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ON NORMS OF FACTORS OF MULTIVARIATE POLYNOMIALS ON CONVEX BODIES*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Estimation of norms of factors of polynomials is a widely investigated extremal problem with numerous applications in functional analysis, number theory, approximation theory. In this note we study the following problem: let K be a convex body in $\mathbb{R}^d$ and consider a product of polynomials qr, where q is arbitrary and r is a monic multivariate polynomial. The goal is to find an upper bound for the uniform norm of q on K provided that such bound for qr is known.

Key words. multivariate polynomials, norms of factors, convex bodies

AMS subject classification. 41A17, 41A63

1. Introduction. Estimation of norms of factors of polynomials is a classical extremal problem which has been widely investigated for various norms. This extremal problem frequently arises in number theory, functional analysis, approximation theory, and therefore it has been studied by many experts in these fields. (For corresponding results and references see [1].) Most of the known results are related to univariate polynomials. In the present note we shall consider this question for multivariate polynomials on convex bodies. A typical problem for norms of factors can be formulated as follows: given a polynomial $p(x)$, $|p(x)| \leq 1$ on $I := [-1, 1]$ with factorization

$$p(x) = \prod_{j=1}^{m}(x - a_j)q_n(x)$$

provide sharp upper bounds for the norm of its factor q_n of degree at most n. In other words we want to estimate the norm of q_n provided that the norm of its product with a monic polynomial is given. The above question in one variable can be resolved using either Remez or Markov inequalities.

A. Solution by the Remez inequality. Set

$$E_\delta := \bigcup_{j=1}^{m}\left(a_j - \frac{\delta}{2m}, a_j + \frac{\delta}{2m}\right).$$

Clearly, $\mu_1(E_\delta) \leq \delta$, and for any $x \notin I \diagdown E_\delta$ and q_n as above we have

$$|q_n(x)| \leq \left(\frac{2m}{\delta}\right)^m.$$

Here and in what follows $\mu_d(\ldots)$ stands for the Lebesgue measure in $\mathbb{R}^d$, while $\|\cdots\|_K$ indicates the uniform norm on K. Then by the Remez inequality (see [1]) with a proper absolute constant $0 \leq c$ we have

$$\|q_n\|_I \leq e^{cn\delta^{\frac{1}{2}}}\left(\frac{2m}{\delta}\right)^m.$$

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Thus setting $\delta = n^{-2}$ yields $\|q_n\|_I = O(n^{2m})$.

B. Solution by the Markov inequality. Let $a \in I$ be such that q_n attains its norm at this point. Then by the Markov inequality [1] for any $x \in I_1 := I \cap \left(a - \frac{1}{2n^2}, a + \frac{1}{2n^2}\right)$

$$|q_n(x)| \geq |q_n(a)| - |x-a| \|q_n'\|_I \geq \|q_n\|_I - |x-a| n^2 \|q_n\|_I \geq \frac{1}{2}\|q_n\|_I. \tag{1.1}$$

Furthermore, since I_1 is an interval of length at least $\frac{1}{2n^2}$ and $g(x) := \prod_{j=1}^{m}(x - a_j)$ is a monic polynomial we have by the well known Chebyshev theorem

$$\|g\|_{I_1} \geq 2^{-m+1}\left(\frac{1}{4n^2}\right)^m \geq 2^{-3m+1}n^{-2m}.$$

Finally, this and (1.1) yield that whenever $y \in I_1$ is such that $g(x)$ attains its uniform norm on I_1 at this point we have

$$\|q_n\|_I \leq 2|q_n(y)| \leq \frac{2}{|g(y)|} \leq 2^{3m}n^{2m} = O(n^{2m}).$$

The asymptotic estimate $O(n^{2m})$ obtained above for $\|q_n\|_I$ in case of a fixed m is in general the best possible. Indeed, it is shown in [3] that for every $\beta > 0$ there exist polynomials q_n of degree n such that $|(x-1)^\beta q_n(x)| \leq 1$ whenever $x \in I$ and at the same time $\|q_n\|_I \geq cn^{2\beta}$ with a proper positive constant c depending only on β. However, for large m the estimate $O(n^{2m})$ becomes inefficient and it can be replaced (using another method) by $O(A^{n+m})$ with a suitable constant $A \geq 1$ (see [1]).

2. New Results. Our goal is to solve the problem discussed above for multivariate polynomials on convex bodies. Both Remez and Markov inequalities are well developed in the multivariate setting but the approach using Markov inequality seems to be more suitable for several variables.

Thus we consider a convex body K in $\mathbb{R}^d$ and the space of real multivariate polynomials in $\mathbb{R}^d$ of total degree at most n denoted as usual by P_n^d. As in the univariate case we shall consider products of $q_n \in P_n^d$ with "monic" polynomials. We shall call a polynomial

$$r(x) = \sum_{k_1+\cdots+k_d \leq n} a_{\mathbf{k}} x^{\mathbf{k}} \in P_n^d$$

monic if the sum of absolute values of its leading coefficients is 1, i.e., $\sum_{k_1+\cdots+k_d = n} |a_{\mathbf{k}}| = 1.$

Furthermore, let $B(a,r)$ denote the ball in $\mathbb{R}^d$ with center at a and radius r, and for the convex body K set

$$r(K) := \sup\{r : B(a,r) \subset K, a \in K\},$$

$$R(K) := \inf\{R : B(a,R) \supset K, a \in K\}.$$

As above $\|\cdots\|_K$ is the uniform norm on K.

THEOREM 2.1. *Let $K \subset \mathbb{R}^d$ be a convex body, $p = q_n r_m$, where q_n and r_m are polynomials of degree n and m, respectively, and in addition r_m is monic. Then*

$$\|q_n\|_K \leq 2(m+1)\left(\frac{16dR(K)}{r(K)^2}\right)^m n^{2m}\|p\|_K. \tag{2.1}$$

Note that just as in the univariate case estimate (2.1) yields an upper bound for q_n of order $O(n^{2m})$. This bound is sharp, in general, for fixed m which is small relative to n, but for large m the following statement provides a more efficient upper bound. In order to formulate it we need to recall a classical inequality by Kneser (see [1]) stating that for any univariate polynomials $q, r, \deg qr = n$ and any interval J we have

$$||q||_J ||r||_J \leq K_n ||qr||_J$$

where K_n is the Kneser constant. The Kneser constant is sharp in the above inequality and its exact value can be found in [1]. Roughly, it grows as 3.3^n.

THEOREM 2.2. *Under conditions of Theorem 2.1, the following estimate holds*

$$||q_n||_K \leq K_{n+m} \left(\frac{2d}{r(K)}\right)^m ||p||_K.$$

REMARK. It should be noted that the estimate $||q_n||_K = O(n^{2m})$ of Theorem 2.1 does not hold, in general, when K is not convex. For instance, set

$$K_a := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x^a, 0 \leq x \leq 1\}, a > 1.$$

Consider the polynomial $q_n \in P_n^1$ constructed in [3] such that

$$|x^{am} q_n(x)| \leq 1, x \in [0, 1], |q_n(0)| \geq c_m n^{2am}.$$

Set $p(x, y) := y^m q_n(x) \in P_{n+m}^2$. Note that y^m is monic. Then clearly

$$||p||_{K_a} \leq 1, ||q_n||_{K_a} \geq c_m n^{2am}.$$

Since $a > 1$ the $O(n^{2m})$ bound fails here for the nonconvex set K_a.

3. Proofs. In order to verify the new results we shall need some lemmas. The first lemma is related to the geometry of convex bodies.

LEMMA 3.1. *Let K be a convex body in $\mathbb{R}^d$ and $0 < \delta \leq r(K)$. Then for any $x \in K$ the set $B(x, \delta) \bigcap K$ contains a ball of radius $\beta := \delta r(K)/4R(K)$.*

Proof. Without loss of generality we may assume that $B(0, r(K)) \in K$ and $x \neq 0$. Set $y := x(1 - \delta/(2|x|))$. We claim that

$$B(y, \beta) \subset (B(x, \delta) \cap K). \tag{3.1}$$

Assume first that $|x| \leq r(K)$. Then for any $z \in B(y, \delta/2)$ we have

$$|z| \leq \delta/2 + |x||1 - \delta/(2|x|)| \leq \max\{|x|, \delta\} \leq r(K).$$

Hence $B(y, \delta/2) \subset B(0, r(K)) \subset K$. Moreover, for every $z \in B(y, \delta/2)$ we clearly have

$$|x - z| \leq \delta/2 + |x|\frac{\delta}{2|x|} = \delta.$$

Thus we also have the inclusion $B(y, \delta/2) \subset B(x, \delta)$. Therefore relation (3.1) holds whenever $|x| \leq r(K)$.

Assume now that $|x| > r(K)$. Note that $|x| \leq 2R(K)$. For any $z \in B(y, \beta)$ set $w := 2|x|(z - y)/\delta$. Then

$$|w| \leq 2|x|\beta/\delta \leq r(K).$$

Therefore $w \in B(0, r(K)) \subset K$. Moreover

$$z = y + \frac{\delta}{2|x|} w = \left(1 - \frac{\delta}{2|x|}\right) x + \frac{\delta}{2|x|} w$$

where $x, w \in K$ and $0 < \delta/(2|x|) \leq \delta/(2r(K)) \leq 1/2$. Thus by the convexity of K we obtain that $z \in K$. Hence $B(y, \beta) \subset K$. Moreover, as above $B(y, \beta) \subset B(y, \delta/2) \subset B(x, \delta)$ which verifies (3.1) in this case, as well. The proof of the lemma is now complete. ☐

Our next lemma provides a Chebyshev-type estimate for the minimal norm of a monic multivariate polynomial. Its proof is based on a similar result by Kellogg for homogeneous polynomials and the univariate Chebyshev inequality. It was shown by Kellogg [2] (see also [4]) that for every homogeneous polynomial of degree m given by $h(x) = \sum_{k_1+\cdots+k_d=m} c_{\mathbf{k}} x^{\mathbf{k}}$ we have

$$\sum_{k_1+\cdots+k_d=m} |c_{\mathbf{k}}| \leq d^m \|h\|_{S^{d-1}},$$

where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$.

LEMMA 3.2. *Let $r_m \in P_m^d$ be a monic polynomial. Then for any $\delta > 0$ and $a \in \mathbb{R}^d$ we have*

$$\|r_m\|_{B(a,\delta)} \geq 2^{-m+1} (\delta/d)^m. \tag{3.2}$$

Proof. For any $w \in S^{d-1}$, $t \in I = [-1, 1]$ set

$$g_m(t) := r_m(\delta w t + a) = A_m t^m + A_{m-1} t^{m-1} + \cdots + A_0,$$

where

$$A_m := \delta^m \sum_{k_1+\cdots+k_d=m} a_{\mathbf{k}} w^{\mathbf{k}},$$

and $a_{\mathbf{k}}$'s are the leading coefficients of r_m. By the well-known Chebyshev theorem we have

$$|A_m| \leq 2^{m-1} \|g_m\|_I \leq 2^{m-1} \|r_m\|_{B(a,\delta)}.$$

Thus using the Kellogg inequality mentioned above and recalling that r_m is monic

$$1 = \sum_{k_1+\cdots+k_d=m} |a_{\mathbf{k}}| \leq (d/\delta)^m 2^{m-1} \|r_m\|_{B(a,\delta)}.$$

This obviously yields (3.2). ☐

Proof of Theorem 2.1. Let $\|p\|_K = 1, \|q_n\|_K = |q_n(x)| = A, x \in K$. By a Markov-type inequality proved by Wilhelmsen [5] on convex bodies for any $w \in S^{d-1}$, $y \in K$

$$|D_w q_n(y)| \leq \frac{2An^2}{r(K)},$$

where D_w denotes the derivative in direction w. By this inequality for arbitrary $\delta > 0$ and $y \in B(x, \delta) \cap K$

$$|q_n(y)| \geq A - |q_n(x) - q_n(y)| \geq A - \frac{2An^2}{r(K)} \delta. \tag{3.3}$$

Now according to Lemma 3.1 there exists a ball B of radius $\beta := \delta r(K)/4R(K)$ such that B is a subset of $B(x, \delta) \cap K$. Applying Lemma 3.2 for the monic polynomial r_m and ball B we obtain

$$\|r_m\|_B \geq 2^{-m+1}(\beta/d)^m. \tag{3.4}$$

In addition, relation (3.3) holds for every $y \in B$. Thus choosing $y \in B$ so that the norm of r_m on B is attained at y we obtain by (3.3) and (3.4)

$$1 \geq |p(y)| = |q_n(y) r_m(y)| \geq A \left(1 - \frac{2\delta n^2}{r(K)}\right) 2^{-m+1}(\beta/d)^m.$$

Setting now $\delta := \frac{mr(K)}{2(m+1)n^2}$ and recalling the value of β yields the needed estimate for $A = \|q_n\|_K$.

Proof of Theorem 2.2. Let $x, y \in K$ be such that

$$\|q_n\|_K = |q_n(x)|, \|r_m\|_K = |r_m(y)|.$$

By the convexity of K we have $J := [x, y] \subset K$. Hence applying the Kneser inequality mentioned above on the interval J (and for corresponding univariate polynomials) we have

$$\|q_n\|_K \|r_m\|_K \leq K_{n+m} \|p\|_K. \tag{3.5}$$

Since r_m is monic and K contains a ball of radius $r(K)$ we obtain using (3.2) with $\delta := r(K)$ together with (3.5)

$$\|q_n\|_K \leq K_{n+m}(2d/r(K))^m \|p\|_K.$$

This completes the proof of Theorem 2.2.

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