the (M, N) general order multivariate Padé approximant on the set E. In what follows we present an example of a short procedure in Maple, called mpa, to compute the (M, N) general order multivariate Padé approximant for two variable pseudo-multivariate functions.

In the procedure mpa(f, x, m, n), f is the one variable projection function of the pseudo-multivariate function F, x is the variable of f, m and n are non-negative integers. It computes the (M, N) general order multivariate Padé approximant to $F(x, y)$ on the set E, where M, N, and E are defined in Theorem 1.3.

```
> with(numapprox):
> mpa:=proc(f,x,m,n)
> local g,px,py,qx,qy,PP,QQ,PQ;
> g:=pade(f,x,[m,n]);
> px:=numer(g); qx:=denom(g);
> py:=subs(x=y,px); qy:=subs(x=y,qx);
> PP:=simplify((x*px*qy-y*qx*py)/(x-y));
> QQ:=simplify(qx*qy);
> PQ:=simplify(PP/QQ);
> return (PQ);
> end proc:
```

EXAMPLE 2.1. The function

$$S(x, y) = (x + y) \sum_{i,j=0}^{\infty} (-1)^{i+j} \frac{x^{2i} y^{2j}}{(2(i + j) + 1)!}$$

is a pseudo-multivariate function with

$$h(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n + 1)!} = \sin z.$$

Running mpa($\sin(x), x, 3, 2$) in Maple gives

$$-\frac{140x^3 + 7x^3y^2 + 140x^2y + 7x^2y^3 - 120x + 140xy^2 - 1200y + 140y^3}{3\left(20 + x^2\right)\left(20 + y^2\right)},$$

which is the (M, N) general order multivariate Padé approximant to the function $S(x, y)$ on the set E, with

$$\begin{aligned} N &= \{(i,j) : 0 \le i, j \le 2\}, \\ M &= \{(i,j) : 0 \le i, j \le 3,\ i + j \le 5\}, \\ E &= \{(i,j) : 0 \le i + j \le 5\}. \end{aligned}$$

EXAMPLE 2.2. A multivariate form of the logarithm series is

$$L(x, y) = \sum_{i+j \ge 1} \frac{x^i y^j}{i + j}.$$

It is a pseudo-multivariate function with

$$h(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n + 1} = \ln(1 - z).$$

Running mpa($\ln(1 - z), z, 3, 1$) gives

$$\begin{aligned} &-\frac{-4x^3 + 3x^3y - 24x^2 + 14x^2y + 3x^2y^2 + 96x}{6(3x - 4)(3y - 4)} \\ &\qquad -\frac{96xy + 14xy^2 + 3y^3x - 4y^3 - 24y^2 + 96y}{6(3x - 4)(3y - 4)}, \end{aligned}$$

which is the (M, N) general order multivariate Padé approximant to the function $L(x, y)$ on the set E, with

$$\begin{aligned} N &= \{(i,j) : 0 \le i, j \le 1\}, \\ M &= \{(i,j) : 0 \le i, j \le 3,\ i + j \le 4\}, \\ E &= \{(i,j) : 0 \le i + j \le 4\}. \end{aligned}$$

EXAMPLE 2.3. The function

$$R(x, y) = \sum_{i,j=0}^{\infty} (-1)^{i+j} \frac{1 \cdot 3 \cdot \cdots \cdot (2(i + j) - 1)}{(i + j)!} x^i y^j,\ |x|, |y| < \frac{1}{2}$$

is a pseudo-multivariate function with

$$\begin{aligned} h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{n!} z^n \\ &= \sqrt{1 + 2z},\ |z| < \frac{1}{2}. \end{aligned}$$

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Similarly, mpa$(sqr\,(1+2z)\,,z,3,2)$ gives the (M,N) general order multivariate Padé approximant $P\,(x,y)\,/Q\,(x,y)$ to $R(x,y)$ on the index set E for $|x|\,,|y|<1/2$, where

$$P\,(x,y)=4x^3+8x^3y+3x^3y^2+36x^2+76x^2y+35x^2y^2+3x^2y^3$$
$$+48x+120xy+76xy^2+8xy^3+48y+36y^2+4y^3+16,$$

and

$$Q\,(x,y)=\left(4+8x+3x^2\right)\left(4+8y+3y^2\right),$$

with the same M,N,E as given in Example 2.1.

EXAMPLE 2.4. The function

$$T\,(x,y)=(x+y)\sum_{i,j=0}^{\infty}(-1)^{i+j}\,\frac{x^{2i}y^{2j}}{2\,(i+j)+1}$$

is a pseudo-multivariate function with

$$h\,(z)=\sum_{n=0}^{\infty}(-1)^n\,\frac{z^{2n+1}}{2n+1}=\tan^{-1}x.$$

Then we can use mpa$(\arctan(x),x,m,n)$ to compute the (M,N) general order multivariate Padé approximant to $T(x,y)$ on the index set E for any given positive integers m,n.

EXAMPLE 2.5. The function

$$H\,(x,y)=(x+y)\sum_{i,j=0}^{\infty}\frac{1\cdot 3\cdot\dots\cdot(2\,(i+j)-1)}{2^{i+j}\,(2\,(i+j)+1)\,(i+j)!}x^{2i}y^{2j}$$

is a pseudo-multivariate function with

$$h\,(z)=\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{2^n\,(2n+1)\,n!}x^{2n+1}=\sin^{-1}x-x.$$

Then we can use mpa$(\arcsin(x)-x,x,m,n)$ to compute the (M,N) general order multivariate Padé approximant to $H(x,y)$ on the index set E for any given positive integers m,n.

For some pseudo-multivariate series, we might not be able to have the explicit functions for the sum of their projection series of one variable. In this case, we can use Maple to write the partial sum up to the degree of $m+n+1$ in the variable, and compute the (m,n) Padé approximant to the partial sum and then use mpa(f,x,m,n) to compute the (M,N) general order multivariate Padé approximant to the multivariate series on the index set E for any given positive integers m,n.

EXAMPLE 2.6. The function

$$F\,(x,y)=\sum_{i,j=0}^{\infty}\frac{x^iy^j}{(i+j)!+(i+j)^3+1}$$

is a pseudo-multivariate function with

$$h\,(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!+n^3+1}.$$

First, we write a short procedure Sn(n,x) to write the sum of the first n terms of the series in $h\left(z\right)$:

```
> Sn:=proc(n,x)
> local i,s;
> s:=0;
> for i from 0 to n do:
> s:=s+x^i/(factorial(i)+i^3+1); od;
> return (s);
> end proc:
```

Then running $\mathrm{mpa}(\mathrm{Sn}(6,x),x,3,2)$ gives the the (M,N) general order multivariate Padé approximant $P\left(x,y\right)/Q\left(x,y\right)$ to $R(x,y)$ on the index set E, where

$$\begin{aligned}P\left(x,y\right) = & -31826219438313x^3+7952534736320x^3y+1360698798107x^3y^2\\ & -17343781985833x^2y-57959034150357x^2+10430516397343x^2y^2\\ & +1360698798107y^3x^2-187261984313898xy+877849421508774x\\ & +7952534736320y^3x-17343781985833xy^2-57959034150357y^2\\ & +2106200105319321-31826219438313y^3+877849421508774y,\end{aligned}$$

and

$$Q\left(x,y\right)=1122\left(-1937619+484160x+82841x^2\right)\left(-1937619+484160y+82841y^2\right),$$

with the same M,N,E as given in Example 2.1. We can use $\mathrm{mpa}(\mathrm{Sn}(m+n+1,x),x,m,n)$ to compute the (M,N) general order multivariate Padé approximant to $F(x,y)$ on the index set E for any given positive integers m,n.

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