Electronic Transactions on Numerical Analysis.
Volume 25, pp. 284-301, 2006.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

ANALYSIS ON THE UNIT BALL AND ON THE SIMPLEX*

YUAN XU†

Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Many results on the unit ball and those on the simplex can be deduced from each other or from the corresponding results on the unit sphere. The areas in which such a connection appears include orthogonal polynomials, approximation, cubature formulas and polynomial interpolation. We explain this phenomenon in some detail.

Key words. analysis, ball, simplex, orthogonal polynomials, approximation, cubature

AMS subject classifications. 41A10, 42C10, 41A63

1. Introduction. Recent studies show that, for several problems in analysis, results on the unit ball $B^d = \{x : \|x\| \le 1\}$ in $\mathbb{R}^d$ and those on the standard simplex

$$T^d = \{x : x_1 \ge 0, \dots, x_d \ge 0, 1 - x_1 - \cdots - x_d \ge 0\}$$

in $\mathbb{R}^d$ can often be deduced from each other or deduced from results on the unit sphere $S^d = \{x : \|x\| = 1\}$ in $\mathbb{R}^{d+1}$, making use of elementary maps between the three domains and symmetry of the polynomial spaces on these domains. Here and in the following, $\|x\|$ denote the Euclidean norm. Problems for which that has occurred all involve polynomials in one form or other. They appear in the areas of orthogonal polynomials, approximation theory, cubature formulas, and polynomial interpolation. The purpose of this paper is to explain this phenomenon in some detail. We will mainly take orthogonal polynomials and best approximation by polynomials as examples, but will mention what else is known in this regard.

The unit sphere is a manifold without a boundary, it is homogeneous in the sense that any point can be translated to any other point by a simple rotation. In contrast, the unit ball and the simplex are manifolds with a boundary, points near the boundary are different from points inside. Analysis on these two domains will have to catch the boundary behavior. This consideration seems to indicate that the results on B^d and T^d should not be deducible from those on S^d. The key, however, lies in the notion of weighted spaces. More specifically, we will work with weighted L^p spaces on these domains. For the domain S^d we will consider mainly the weight function h_κ^2, where

$$h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \qquad \kappa_i \ge 0, \tag{1.1}$$

which becomes zero on the coordinate plane $x_i = 0$ if $\kappa_i > 0$. Consequently, a function $f \in L^p(S^d; h_\kappa^2)$ can have singularities on the intersections of the sphere and the coordinates planes. When we work with the weighted spaces, these intersections play the role of the boundary on the sphere S^d. Corresponding to h_κ^2, we have the weight function

$$W_{\kappa,\mu}^B(x) = \prod_{i=1}^{d} |x_i|^{\kappa_i} (1 - \|x\|^2)^{\mu - 1/2}, \quad \kappa_i \ge 0, \ \mu \ge 0, \tag{1.2}$$

*Received April 19, 2005. Accepted for publication September 22, 2005. The author was supported in part by the NSF Grant DMS-0201669.

†Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222 (yuan@math.uoregon.edu).

ETNA
Kent State University
etna@mcs.kent.edu

defined on B^d, where $\mu = \kappa_{d+1}$, and the weight function

$$W^T_{\kappa,\mu}(x) = \prod_{i=1}^{d} x_i^{\kappa_i - 1/2}(1 - x_1 - \cdots - x_d)^{\mu-1/2}, \quad \kappa_i \geq 0, \ \mu \geq 0, \tag{1.3}$$

defined on T^d. These are the weight functions that will be used in this paper. Many results that we will discuss hold for more general weight functions, mainly for those weight functions that are invariant under a finite reflection group. The weight function h_κ in (1.1) is a special case, which is invariant under the group $\mathbb{Z}_2^{d+1}$, and $W^B_{\kappa,\mu}$ in (1.2) is invariant under the group $\mathbb{Z}_2^d$. We will not discuss the most general case in order to keep our exposition simple and keep the main idea clear.

These three weight functions are closely related and the relation extends to orthogonal polynomials and cubature formulas with respect to these weight functions, as explored in [26, 27]. More recently, it has been realized that we can get a complete characterization for the best approximation on B^d and on T^d this way [34, 35, 36]. It is this latter development that we choose as examples for the main idea. Our goal is to explain how results on the ball B^d and on the simplex T^d can be derived. We will not give a complete survey of the results known on these domains, neither will we state the results in their most general form.

The paper is organized as follows. The background and the basic relation between the three domains are given in the next section. The results on orthogonal polynomials and approximation on the unit sphere are discussed in Section 3. The way to obtain results on the unit ball and on the simplex is explained in Section 4 and in Section 5, respectively. Finally, in Section 6, we give a brief account on other problems for which results on B^d and on T^d can be obtained from each other or from those on S^d.

2. Basic relations. Let $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$ be the space of polynomials of d real variables and let Π^d_n be the subspace of polynomials of degree at most n. We also denote by $\mathcal{P}^d_n$ the space of homogeneous polynomials of degree n. It is known that

$$\dim \mathcal{P}^d_n = \binom{n+d-1}{d-1} \quad \text{and} \quad \dim \Pi^d_n = \binom{n+d}{d}.$$

2.1. Polynomial spaces on S^d and on B^d. Denote by $\mathcal{P}_n(S^d)$ and $\Pi_n(S^d)$ the restriction of $\mathcal{P}^{d+1}_n$ and Π^{d+1}_n on S^d, respectively. The polynomials in $\mathcal{P}_n(S^d)$ may not be homogeneous. In fact, we have

$$\mathcal{P}^d_n = \sum_{0 \leq 2j \leq n} \|x\|^{n-2j}\mathcal{P}_{n-2j}(S^d)$$

so that $\|x\|^{n-2j}\mathcal{P}_{n-2j}(S^d) \subset \mathcal{P}^d_n$. Let S^d_+ denote the upper hemisphere of S^d. A simple-minded relation between S^d_+ and B^d is as follows:

$$x \in B^d \Longleftrightarrow (x, x_{d+1}) \in S^d_+, \quad x_{d+1} = \sqrt{1 - \|x\|^2}. \tag{2.1}$$

Clearly, a similar relation holds for the lower hemisphere. The domain S^d_+ induces a symmetry in the polynomial space. Let $\mathcal{P}^+_n(S^d)$ denote the subspace of elements in $\mathcal{P}_n(S^d)$ that are even in its $(d+1)$-th coordinates. The mapping (2.1) leads immediately to the following basic result:

LEMMA 2.1. *For each $n \geq 0$ the equation*

$$\mathcal{P}_n(S^d) = \Pi^d_n \cup x_{d+1}\Pi^d_{n-1} \tag{2.2}$$

ETNA
Kent State University
etna@mcs.kent.edu

holds in the sense that for each $P \in \mathcal{P}_n(S^d)$ there exist unique elements $p \in \Pi_n^d$ and $q \in \Pi_{n-1}^d$ such that

$$P(x, x_{d+1}) = p(x) + x_{d+1} q(x), \qquad (x, x_{d+1}) \in S^d.$$

In particular, there is a one-to-one correspondence between Π_n^d and $\mathcal{P}_n^+(S^d)$.

Proof. Let $P \in \mathcal{P}_n(S^d)$. We can write $P(x, x_{d+1}) = \sum p_j(x) x_{d+1}^j$ for some $p_j \in \Pi_{n-j}^d$. Using the fact that $x_{d+1}^2 = 1 - \|x\|^2$, we have $P(x, x_{d+1}) = p(x) + x_{d+1} q(x)$, where $p \in \Pi_n^d$ and $q \in \Pi_{n-1}^d$. Clearly p and q are unique. $\square$

Using (2.1) as a change of variables leads immediately to the relation

$$\int_{S^d} f(y) d\omega(y) = \frac{1}{2} \int_{B^d} \left[f\left(x, \sqrt{1 - \|x\|^2}\right) + f\left(x, -\sqrt{1 - \|x\|^2}\right) \right] \frac{dx}{\sqrt{1 - \|x\|^2}}, \tag{2.3}$$

where $d\omega$ is the surface measure on the sphere S^d.

These simple observations have important applications for orthogonal polynomials and approximation by polynomials, as will be discussed in Section 4.

2.2. Polynomial spaces on B^d and on T^d. We start with a simple mapping between B^d and T^d. Let $B_+^d := \{x \in B^d : x_1 \geq 0, \ldots, x_d \geq 0\}$ be the positive quadrant of B^d. A simpleminded relation between B^d and T^d is as follows:

$$(x_1, \ldots, x_d) \in B_+^d \iff (x_1^2, \ldots, x_d^2) \in T^d. \tag{2.4}$$

A polynomial P of the form $P(x) = p(x_1^2, \ldots, x_d^2)$ is invariant under sign changes of its coordinates; that is, it is invariant under the group $G = \mathbb{Z}_2^d$. Let ψ denote the map

$$\psi : (x_1, \ldots, x_d) \in B^d \mapsto (x_1^2, \ldots, x_d^2) \in T^d. \tag{2.5}$$

The domain B_+^d can be looked upon as the fundamental domain for the polynomials invariant under $\mathbb{Z}_2^d$. Let us define

$$G\Pi_{2n}^d := \{p \in \Pi_{2n}^d : p \text{ invariant under } \mathbb{Z}_2^d\}.$$

The relation (2.4) leads to a correspondence between polynomial spaces:

LEMMA 2.2. *The map ψ introduces a one-to-one correspondence between Π_n^d and $G\Pi_{2n}^d$; more precisely, $p \in \Pi_n^d$ corresponds to $p \circ \psi \in G\Pi_{2n}^d$.*

Proof. If $P \in G\Pi_{2n}^d$ then P is even in each of its variables. Hence, it is easy to see that $P(x) = p(x_1^2, \ldots, x_d^2) = (p \circ \psi)(x)$ for some $p \in \Pi_n^d$. The correspondence between P and p is evidently one-to-one. $\square$

Using (2.4) as a change of variables leads immediately to the relation

$$\int_{B^d} f(x_1^2, \ldots, x_d^2) dx = \int_{T^d} f(x_1, \ldots, x_d) \frac{dx}{\sqrt{x_1 \cdots x_d}}. \tag{2.6}$$

These observations will play important roles in the study of orthogonal polynomials and approximation by polynomials, which will be discussed in Section 5.

3. Analysis on the unit sphere. In this section we review results for orthogonal polynomials and approximation with respect to the weight function h_κ^2 on the unit sphere S^d. The weight function h_κ is given in (1.1), which has singularity at the intersections of the sphere and coordinate planes.

3.1. Orthogonal polynomials on the sphere. Let h_κ be defined as in (1.1). We consider orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_\kappa := a_\kappa \int_{S^d} f(x)g(x)h_\kappa^2(x)d\omega(x),$$

where a_k is a constant such that $a_\kappa^{-1}\int_{S^d} h_\kappa^2(x)d\omega = 1$. Let $\mathcal{H}_n(h_\kappa^2)$ denote the subspace of orthogonal homogeneous polynomials of degree n with respect to this inner product. It is known that $\dim \mathcal{H}_n(h_\kappa^2) = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d$. The elements of this space are called h-harmonics. If $h_\kappa(x) \equiv 1$, $\mathcal{H}_n(h_\kappa^2)$ is the space of ordinary spherical harmonics of degree n. The weight function h_κ in (1.1) is an example of a family of weight functions invariant under reflection groups, for which the corresponding h-harmonics enjoy properties similar to those of ordinary spherical harmonics; see [9, 10] and the references therein.

We state the basic properties of h-harmonics below. The essential ingredient is the Dunkl operator $\mathcal{D}_j$ which, for h_κ in (1.1), is defined by [9]

$$\mathcal{D}_j f(x) = \partial_j f(x) + \kappa_j \frac{f(x) - f(x_1, \ldots, -x_j, \ldots, x_{d+1})}{x_j}, \quad 1 \le j \le d+1.$$

These are first order differential-difference operators that maps $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$ and they commute with each other; that is, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$ for $1 \le i,j \le d+1$. The h-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_{d+1}^2$, which plays the role of the usual Laplacian: If $P \in \mathcal{H}_n(h_\kappa^2)$, then $\Delta_h P = 0$. Furthermore, in spherical-polar coordinates $x = rx'$, $r > 0$, $x' \in S^d$, the h-Laplacian takes the form [30]

$$\Delta_h = \frac{\partial^2}{\partial r^2} + \frac{2\lambda+1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{h,0}, \quad \text{where} \quad \lambda := |\kappa| + \frac{d-1}{2}, \tag{3.1}$$

a formula similar to the spherical-polar form of the usual Laplacian. The operator $\Delta_{h,0}$ is called the spherical h-Laplacian. When restricted to S^d, h-harmonics are eigenfunctions of $\Delta_{h,0}$, that is,

$$\Delta_{h,0}Y(x) = -n(n+2\lambda)Y(x), \qquad x \in S^d, \quad Y \in \mathcal{H}_n(h_\kappa^2). \tag{3.2}$$

The basic Hilbert space theory shows that $L^2(h_\kappa^2; S^d)$ can be decomposed as

$$L^2(h_\kappa^2; S^d) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(h_\kappa^2): \qquad f = \sum_{n=0}^{\infty} \mathrm{proj}_{\mathcal{H}_n} f,$$

where $\mathrm{proj}_{\mathcal{H}_n}$ is the projection operator from $L^2(h_\kappa^2)$ onto $\mathcal{H}_n(h_\kappa^2)$. It is known that

$$\mathrm{proj}_{\mathcal{H}_n} f = a_\kappa \int_{S^d} f(y)P_n(h_\kappa^2; x, y)W_\kappa^2(y)dy, \tag{3.3}$$

where $P_n(h_\kappa^2)$ is the reproducing kernel of $\mathcal{H}_n(h_\kappa^2)$ (zonal h-harmonic). The reproducing kernel turns out to satisfy a compact formula [25]

$$P_n(h_\kappa^2; x, y) = \frac{\lambda+n}{\lambda} V_\kappa\left[C_n^\lambda(\langle x, \cdot\rangle)\right](y), \tag{3.4}$$

where C_n^λ is the Gegenbauer polynomial of degree n and V_κ is the intertwining operator, which is a linear operator uniquely determined by $V_\kappa 1 = 1$ and $\mathcal{D}_j V_\kappa = V_\kappa \partial_j$, $1 \le j \le d+1$. For h_κ in (1.1), it is known that V_κ satisfies [24]

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^{d+1}} f(x_1t_1, \ldots, x_{d+1}t_{d+1}) \prod_{i=1}^{d+1}(1+t_i)(1-t_i^2)^{\kappa_i - 1}dt. \tag{3.5}$$

The formula (3.4), just like the classical zonal harmonics, plays an important role in the study of h-harmonic expansions, which points out a connection to Gegenbauer expansions and indicates a possible connection to functions of one variable.

3.2. Weighted approximation. We work with the space $L^p(h_\kappa^2; S^d)$ that is equipped with the norm

$$\|f\|_{\kappa,p} := \left(a_\kappa \int_{S^d} |f(x)|^p h_\kappa^2(x) d\omega\right)^{1/p}$$

for $1 \le p < \infty$ and $\|f\|_\infty = \sup_{x \in S^d} |f(x)|$ for $p = \infty$.

The equation (3.3) and the explicit formula (4.6) suggests the definition of the following weighted convolution: For $f \in L^1(h_\kappa^2; S^d)$ and $g \in L^1(w_\lambda, [-1,1])$,

$$(f *_\kappa g)(x) = a_\kappa \int_{S^d} f(y) V_\kappa[g(\langle \cdot, x \rangle)](y) h_\kappa^2(y) d\omega. \tag{3.6}$$

For the surface measure ($h_\kappa(x) = 1$), this is the spherical convolution in [8]. It satisfies the usual properties of convolution. In particular, it satisfies Young's inequality:

PROPOSITION 3.1. *For $f \in L^q(h_\kappa^2)$ and $g \in L^r(w_\lambda; [-1,1])$,*

$$\|f *_\kappa g\|_{\kappa,p} \le \|f\|_{\kappa,q} \|g\|_{w_\lambda,r}.$$

where $p, q, r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$.

This shows that $(f *_\kappa g)(x)$ is finite for $f \in L^1(h_\kappa^2)$ and $g \in L^1(w_\lambda, [-1,1])$. We note that the projection operator $\mathrm{proj}_{\mathcal{H}_n}$ in (3.3) can be written as a convolution of f with the Gegenbauer polynomial C_n^λ, which indicates a possible reduction in the study of h-harmonic expansions to that of Gegenbauer expansions. We shall not pursue this line of study here; see, for example, [29]. Instead, we use the convolution to define a weighted spherical means, T_θ^κ, which is defined *implicitly* as follows:

$$c_\lambda \int_0^\pi T_\theta^\kappa f(x) g(\cos\theta)(\sin\theta)^{2\lambda} d\theta := (f *_\kappa g)(x),$$

where g is any $L^\infty([-1,1])$ function and $0 \le \theta \le \pi$. Young's inequality can be used to show that T_θ^κ is well-defined. In the case of $\kappa = 0$, the weighted spherical means coincides with the classical spherical means

$$T_\theta f(x) = \frac{1}{\sigma_{d-1}(\sin\theta)^{d-1}} \int_{\langle x,y \rangle = \cos\theta} f(y) d\omega, \tag{3.7}$$

which was studied in [5, 18]. The weighted spherical means shares essentially all properties of the classical spherical means [34], including those listed below:

PROPOSITION 3.2. *The means $T_\theta^\kappa f$ satisfy the following properties:*

1. *Let $f_0(x) = 1$; then $T_\theta^\kappa f_0(x) = 1$.*
2. *If $f \sim \sum_{n=0}^\infty \mathrm{proj}_{\mathcal{H}_n} f$, then*

$$T_\theta^\kappa f \sim \sum_{n=0}^\infty \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} \mathrm{proj}_{\mathcal{H}_n} f.$$

3. *For $f \in L^p(h_\kappa^2)$, $1 \le p < \infty$, or $f \in C(S^d)$,*

$$\|T_\theta^\kappa f\|_{\kappa,p} \le \|f\|_{\kappa,p} \qquad \textit{and} \qquad \lim_{\theta \to 0} \|T_\theta^\kappa f - f\|_{\kappa,p} = 0.$$

ETNA
Kent State University
etna@mcs.kent.edu

The last property suggests immediately the following definition of a weighted modulus of smoothness: For $r > 0$ and $1 \le p \le \infty$,

$$\omega_r(f,t)_{\kappa,p} := \sup_{0<\theta\le t} \|(I - T_\theta^\kappa)^{r/2} f\|_{\kappa,p}. \tag{3.8}$$

For $\kappa = 0$, this coincides with the classical modulus of smoothness on the sphere that has been used by many authors; see, for example, [5, 17, 18, 20] and the references therein. It satisfies the usual properties of modulus of smoothness.

There is also a weighted K-functional, defined using the spherical h-Laplacian $\Delta_{h,0}$ in (3.1): For $r > 0$ and $1 \le p \le \infty$,

$$K_r(f,t)_{\kappa,p} := \inf_g \left\{ \|f - g\|_{\kappa,p} + t^r \|(-\Delta_{h,0})^{r/2} g\|_{\kappa,p} \right\}, \tag{3.9}$$

where the infimum is taken over the space of all $g \in L^p(h_\kappa^2)$ for which $\|\Delta_{h,0} g\|_p$ is finite.

Just as in the classical approximation theory, the weighted modulus of smoothness and the weighted K-functional are equivalent [34].

THEOREM 3.3. *For $r > 0$, $1 \le p \le \infty$, and $f \in L^p(h_\kappa^2; S^d)$,*

$$c_1 \omega_r(f;t)_{\kappa,p} \le K_r(f;t)_{\kappa,p} \le c_2\, \omega_r(f;t)_{\kappa,p}, \qquad 0 < t < \pi/2,$$

where c_1 and c_2 are constants independent of f.

These two equivalent gadgets can be used to characterize the best approximation by polynomials [34]. For $f \in L^p(h_\kappa^2; S^d)$, $1 \le p \le \infty$, we denote by

$$E_n(f)_{\kappa,p} := \inf \left\{ \|f - P\|_{\kappa,p} : P \in \Pi_n(S^d) \right\} \tag{3.10}$$

the error of best approximation by polynomials in the weighted L^p space.

THEOREM 3.4. *For $f \in L^p(h_\kappa^2; S^d)$, $1 \le p \le \infty$,*

$$E_n(f)_{\kappa,p} \le c\, \omega_r(f; n^{-1})_{\kappa,p} \tag{3.11}$$

and, on the other hand,

$$\omega_r(f, n^{-1})_{\kappa,p} \le c\, n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{\kappa,p}.$$

In other words, both direct and inverse theorems for the best approximation hold.

These results provide a complete characterization of the best approximation by polynomials. For the surface measure on S^d ($\kappa = 0$), they were proved in [20], which brings a long investigation with various early results obtained by many other authors to a completion. See the results in [5, 15, 17, 18, 20] and the references therein. The proof of these two theorems are rather involved.

Let us mention one result that will be used to explain how to obtain results on the ball and on the simplex. Let $\eta \in C^k[0, +\infty)$ be a function defined by $\eta(x) = 1$ for $0 \le x \le 1$ and $\eta(x) = 0$ if $x \ge 2$. Define a sequence of operators η_n for $n > 0$ by

$$\eta_n f := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \mathrm{proj}_{\mathcal{H}_k} f. \tag{3.12}$$

Since $\eta(k/n) = 0$ if $k \ge 2n$, the series is finite and $\eta_n f$ is a spherical polynomial of degree at most $2n - 1$. Furthermore, the operator η_n preserves polynomials of degree n. The main properties of η_n are given in the following proposition:

PROPOSITION 3.5. *Let $f \in L^p(h_\kappa^2)$, $1 \le p \le \infty$. If $k \ge \lfloor \lambda \rfloor + 1$ then*

1. $\eta_n f \in \Pi_{2n-1}(S^d)$ *and* $\eta_n P = P$ *for* $P \in \Pi_n(S^d)$*;*
2. *for* $n > 0$, $\|\eta_n f\|_{\kappa,p} \le c\|f\|_{\kappa,p}$*;*
3. *for* $n > 0$, $\|f - \eta_n f\|_{\kappa,p} \le cE_n(f)_{\kappa,p}$.

This proposition was proved in [34], where we assume that η is in C^∞. The proof is based on the boundedness of the Cesàro (C,δ)-means for the h-harmonic expansions, which holds if $\delta > \lambda$ as shown in [24]. This gives the condition $k \ge \lfloor \lambda \rfloor + 1$. For the Lebesgue measure ($\kappa = 0$), the definition of the operator $\eta_n f$ appeared first in [14] and it played an important role in [20].

4. Analysis on the unit ball. Our goal in this section is to show how the results on B^d can be deduced from those on S^d. We consider analysis in the weighted space $L^p(W^B_{\kappa,\mu})$ on B^d, where $W^B_{\kappa,\mu}$ is given in (1.2), which has the norm

$$\|f\|_{W^B_{\kappa,\mu},p} := \left(a^B_{\kappa,\mu} \int_{B^d} |f(x)|^p W^B_{\kappa,\mu}(x)dx \right)^{1/p}$$

for $1 \le p < \infty$, and $\|f\|_\infty = \sup_{x\in B^d} |f(x)|$ for $p = \infty$, where $a^B_{\kappa,\mu}$ is the normalization constant of the weight function $W^B_{\kappa,\mu}$.

Frequently we will refer to the results in the previous section. For this purpose it is more convenient for us to refer to those results in terms of the weight function

$$h_{\kappa,\mu}(x) := \prod_{i=1}^d |x_i|^{\kappa_i} |x_{d+1}|^\mu = \widetilde{h}_\kappa(x_1,\dots,x_d)|x_{d+1}|^\mu,$$

which is the weight function h_κ defined in (1.1) with $\kappa_{d+1} = \mu$, where we use the notation $\widetilde{h}_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$ for $x \in \mathbb{R}^d$. Note that $\widetilde{h}_\kappa$ is h_κ in (1.1) with $d+1$ replace by d. Thus, when we refer to the results in the previous section, we will replace h_κ by $h_{\kappa,\mu}$. Furthermore, whenever we refer to a notion that appeared in the previous section and denoted by a notation that contains a subindex κ, we will then replace κ by κ,μ. For example, we will use $V_{\kappa,\mu}$ to denote the intertwining operator in (3.5) associated with $h_{\kappa,\mu}$ and use $*_{\kappa,\mu}$ to denote the convolution in (3.6) associated with $h_{\kappa,\mu}$.

4.1. Orthogonal polynomials on B^d. Under the mapping (2.1), the weight functions $W^B_{k,\mu}$ at (1.2) is related to the weight function $h_{\kappa,\mu}$ by the following relation:

$$h^2_{\kappa,\mu}(x,x_{d+1}), \quad (x,x_{d+1}) \in S^d_+ \quad \Longleftrightarrow \quad W^B_{\kappa,\mu}(x), \quad x \in B^d. \tag{4.1}$$

We consider the inner product on the unit ball

$$\langle f,g\rangle_B = a_{\kappa,\mu} \int_{B^d} f(x)g(x)W^B_{\kappa,\mu}(x)dx,$$

where $a_{\kappa,\mu}$ is the normalization constant of $W_{\kappa,\mu}$. Let $\mathcal{V}_n(W^B_{\kappa,\mu})$ denote the space of orthogonal polynomials of degree n with respect to $\langle f,g\rangle_B$. Several explicit bases of $\mathcal{V}_n(W^B_{\kappa,\mu})$ are known explicitly; see, for example, [10]. In the case of the classical weight function

$$W_\mu(x) = (1-\|x\|^2)^{\mu-1/2}, \qquad x \in B^d,$$

which is the same as $W^B_{0,\mu}(x)$, some of these bases can be traced back to Hermite; see [3, 11]. What we need, however, is the relation between orthogonal polynomials on B^d and those on S^d. This relation follows as a consequence of Lemma 2.1.

PROPOSITION 4.1. *Write* $y = r(x,x_{d+1})$, $r = \|y\|$ *and* $x \in B^d$. *Then*

$$\mathcal{H}_n(h^2_{\kappa,\mu}; S^d) = \mathcal{V}_n(W^B_{\kappa,\mu}) \oplus x_{d+1}\mathcal{V}_{n-1}(W^B_{\kappa,\mu+1}).$$

ETNA
Kent State University
etna@mcs.kent.edu

More precisely, if $\{P_\alpha^n\}$ *is a basis of* $\mathcal{V}_n(W^B_{\kappa,\mu})$ *and* $\{Q_\alpha^{n-1}\}$ *is a basis of* $\mathcal{V}_{n-1}(W^B_{\kappa,\mu+1})$*, then the functions* $r^n P_\alpha^n(y_1,\ldots,y_d)$ *and* $r^n y_{d+1} Q_\alpha^{n-1}(y_1,\ldots,y_d)$ *are homogeneous polynomials and their restriction on* S^d *form a basis for* $\mathcal{H}_n(h^2_{\kappa,\mu}, S^d)$.

The proposition establishes the relation between orthogonal polynomials on the sphere and those on the unit ball. It shows, in particular, that

$$\mathcal{V}_n(W^B_{\kappa,\mu}) = \operatorname{span}\Big\{P\Big(x,\sqrt{1-\|x\|^2}\Big):\ P\in\mathcal{H}_n(h^2_{\kappa,\mu};S^d), P \text{ is even in } x_{d+1}\Big\}.$$

In other words, orthogonal polynomials with respect to $W^B_{\kappa,\mu}$ on B^d correspond one-to-one to spherical h-harmonics associated with $h^2_{\kappa,\mu}$ that are even in x_{d+1}. Note that the ordinary spherical harmonics on S^d corresponds to the orthogonal polynomials with respect to $W_0(x) = (1-\|x\|^2)^{-1/2}$ on B^d, while the orthogonal polynomials with respect to the Lebesgue measure dx on B^d correspond to the h-spherical harmonics associated to $|x_{d+1}|d\omega$ on S^d.

The mapping (2.1) goes deeper than just inducing a correspondence. It turns out that, under this mapping, the spherical h-Laplacian $\Delta_{h,0}$ in (3.1) becomes [30]

$$D^B_{\kappa,\mu} := \Delta_h - \langle x,\nabla\rangle^2 - 2\lambda_\mu\langle x,\nabla\rangle, \qquad \lambda_\mu := |\kappa| + \mu + \frac{d-1}{2}, \tag{4.2}$$

where $\nabla = (\partial_1,\ldots,\partial_d)$ is the gradient and Δ_h is the h-Laplacian associated to $\widetilde{h}_\kappa$. The orthogonal polynomials in $\mathcal{V}_n(W^B_{\kappa,\mu})$ become eigenfunctions of $D^B_{\kappa,\mu}$, see (3.2),

$$D^B_{\kappa,\mu}P = -n(n+2\lambda_\mu)P, \qquad P\in\mathcal{V}_n(W^B_{\kappa,\mu}). \tag{4.3}$$

For the classical weight function W_μ, $\kappa = 0$, this is the second order partial differential equation satisfied by the classical orthogonal polynomials on B^d.

Because of the equation (2.3) and the mapping (2.1), we define an operator

$$V^B_{\kappa,\mu}f(x,x_{d+1}) = c_\mu\int_{-1}^1 \widetilde{V}_\kappa\left[f(\cdot,x_{d+1}t)\right](x)(1-t^2)^{\mu-1}dt, \quad x\in\mathbb{R}^d, \tag{4.4}$$

where $\widetilde{V}_\kappa$ is the intertwining operator for $\widetilde{h}_\kappa$ and it is given explicitly in (3.5) with $d+1$ replaced by d. Recall that $V_{\kappa,\mu}$ denote the intertwining operator for $h_{\kappa,\mu}$. The new operator is simply $[V_{\kappa,\mu}(x,x_{d+1}) + V_{\kappa,\mu}(x,-x_{d+1})]/2$.

Using (2.3) and (3.4), the reproducing kernel $P_n(W^B_{\kappa,\mu};x,y)$ of $\mathcal{V}_n(W^B_{\kappa,\mu})$ becomes

$$P_n(W^B_{\kappa,\mu};x,y) = \frac{n+\lambda_\mu}{\lambda_\mu}V^B_{\kappa,\mu}\left[C_n^\lambda(\langle\cdot,Y\rangle)\right](X), \tag{4.5}$$

where $X = (x,\sqrt{1-\|x\|^2})$ and $Y = (y,\sqrt{1-\|y\|^2})$ with $x,y\in B^d$, which is an integral formula according to the explicit formula (3.5). In particular, for the classical weight function W_μ, the explicit integral formula becomes [28]

$$P_n(W_\mu;x,y) = c_\mu\frac{n+\lambda_\mu}{\lambda_\mu}\int_{-1}^1 C_n^{\lambda_\mu}\left(\langle x,y\rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\,t\right)(1-t^2)^{\mu-1}dt,$$

where $\lambda_\mu = \mu + \frac{d-1}{2}$. The projection operator $L^2(W^B_{\kappa,\mu}) \to \mathcal{V}_n(W^B_{\kappa\mu})$ is defined by

$$\operatorname{proj}_{\mathcal{V}_n(B)}f(x) := a_{\kappa,\mu}\int_{B^d} f(y)P_n(W^B_{\kappa,\mu};x,y)W^B_{\kappa,\mu}(y)dy. \tag{4.6}$$

The explicit formula (4.5) of the reproducing kernel plays an important role in the study of the convergence of orthogonal expansions on B^d.

ETNA
Kent State University
etna@mcs.kent.edu

4.2. Weighted approximation on the ball. The operator $V_{\kappa,\mu}^B$ can be used to define, setting $x_{d+1} = \sqrt{1-\|x\|^2}$, a convolution structure $*_{\kappa,\mu}^B$ between $f \in L^1(W_\kappa)$ and $g \in L^1(w_{\lambda_\mu}; [-1,1])$, just as in (3.6),

$$(4.7)\quad (f *_{\kappa,\mu}^B g)(x) = \int_{B^d} f(y) V_{\kappa,\mu}^B [g(\langle \cdot, X \rangle)] (Y) W_{\kappa,\mu}^B (y) dy, \quad X = \left(x, \sqrt{1-\|x\|^2}\right).$$

The equation (4.5) and (4.6) show that $\mathrm{proj}_{\mathcal{V}_n(B)}$ can be written as a convolution of f with the Gagenbauer polynomial $C_n^{\lambda_\mu}$ under $*_{\kappa,\mu}^B$. It turns out that this convolution structure is related to the convolution structure $*_{\kappa,\mu}$ on the unit sphere:

$$(4.8)\qquad (f *_{\kappa,\mu}^B g)(x) = (F *_{\kappa,\mu} g)\left(x, \sqrt{1-\|x\|^2}\right)$$

where F is defined by $F(x, x_{d+1}) := f(x)$. Clearly we can also take this equation as the definition of $*_{\kappa,\mu}^B$. It follows immediately from (2.3) that $f *_{\kappa,\mu}^B g$ also satisfies Young's inequality. We can define an analogue of the weighed spherical means, T_θ^B, as follows: For $f \in L^1(W_{\kappa,\mu}^B)$, the operator $T_\theta^B f$ is defined implicitly by

$$(4.9)\qquad b_\lambda \int_0^\pi T_\theta^B f(x) g(\cos\theta)(\sin\theta)^{2\lambda+2\mu} d\theta = (f *_{\kappa,\mu}^B g)(x)$$

for every $g \in L^1(w_\lambda, [-1,1])$. Since the convolutions on B^d and on S^d are related by (4.8), it follows readily that the following relation holds:

$$(4.10)\qquad T_\theta^B f(x) = T_\theta^\kappa F\left(x, \sqrt{1-\|x\|^2}\right), \qquad x \in B^d,$$

where $F(x, x_{d+1}) = f(x)$, which can also be taken as the definition of T_θ^B. As a consequence of this relation, the properties of this operator follows immediately from those of $T_\theta^{\kappa,\mu}$ in Proposition 3.2:

PROPOSITION 4.2. *The means $T_\theta^B f$ satisfy the following properties:*

1. *Let $f_0(x) = 1$; then $T_\theta^B f_0(x) = 1$.*
2. *If $f \sim \sum_{n=0}^\infty \mathrm{proj}_{\mathcal{V}_n(B)} f$, then*

$$T_\theta^B f \sim \sum_{n=0}^\infty \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} \mathrm{proj}_{\mathcal{V}_n(B)} f.$$

3. *For $f \in L^p(W_{\kappa,\mu}^B)$, $1 \le p < \infty$, or $f \in C(B^d)$,*

$$\|T_\theta^B f\|_{W_{\kappa,\mu}^B, p} \le \|f\|_{W_{\kappa,\mu}^B, p} \qquad \textit{and} \qquad \lim_{\theta\to 0} \|T_\theta^B f - f\|_{W_{\kappa,\mu}^B, p} = 0.$$

The operator T_θ^B is called the generalized translation operator in [36], since the property 2 in the Proposition 4.2 implies that

$$\mathrm{proj}_{\mathcal{V}_n(B)} T_\theta^B f = \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} \mathrm{proj}_{\mathcal{V}_n(B)} f$$

which becomes, when $d = 1$ and $\kappa = 0$, the property satisfied by the translation operator T_s for the Gegenbauer weight function $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ on $[-1,1]$. The translation operator T_s is usually defined by

$$(4.11)\qquad T_s f(t) = b_{\lambda-1/2} \int_{-1}^1 f\left(st + u\sqrt{1-s^2}\sqrt{1-t^2}\right)(1-u^2)^{\lambda-1} du,$$

ETNA
Kent State University
etna@mcs.kent.edu

which plays an important role in the study of orthogonal expansions in Gegenbauer polynomials; see, for example, [4, 5, 7, 18, 22]. For $d = 1$ and $\kappa = 0$, we have $T_\theta^B = T_{\cos\theta}$. Furthermore, in [36] an analogue of (4.11) is found for the classical weight function W_μ:

PROPOSITION 4.3. *Let $U(x)$ be the unitary matrix whose first column is $x/\|x\|$ and $D(x) = \mathrm{diag}\{\sqrt{1-\|x\|^2}, 1, \ldots, 1\}$. Then the generalized translation operator for W_μ is an integral transform*

$$T_\theta^B f(x) = a_\mu \int_{B^d} f\left(\cos\theta x + \sin\theta y D(x) U(x)\right)(1-\|y\|^2)^{\mu-1} dy,$$

where y is considered as a row vector and $yD(x)$ is the matrix multiplication.

The mapping (2.1) and (4.10) can be used to give an integral equation for the weighted spherical means T_θ^κ for the weight function $h_\kappa(x) = |x_{d+1}|^\mu$ on the sphere S^d. We do not know if such an integral formula holds for T_θ^B with respect to the weight function $W_{\kappa,\mu}^B$ or equivalently for T_θ^κ for h_κ^2 defined in (1.1).

As a consequence of the property 3 of the Proposition 4.2 we can define a modulus of smoothness on B^d as follows: For $r > 0$, $1 \le p \le \infty$, and $f \in L^p(W_{\kappa,\mu}^B)$,

$$\omega_r(f,t)_{W_{\kappa,\mu}^B,p} = \sup_{0<\theta\le t} \|(I - T_\theta^B)^{r/2} f\|_{W_{\kappa,\mu}^B,p}. \tag{4.12}$$

By (4.10), this modulus of smoothness is related to the modulus $\omega_r(f;t)_{\kappa,\mu,p}$ associated with $h_{\kappa,\mu}$ as defined in (3.8). In fact, we have

$$\omega(f;t)_{W_{\kappa,\mu}^B,p} = \omega_r(F;t)_{\kappa,\mu,p}, \qquad F(x,x_{d+1}) = f(x). \tag{4.13}$$

Hence, properties of $\omega_r(f;t)_{W_{\kappa,\mu}^B,p}$ follow from those satisfied by $\omega_r(f;t)_{\kappa,\mu,p}$.

Using the differential-difference operator $D_{\kappa,\mu}^B$ in (4.2), we can also define a K-functional as follows: For $f \in L^p(W_{\kappa,\mu}^B)$, $r > 0$,

$$K_r(f;t)_{W_{\kappa,\mu}^B,p} := \inf\left\{\|f-g\|_{W_{\kappa,\mu}^B,p} + t^r \|(-D_{\kappa,\mu}^B)^{r/2} g\|_{W_{\kappa,\mu}^B,p}\right\}, \tag{4.14}$$

where the infimum is taken over all $g \in L^p(W_{\kappa,\mu}^B)$ for which $\|(-D_{\kappa,\mu}^B)^{r/2} g\|_{W_{\kappa,\mu}^B,p}$ is finite. Since the operator $D_{\kappa,\mu}^B$ is deduced from that of $\Delta_{h,0}$, there is also a connection between the K-functionals $K_r(f;t)_{W_{\kappa,\mu}^B,p}$ and $K_r(f;t)_{\kappa,\mu,p}$ associated with $h_{\kappa,\mu}$ as defined in (3.9). If fact, we also have

$$K(f;t)_{W_{\kappa,\mu}^B,p} = K_r(F;t)_{\kappa,\mu,p}, \qquad F(x,x_{d+1}) = f(x). \tag{4.15}$$

Consequently, the following equivalence follows from Theorem 3.3 right away:

THEOREM 4.4. *For $f \in L^p(W_{\kappa,\mu}^B)$, $1 \le p \le \infty$,*

$$c_1 \omega_r(f;t)_{W_{\kappa,\mu}^B,p} \le K_r(f;t)_{W_{\kappa,\mu}^B,p} \le c_2 \omega_r(f;t)_{W_{\kappa,\mu}^B,p},$$

where c_1 and c_2 are constants independent of f.

Again these two gadgets can be used to characterize the best approximation by polynomials. For $f \in L^p(W_{\kappa,\mu}^B)$, $1 \le p \le \infty$, let

$$E_n(f)_{W_{\kappa,\mu}^B,p} := \inf\left\{\|f - P\|_{W_{\kappa,\mu}^B,p} : P \in \Pi_n^d\right\}$$

denote the error of the best approximation by polynomials of degree at most n. Using the basic relations (2.1) and (2.2), we can prove that

$$E_n(f)_{W_{\kappa,\mu}^B,p} = E_n(F)_{\kappa,\mu,p}, \qquad F(x,x_{d+1}) = f(x).$$

ETNA
Kent State University
etna@mcs.kent.edu

Consequently, the following theorem follows immediately from Theorem 3.4.

THEOREM 4.5. *For $f \in L^p(W^B_{\kappa,\mu})$, $1 \le p \le \infty$,*

$$E_n(f)_{W^B_{\kappa,\mu},p} \le c\,\omega_r(f;n^{-1})_{W^B_{\kappa,\mu},p}.$$

On the other hand,

$$\omega_r(f;n^{-1})_{W^B_{\kappa,\mu},p} \le c\,n^{-r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{W^B_{\kappa,\mu},p}.$$

To further illustrate how the results on the unit ball can be derived from those on the sphere, we state and prove a theorem analogous to Proposition 3.5. Let $\eta \in C^k[0,\infty)$ as in (3.12). We define a sequence of operators η_n^B by

$$\eta_n^B f := \sum_{k=0}^{\infty}\eta\left(\frac{k}{n}\right)\mathrm{proj}_{\mathcal{V}_k(B)}f. \tag{4.16}$$

PROPOSITION 4.6. *Let $f \in L^p(W^B_{\kappa,\mu})$, $1 \le p < \infty$, and $f \in C(B^d)$ if $p = \infty$. If $k \ge \lfloor\lambda\rfloor + 1$ then*

1. *$\eta_n^B f \in \Pi_{2n-1}$ and $\eta_n^B P = P$ for $P \in \Pi_n$;*
2. *for $n > 0$, $\|\eta_n^B f\|_{W^B_{\kappa,\mu},p} \le c\|f\|_{W^B_{\kappa,\mu},p}$;*
3. *for $n > 0$, $\|f - \eta_n^B f\|_{W^B_{\kappa,\mu},p} \le cE_n(f)_{W^B_{\kappa,\mu},p}$.*

Proof. Using the definition of the operator V_κ^B, it follows easily from (3.4) and (4.5) that

$$\begin{aligned} P_n(W^B_{\kappa,\mu};x,y) = \frac{1}{2}\Big[&P_n(h^2_{\kappa,\mu};(x,\sqrt{1-\|x\|^2}),(y,\sqrt{1-\|y\|^2})) \\ &+P_n(h^2_{\kappa,\mu};(x,\sqrt{1-\|x\|^2}),(y,-\sqrt{1-\|y\|^2}))\Big], \end{aligned} \tag{4.17}$$

from which we derive from (3.3) and (4.6) that the following relation holds:

$$\mathrm{proj}_{\mathcal{V}_n(B)}f(x) = \mathrm{proj}_{\mathcal{H}_n}F\left(x,\sqrt{1-\|x\|^2}\right), \qquad F(x,x_{d+1}) := f(x).$$

Consequently, $\eta_n^B f(x) = \eta_n F(x,\sqrt{1-\|x\|^2})$, from which the stated results follow from the equation (2.3) and Proposition 4.6. ◻

5. Analysis on the simplex.

In this section we show how results on the simplex can be deduced from those on the ball. Since the basic relation (2.4) amounts to a non-linear change of variables, the deduction is more complicated than the deduction from the sphere to the ball.

Recall the weight function $W^T_{\kappa,\mu}$ given in (1.3). We will consider $L^p(W^T_{\kappa,\mu})$ space with norm $\|\cdot\|_{W^T_{\kappa,\mu},p}$ defined similarly as $\|\cdot\|_{W^B_{\kappa,\mu},p}$. Recall the map ψ in (2.5). Using (2.6), it is easy to see that $f \in L^p(W^T_{\kappa,\mu})$ is equivalent to $f\circ\psi \in L^p(W^B_{\kappa,\mu})$; furthermore, we have

$$\|f\|_{W^T_{\kappa,\mu},p} = \|f\circ\psi\|_{W^B_{\kappa,\mu},p}$$

5.1. Orthogonal polynomials on the simplex.

Under the mapping ψ, $W^T_{\kappa,\mu}(x)$, $x \in T^d$, becomes $W^B_{\kappa,\mu}(x)$, $x \in B^d$, since the Jacobian of changing variables from $x \mapsto \psi(x)$ is $2^{-d}(x_1\cdots x_d)^{-1/2}$. Let $\mathcal{V}_n(W^T_{\kappa,\mu})$ denote the space of orthogonal polynomials of degree n with respect to the inner product

$$\langle f,g\rangle_T := a_{\kappa,\mu}\int_{T^d}f(x)g(x)W^T_{\kappa,\mu}(x)dx$$

ETNA
Kent State University
etna@mcs.kent.edu

on T^d. Under the mapping (2.5), the inner product $\langle\cdot,\cdot\rangle_T$ is related to $\langle\cdot,\cdot\rangle_B$ by

$$\langle f, g\rangle_T = \langle f\circ\psi, g\circ\psi\rangle_B,$$

from which the relation between $\mathcal{V}_n(W^T_{\kappa,\mu})$ and $\mathcal{V}_n(W^B_{\kappa,\mu})$ follows immediately. Let us define $G\mathcal{V}_{2n}(W^B_{\kappa,\mu}) := \mathcal{V}_{2n}(W^B_{\kappa,\mu}) \cap G\Pi_{2n}$ on B^d, which contains polynomials in $\mathcal{V}_{2n}(W^B_{\kappa,\mu})$ that are invariant under $\mathbb{Z}_2^d$ (invariant under sign changes).

PROPOSITION 5.1. *The mapping* (2.5) *induces an one-to-one correspondence between* $R \in \mathcal{V}_n(W^T_\kappa)$ *and* $R\circ\psi \in G\mathcal{V}_{2n}(W_\kappa)$.

Since $f\circ\psi$ is invariant under $\mathbb{Z}_2^d$, the mapping ψ also translates the operator $D^B_{\kappa,\mu}$ defined in (4.2) to the differential operator $D^T_{\kappa,\mu}$ [30] defined by

$$D^T_{\kappa,\mu} := \sum_{i=1}^d x_i(1-x_i)\partial_i^2 - 2\sum_{1\le i<j\le d} x_i x_j \partial_i\partial_j + \sum_{i=1}^d \left(\left(\kappa_i + \frac{1}{2}\right) - \lambda_\mu x_i\right)\partial_i, \tag{5.1}$$

and the orthogonal polynomials in $\mathcal{V}_n(W^T_{\kappa,\mu})$ are the eigenfunctions of $D^T_{\kappa,\mu}$,

$$D^T_\kappa P = -n(n+\lambda_\mu)P, \qquad P \in \mathcal{V}_n(W^T_{\kappa,\mu}).$$

This is the classical partial differential equations satisfied by orthogonal polynomials on T^d. Since the elements in $G\mathcal{V}_{2n}(W^T_{\kappa,\mu})$ are of the form $R\circ\psi$ with $R\in\mathcal{V}_n(W^T_{\kappa,\mu})$, the reproducing kernel of $\mathcal{V}_n(W^T_{\kappa,\mu})$ satisfies

$$P_n(W^T_{\kappa,\mu}; x, y) = 2^{-d}\sum_{\varepsilon\in\mathbb{Z}_2^d} P_{2n}(W^B_{\kappa,\mu}; x^{1/2}, \varepsilon y^{1/2}) \tag{5.2}$$

where $x^{1/2} := (\sqrt{x_1},\dots,\sqrt{x_d})$ and $\varepsilon u = (\varepsilon_1 u_1, \dots, \varepsilon_d u_d)$. This equation suggests the definition of the following operator defined on functions on $\mathbb{R}^{d+1}$,

$$V^T_{\kappa,\mu} F(x, x_{d+1}) = 2^{-d}\sum_{\varepsilon\in\mathbb{Z}_2^d} V^B_{\kappa,\mu} F(\varepsilon x, x_{d+1}), \tag{5.3}$$

where $V^B_{\kappa,\mu}$ is defined in (4.4). Using the fact that $C_n^\lambda(t) = c p_n^{(\lambda-1/2,-1/2)}(2t^2-1)$, where c is a constant and $p_n^{(\alpha,\beta)}$ denotes the orthonormal Jacobi polynomial of degree n, it follows from (5.2) and (5.3) that

$$P_n(W^T_{\kappa,\mu}; x, y) = p_n^{(\lambda_\mu-\frac{1}{2},-\frac{1}{2})}(1) V^T_{\kappa,\mu}\left[p_n^{(\lambda_\mu-\frac{1}{2},-\frac{1}{2})}\left(2\langle\cdot, Y^{1/2}\rangle^2 - 1\right)\right](X^{1/2}), \tag{5.4}$$

where $X^{1/2} = \left(\sqrt{x_1},\dots,\sqrt{x_d},\sqrt{1-x_1-\cdots-x_d}\right)$ and $Y^{1/2}$ is defined similarly. By the definition of $V^B_{\kappa,\mu}$ and the equation (3.5), this gives an explicit compact formula for the kernel. Again, the projection operator $L^2(W^T_{\kappa,\mu}) \to \mathcal{V}_n(W^T_{\kappa,\mu})$ is defined by

$$\operatorname{proj}_{\mathcal{V}_n(T)} f(x) := a_{\kappa,\mu}\int_{T^d} f(y) P_n(W^T_{\kappa,\mu}; x, y) W^T_{\kappa,\mu}(y)dy.$$

The formula (5.4) of the reproducing kernel plays an essential role in the study of orthogonal expansions on T^d. It shows, in particular, that the expansions on T^d are connected to Jacobi expansions, rather than Gegenbauer expansions.

ETNA
Kent State University
etna@mcs.kent.edu

5.2. Weighted approximation on the simplex. Using the operator $V_{\kappa,\mu}^T$, we can define a convolution $f *_{\kappa,\mu}^T g$ for $f \in L^1(W_\kappa)$ and $g(2\{\cdot\}^2-1) \in L^1(w_\lambda, [-1,1])$ on T^d as in (4.7). The basic mapping (2.5) shows that

$$((f *_{\kappa,\mu}^T g) \circ \psi)(x) = \left((f\circ\psi) *_{\kappa,\mu}^B g\left(2\{\cdot\}^2-1\right)\right)(x), \tag{5.5}$$

which can also be taken as a definition of $*_{\kappa,\mu}^T$. The equations (5.4) shows that $\mathrm{proj}_{\mathcal{V}_n(T)}$ can be written as a convolution of f and the Jacobi polynomial. Using the convolution, we can define an analogue of a generalized translation operator T_θ^T by

$$b_\lambda \int_0^\pi T_\theta^T f(x) g(\cos 2\theta)(\sin\theta)^{2\lambda_\mu} d\theta = (f *_{\kappa,\mu}^T g)(x) \tag{5.6}$$

for every $g \in L^1(w_\lambda, [-1,1])$. Note that we have $g(\cos 2\theta)$ in contrast to $g(\cos\theta)$ in (4.9), which comes from $2\cos^2\theta - 1 = \cos 2\theta$. From the relation (5.5) it follows that

$$((T_\theta^T f)\circ\psi)(x) = T_\theta^B(f\circ\psi)(x), \qquad x \in T^d, \tag{5.7}$$

from which the properties of T_θ^T follows from those in Proposition 4.2.

PROPOSITION 5.2. *The means $T_\theta^T f$ satisfy the following properties:*

1. *Let $f_0(x) = 1$; then $T_\theta^T f_0(x) = 1$.*
2. *If $f \sim \sum_{n=0}^\infty \mathrm{proj}_{\mathcal{V}_n(T)} f$, then*

$$T_\theta^T f \sim \sum_{n=0}^\infty \frac{p_n^{(\lambda-\frac12,-\frac12)}(\cos 2\theta)}{p_n^{(\lambda-\frac12,-\frac12)}(1)} \mathrm{proj}_{\mathcal{V}_n(T)} f.$$

3. *For $f \in L^p(W_{\kappa,\mu}^T)$, $1 \le p < \infty$, or $f \in C(T^d)$,*

$$\|T_\theta^T f\|_{W_{\kappa,\mu}^T,p} \le \|f\|_{W_{\kappa,\mu}^T,p} \qquad \textit{and} \qquad \lim_{\theta\to 0} \|T_\theta^T f - f\|_{W_{\kappa,\mu}^T,p} = 0.$$

Just like the case of B^d, the last property of this proposition suggests the following definition of a modulus of smoothness on T^d: For $r > 0$ and $1 \le p \le \infty$,

$$\omega_r(f,t)_{W_{\kappa,\mu}^T,p} = \sup_{0<\theta\le t} \|(I - T_\theta^T)^{r/2} f\|_{W_{\kappa,\mu}^T,p}. \tag{5.8}$$

Under the mapping (2.5) the relation (5.7) and (4.12) immediately show that

$$\omega_r(f,t)_{W_{\kappa,\mu}^T,p} = \omega_r(f\circ\psi, t)_{W_{\kappa,\mu}^B,p}. \tag{5.9}$$

As in the case of B^d, we can use the operator $D_{\kappa,\mu}^T$ in (5.1) to define a K-functional as follows: For $f \in L^p(W_{\kappa,\mu}^T)$, $r > 0$,

$$K_r(f;t)_{W_{\kappa,\mu}^T,p} := \inf\left\{\|f-g\|_{W_{\kappa,\mu}^T,p} + t^r \|(-D_{\kappa,\mu}^T)^{r/2} g\|_{W_{\kappa,\mu}^T,p}\right\}, \tag{5.10}$$

where the infimum is taken over all $g \in L^p(W_{\kappa,\mu}^T)$ for which $\|(-D_{\kappa,\mu}^T)^{r/2} g\|_{W_{\kappa,\mu}^T,p}$ is finite. Since $D_{\kappa,\mu}^T$ is obtained from $D_{\kappa,\mu}^B$ by a change of variable (2.5), it follows that

$$K_r(f;t)_{W_{\kappa,\mu}^T,p} = K_r(f\circ\psi; 2t)_{W_{\kappa,\mu}^B,p}, \tag{5.11}$$

ETNA
Kent State University
etna@mcs.kent.edu

Consequently, the following equivalence follows from Theorem 3.3 right away:

THEOREM 5.3. *For $f \in L^p(W^T_{\kappa,\mu})$, $1 \le p \le \infty$,*

$$c_1\omega_r(f;t)_{W^T_{\kappa,\mu},p} \le K_r(f;t)_{W^T_{\kappa,\mu},p} \le c_2\omega_r(f;t)_{W^T_{\kappa,\mu},p},$$

where c_1 and c_2 are constants independent of f.

As before, the two gadgets can be used to characterize the best approximation by polynomials. For $f \in L^p(W^T_{\kappa,\mu})$, $1 \le p \le \infty$, let

$$E_n(f)_{W^T_{\kappa,\mu},p} := \inf\left\{\|f-P\|_{W^T_{\kappa,\mu},p} : P \in \Pi^d_n\right\}$$

denote the error of the best approximation by polynomials of degree at most n. Using (2.5) and taking into consideration of the symmetry of $P \circ \psi$, we can show that

$$E_n(f)_{W^T_{\kappa,\mu},p} = E_n(f\circ\psi)_{W^B_{\kappa,\mu},p}.$$

Hence, the following characterization follows immediately from Theorem 4.5 and (5.9):

THEOREM 5.4. *For $f \in L^p(W^T_{\kappa,\mu})$, $1 \le p \le \infty$,*

$$E_n(f)_{W^T_{\kappa,\mu},p} \le c\,\omega_r(f;n^{-1})_{W^T_{\kappa,\mu},p}.$$

On the other hand,

$$\omega_r(f;n^{-1})_{W^T_{\kappa,\mu},p} \le c\,n^{-r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{W^T_{\kappa,\mu},p}.$$

5.3. Additional difficulty for analysis on the simplex. In the above discussion we put our emphasis on the similarity between results on the ball and on the simplex. In fact, most of the results on these two domains appear to be equivalent in the sense that they can be deduced from each other, and both can be deduced from the results on the sphere. However, for certain problems, the simplex is more difficult to work with. The difficulty appears in the connection between $P_n(W^T_{\kappa,\mu};\cdot,\cdot)$ and $P_{2n}(W^B_{\kappa,\mu};\cdot,\cdot)$ shown in (5.2), which forces us to switch from $C^\lambda_n(t)$ to $p_n^{(\lambda-1/2,1/2)}(t)$ as in (5.4). As a consequence, the results for certain problems on T^d will not follow as an exact consequence of those on B^d. This is so especially for the study of orthogonal expansions.

To illustrate this point, let $\eta \in C^k[0,\infty)$ as in (3.12) and define operators η^T_n by

$$\eta^T_n f := \sum_{k=0}^{\infty}\eta\left(\frac{k}{n}\right)\operatorname{proj}_{\mathcal{V}_k(T)}f. \tag{5.12}$$

The main properties of $\eta^T_n f$ is the following theorem analogous to Proposition 4.6:

PROPOSITION 5.5. *Let $f \in L^p(W^T_{\kappa,\mu})$, $1 \le p < \infty$, and $f \in C(T^d)$ if $p = \infty$. If $k \ge \lfloor\lambda\rfloor + 1$ then*

1. *$\eta^T_n f \in \Pi_{2n-1}$ and $\eta^T_n P = P$ for $P \in \Pi_n$;*
2. *for $n > 0$, $\|\eta^T_n f\|_{W^T_{\kappa,\mu},p} \le c\|f\|_{W^T_{\kappa,\mu},p}$;*
3. *for $n > 0$, $\|f - \eta^T_n f\|_{W^T_{\kappa,\mu},p} \le cE_n(f)_{W^T_{\kappa,\mu},p}$.*

This theorem, however, does not follow as a consequence of Proposition 4.6. In fact, the relation (5.4) shows that $\operatorname{proj}_{\mathcal{V}_k(T)}$ is related to $\operatorname{proj}_{\mathcal{V}_{2k}(B)}$, which shows that there is no direct relation between η^T_n and η^B_n, as each is a sum over k from 0 to n. The proof of this theorem can be modeled after the proof of Proposition 3.5 in [34], which goes back to [14].

The same phenomenon also appears when we try to find the critical index of the Cesàro (C,δ)-means of the orthogonal expansions. In fact, for $W^B_{\kappa,\mu}$ on B^d, the sharp critical index was established in [16], whiles for $W^T_{\kappa,\mu}$ on T^d the result was not established for all parameter ranges. The study of (C,δ)-means of orthogonal expansions on T^d does not follow from the one on B^d. See [16] for details.

For $d = 1$, $W^T_\kappa(x) = x^{\kappa_1-1/2}(1-x)^{\kappa_1-1/2}$ is the Jacobi weight function on $[0,1]$. However, T^T_θ is not the usual translation operator associated with the product formula of the Jacobi series. In fact, it corresponds to the "wrong" product formula

$$P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y) = c_n \int_{-1}^{1}\int_{-1}^{1} C_{2n}^{\alpha+\beta+1}(z(t,s,x,y))(1-s^2)^{\alpha-1/2}(1-t^2)^{\beta-1/2}ds\,dt$$

where $z(t,s,\cos\theta,\cos\phi) = \cos\theta\cos\phi s + \sin\theta\sin\phi t$. Finally, we mention that it would be interesting to find if T^T_θ can be written as an integral transform, like the formula of T^B_θ for W_μ in Proposition 4.3.

6. Other problems on the unit ball and on the simplex. Besides orthogonal polynomials and approximation discussed in the previous sections, the connection between S^d, B^d and T^d can be useful in several other problems in analysis. In this section we briefly discuss three other problems.

6.1. Polynomial of least deviation from zero. For $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}_0^d$, we define the monomial $x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$. The degree of the monomial x^α is $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Let Ω be a region in $\mathbb{R}^d$. If $p^*(x)$ is a polynomial of best approximation to the monomial x^α in the uniform norm on Ω, then $x^\alpha - p^*(x)$ is called the polynomial of least deviation from zero. We shall also call $p^*(x)$ a least polynomial.

Using the basic relations (2.1) and (2.5) between the three domains and the relations between polynomial spaces as given in Lemma 2.1 and Lemma 2.2, one can often reduce the problem of finding least polynomials on B^d to that of S^d and to that of T^d. As an illustration we state one such result [32].

THEOREM 6.1. *Let $\alpha \in \mathbb{N}_0^d$ and write $2\alpha = (2\alpha_1,\ldots,2\alpha_d)$ and $|\alpha| = n$. If $p^*(x)$ is a least polynomial for x^α on T^d, then $p^*(x_1^2,\ldots,x_d^2)$ is a least polynomial for $x^{2\alpha}$ on B^d; conversely, if q^* is a least polynomial for $x^{2\alpha}$ on B^d in the form $q^*(x) = p^*(x_1^2,\ldots,x_d^2)$, then $p^*(x)$ is a least polynomial for x^α on T^d.*

For $d = 2$, the least polynomials to $x^n y^m$ from Π^2_{n+m-1} on the domain B^2 and T^2 were known. Their relation as stated in the theorem was used in [6]. For $d > 2$, only a few examples of least polynomials were known, see [1, 2, 19, 21, 32]. The above theorem can be used to find least polynomials for monomials of lower degrees. It shows, in particular, that in order to find a least polynomial for $x^{2\alpha}$ on B^d, it is enough to work with x^α, which has lower degree, on T^d. For example, one least polynomial for $x_1x_2x_3$ on T^3 is given by [32]

$$R_3(x) = 72x_1x_2x_3 - 4(x_1+x_2+x_3) + 4(x_1+x_2+x_3)^2 - 8(x_1x_2+x_2x_3+x_1x_3) + 1,$$

which gives immediately a least polynomial for $x_1^2x_2^2x_3^2$ on B^3 using the theorem.

6.2. Cubature formula. The formulas (2.3) and (2.6) relate the integral in three regions. Together with the connection of the polynomials on these domains, they lead to relations between cubature formulas on S^d, B^d and T^d. These relations were discussed in [26, 27] and they were used in [12, 13] to generate new cubature formulas. We state only one theorem that captures the spirit of such a result.

THEOREM 6.2. *If there is a cubature formula of degree M on T^d giving by*

$$\int_{T^d} f(u)W^T_{\kappa,\mu}(u)du = \sum_{i=1}^{N} \lambda_i f(u_i), \tag{6.1}$$

with all $u_i \in T^d$*, then there is a cubature formula of degree* $2M+1$ *on* B^d *given by*

$$\int_{B^d} g(x) W^B_{\kappa,\mu}(x)dx = \sum_{i=1}^{N} \lambda_i 2^{-k(u_i)} \sum_{\varepsilon \in \mathbb{Z}_2^d} f(\varepsilon_1 \sqrt{u_{i,1}}, \dots \varepsilon_d \sqrt{u_{i,d}}) \tag{6.2}$$

where $k(u)$ *denote the number of non-zero components in* u*. Moreover, a cubature formula of degree* $2M+1$ *in the form of* (6.1) *on* B^d *implies a cubature formula of degree* M *in the form of* (6.2) *on* T^d*.*

A similar result holds for cubature formulas on S^d and on B^d, which also extends to a relation between cubature formulas on S^d and on T^d. We note that a cubature for the surface measure on S^d corresponds to a cubature for "Chebyshev" weight function $W_0(x)$ on B^d, which in turn corresponds to a cubature for "Chebyshev" weight function $W_0^T(x)$ on T^d.

6.3. Polynomial Interpolation. The relation (2.1) between polynomial spaces on B^d and those on S^d can also be used in the problem of polynomial interpolation.

Let $M_n^d = \dim \Pi_n^d$. We consider the following interpolation problem on B^d:

PROBLEM 1. *Let* E *be a set of* M_n^d *points on* B^d*. Find conditions on* E *such that, for any given data* $\{f_i\}$*, there is a unique polynomial* $Q \in \Pi_n^d$ *satisfying* $Q(x_i) = f_i$*, for* $x_i \in E$ *and* $1 \le i \le M_n^d$*,*

Let $N_n^d = \dim \Pi_n(S^d)$. The interpolation problem on S^d that we consider is:

PROBLEM 2. *Let* X *be a set of* N_n^d *distinct points on* S^d*. Find conditions on* X *such that, for any given data* $\{f_i\}$*, there is a unique polynomial* $S \in \Pi_n(S^d)$ *satisfying* $S(y_i) = f_i$*, for* $y_i \in X$ *and* $1 \le i \le N_n^d$*.*

We call the point set X on S^d symmetric if $x = (x', x_{d+1}) \in X$ implies that $(x', -x_{d+1}) \in X$, where $x' = (x_1, \dots, x_d)$. From the relation (2.1) between polynomials spaces, solutions of these two problems are related as follows [33]:

THEOREM 6.3. *Let* E *be a set of* M_n^d *points on* B^d *that solves Problem 1. Assume that* E *contains exactly* $M_n^d - M_{n-1}^d$ *points on the boundary* S^{d-1} *of* B^d *and that* $E^\circ := E \setminus (E \cap S^{d-1})$ *solves Problem 1 for* Π_{n-1}^d*. Define*

$$X_E = \{(x', 0) : x' \in E \cap S^{d-1}\} \bigcup \left\{ (x', \pm x_{d+1}) : x_{d+1} = \sqrt{1 - \|x'\|^2}, \quad x' \in E^\circ \right\}.$$

Then X_E *solves Problem 2. On the other hand, if* X *solves Problem 2,* X *is symmetric, and there are exactly* $M_n^d - M_{n-1}^d$ *points on the hyperplane* $\{x \in \mathbb{R}^{d+1} : x_{d+1} = 0\}$*, then* $E_X = \{x' : (x', x_{d+1}) \in X \cap S_+^{d+1}\}$ *solves Problem 1.*

The relation (2.5) and the relation between the polynomial spaces on B^d and T^d as described in Lemma 2.2 can also be used for interpolation problem. However, because the mapping $x \mapsto \psi(x)$ is nonlinear, a polynomial of degree n that interpolates on N_n^d points on T^d corresponds to an interpolation polynomial on B^d that belongs to a subspace of Π_{2n}^d; see [33] for a discussion in the case of $d = 2$.

Acknowledgment. This paper is an extended version of my plenary talk in the conference Constructive Functions, Georgia Tech, Atlanta, November 7-9, 2004. I thank the organizers for their invitation. I'm indebted to an anonymous referee for his careful reading and numerous corrections.

REFERENCES

[1] N. N. ANDREEV AND V. A. YUDIN, *Polynomials of least deviation from zero, and Chebyshev-type cubature formulas*, Proc. Steklov Inst. Math., 232 (2001), pp. 39–51.

[2] N. N. ANDREEV AND V. A. YUDIN, *Best approximation of polynomials on the sphere and on the ball*, in Recent Progress in Multivariate Approximation (Witten-Bommerholz, 2000), pp. 23–30, W. Haussmann, K. Jetter, and M. Reimer, eds., Internat. Ser. Numer. Math., 137, Birkhäuser, Basel, 2001.

[3] P. APPELL AND J. K. DE FÉRIET, *Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[4] R. ASKEY AND S. WAINGER, *On the behavior of special classes of ultraspherical expansions*, I, II, J. Anal. Math., 15 (1965), pp. 193–220, 221–244.

[5] H. BERENS, P. L. BUTZER AND S. PAWELKE, *Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten*, Publ. Res. Inst. Math. Sci. Ser. A., 4 (1968), pp. 201–268.

[6] B. D. BOJANOV, W. HAUSSMANN, AND G. P. NIKOLOV, *Bivariate polynomials of least deviation from zero*, Canad. J. Math., 53 (2001), pp. 489–505.

[7] P. L. BUTZER, *Legendre transform methods in the solution of basic problems in algebraic approximation*, in Functions, series, operators, vol. I, II, Budapest, 1980, pp. 277–301, B. Sz.-Nagy and J. Szabados, eds., Colloq. Math. Soc. Bolyai, 35, North-Holland, Amsterdam, 1983.

[8] A. P. CALDERON AND A. ZYGMUND, *On a problem of Mihlin*, Trans. Amer. Math. Soc., 78 (1955), pp. 209–224. Addentum, 84 (1957), pp. 559–560.

[9] C. F. DUNKL, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc., 311 (1989), pp. 167–183.

[10] C. F. DUNKL AND YUAN XU, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, 81, Cambridge Univ. Press, 2001.

[11] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher transcendental functions*, McGraw-Hill, vol 2, New York, 1953.

[12] S. HEO AND YUAN XU, *Invariant cubature formulae for spheres and balls by combinatorial methods*, SIAM J. Numer. Anal., 38 (2000), pp. 626–638.

[13] S. HEO AND YUAN XU, *Constructing fully symmetric cubature formulae for the sphere*, Math. Comp., 70 (2001), pp. 269–279

[14] A. I. KAMZOLOV, *The best approximation of the classes of functions $W_p^\alpha(S^n)$ by polynomials in spherical harmonics*, Mat. Zametki, 32 (1982), pp. 285–293; English transl in Math Notes, 32 (1983), pp. 622–628.

[15] L. Q. LI AND K. Y. WANG, *Harmonic analysis and approximation on the unit sphere*, Science Press, Beijing, 2000.

[16] ZH.-K, LI AND YUAN XU, *Summability of orthogonal expansions I, on unit sphere, and II, on ball and simplex*, J. Approx. Theory, 122 (2003), pp. 267–333.

[17] P. I. LIZORKIN AND S. M. NIKOLSKII, *Approximation theory on the sphere*, Proc. Steklov Inst. Math., 172 (1987), pp. 295–302.

[18] S. PAWELKE, *Über die Approximationsordnung bei Kugelfunktionen und algebraischen Polynomen*, Tôhoku Math. J., 24 (1972), pp. 473–486.

[19] T. J. RIVLIN AND H. S. SHAPIRO, *A unified approach to certain problems of approximation and minimization*, J. Soc. Indust. Appl. Math., 9 (1961), pp. 670–699.

[20] KH. RUSTAMOV, *On approximation of functions on the sphere*, translation in Russian Acad. Sci. Izv. Math., 43 (1994), pp. 311–329.

[21] H. S. SHAPIRO, *Some theorems on Cebysev approximation II*, J. Math. Anal. Appl., 17 (1967), pp. 262–268.

[22] R. L. STENS AND M. WEHRENS, *Legendre transform methods and best algebraic approximation*, Comment. Math. Prace Mat., 21 (1980), pp. 351–380.

[23] G. SZEGŐ, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Providence, 4th edition, 1975.

[24] YUAN XU, *Orthogonal polynomials for a family of product weight functions on the spheres*, Canad. J. Math., 49 (1997), pp. 175–192.

[25] YUAN XU, *Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups*, Proc. Amer. Math. Soc., 125 (1997), pp. 2963–2973.

[26] YUAN XU, *Orthogonal polynomials and cubature formulae on spheres and on balls*, SIAM J. Math. Anal., 29 (1998), pp. 779–793.

[27] YUAN XU, *Orthogonal polynomials and cubature formulae on spheres and on simplices*, Methods Anal. and Appl., 5 (1998), pp. 169–184.

[28] YUAN XU, *Summability of Fourier orthogonal series for Jacobi weight on a ball in $\mathbb{R}^d$*, Trans. Amer. Math. Soc., 351 (1999), pp. 2439–2458.

[29] YUAN XU, *Orthogonal polynomials and summability in Fourier orthogonal series on spheres and on balls*, Math. Proc. Cambridge Phil. Soc., 31 (2001), pp. 139–155.

[30] YUAN XU, *Orthogonal polynomials on the ball and on the simplex for weight functions with reflection symmetries*, Constr. Approx., 17 (2001), pp. 383–412.

[31] YUAN XU, *Approximation by means of h-harmonics polynomials on the unit sphere*, Adv. in Comp. Math., 21 (2004), pp. 37–58.

[32] YUAN XU, *On polynomials of least deviation from zero in several variables*, Experimental Math., 13 (2004), pp. 103–112.

[33] YUAN XU, *Polynomial interpolation on the unit sphere and on the unit ball*, Adv. in Comp. Math., 20 (2004), pp. 247–260.
[34] YUAN XU, *Weighted approximation of functions on the unit sphere*, Const. Approx., 21 (2005), pp. 1–28.
[35] YUAN XU, *Almost Everywhere Convergence of orthogonal expansion of several variables*, Const. Approx., 22 (2005), pp. 67–93.
[36] YUAN XU, *Generalized translation operator and approximation in several variables*, J. Comp. Appl. Math., 178 (2005), pp. 489–512.