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ON EULER'S DIFFERENTIAL METHODS FOR CONTINUED FRACTIONS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. A differential method discovered by Euler is justified and applied to give simple proofs to formulas relating important continued fractions with Laplace transforms. They include Stieltjes formulas and some Ramanujan formulas. A representation for the remainder of Leibniz's series as a continued fraction is given. We also recover the original Euler's proof for the continued fraction of hyperbolic cotangent.

Key words. continued fractions, Ramanujan formulas, Laplace transform

AMS subject classification. 30B70

1. Introduction. In 1739 in his second paper on Continued Fractions [4, §§49-55] Euler presented a simple and interesting method of developing functions into continued fractions. This method preceded the well-known method of Gauss, see [1, §2.5]. Due to this reason it cannot be found in books on continued fractions and appears to be forgotten. The purpose of this paper is to demonstrate the importance of Euler's method. It leads to very elementary proofs of formulas for continued fractions including some formulas by Ramanujan, see §9.

To justify Euler's method we need the following Test by Markov. It could happen that it was known to Euler. However, Euler said nothing in his papers on this very useful observation.

THEOREM 1.1 (Markov [15]). *Let $b_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} (a_n/b_n)$ be a positive continued fraction and let z_{n+1} defined by*

$$z_1 = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n + z_{n+1}}$$

be positive infinitely often. Then if the continued fraction converges it converges to z_1.

Proof. Let $s_0(w) = b_0 + w$, $s_k = a_k/(b_k + w)$. Then

$$z_1 = S_n(z_{n+1}) = s_0 \circ s_1 \circ \ldots \circ s_n(z_{n+1})$$

with $z_{n+1} \geqslant 0$ infinitely often. Every function $s_k(z)$ is continuous and monotonic on $[0, +\infty)$. Hence the same is true for their composition S_n. Picking two limit values $w = 0$ and $w = +\infty$, we obtain that z_1 must be in the interval with the end-points at

$$\frac{P_n}{Q_n} = S_n(0)\,, \qquad \frac{P_{n-1}}{Q_{n-1}} = S_n(+\infty)\,.$$

Since the continued fraction is assumed to converge, the proof is completed. ∎

Markov's test is directed to overcome the so-called *paradox of quadratic equations*. If one considers the quadratic equation

$$z^2 - 2z - 1 = 0 \Longleftrightarrow z = 2 + \frac{1}{z}$$

and applies iterates to develop $1 - \sqrt{2} < 0$ into a continued fraction, then this results in the formula

$$-\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2 + 1/(1-\sqrt{2})}\,, \tag{1.1}$$

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showing that the limit may not be equal to the expanded value.

According to Euler's formula z_1 and z_{n+1} are related by

$$\boxed{z_1 = \frac{P_n + z_{n+1}P_{n-1}}{Q_n + z_{n+1}Q_{n-1}}\ .} \tag{1.2}$$

In 1655 Brouncker (see [20]) discovered formulas

$$\begin{aligned} P_n &= b_n P_{n-1} + a_n P_{n-2}\ , \\ Q_n &= b_n Q_{n-1} + a_n Q_{n-2}\ , \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} P_{-1} &= 1\ , \qquad & P_0 &= b_0\ , \\ Q_{-1} &= 0\ , \qquad & Q_0 &= 1\ , \end{aligned}$$

and the formula

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}a_1 \dots a_n}{Q_n Q_{n-1}}\ , \quad n = 1, 2, \dots\ . \tag{1.4}$$

Formulas (1.2-1.4) can be easily proved by induction. See for instance [8]. Formulas (1.3) were explicitly stated by Wallis and extensively used by Euler. Therefore they are called the Euler-Wallis formulas. An elementary corollary of (1.3) is the following theorem due to Pringsheim [17] (see for instance [10, Ch. I, §5]).

THEOREM 1.2. *Let $a_n > 0$, $b_n > 0$ and*

$$\sum_{n=1}^{\infty} \left(\frac{b_{n-1}b_n}{a_n} \right)^{1/2} = +\infty\ .$$

Then $b_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right)$ converges.

COROLLARY 1.3. *If $\{x_n\}_{n\geqslant 0}$, $\{y_n\}_{n\geqslant 0}$, $\{x_n\}_{n\geqslant 0}$ are three positive arithmetic progressions, then the continued fraction*

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{x_n y_n}{z_n} \right) \tag{1.5}$$

converges.

It should be mentioned that the convergence of continued fractions considered in this paper follows directly from (1.4) by (1.3).

The first Euler's Differential Method allows one in many cases to sum up continued fractions (1.5) by solving elementary differential equations. It reduces to a simple lemma followed by a beautiful observation.

LEMMA 1.4 (Euler [4]). *Let R and P be two positive functions on $(0, 1)$ which for $n = 0, 1, 2, \dots$ and some positive α, β, γ satisfy*

$$(a + n\alpha) \int_0^1 PR^n dx = (b + n\beta) \int_0^1 PR^{n+1} dx + (c + n\gamma) \int_0^1 PR^{n+2} dx\ , \tag{1.6}$$

then

$$\frac{\int_0^1 PRdx}{\int_0^1 Pdx} \frac{a}{b\,+} \, \frac{(a+\alpha)c}{b+\beta} \, {+} \, \frac{(a+2\alpha)(c+\gamma)}{b+2\beta} \, {+} \, \frac{(a+3\alpha)(c+2\gamma)}{b+3\beta} \, {+} \dots\ . \tag{1.7}$$

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Proof. The condition of the lemma can be obviously written as follows

$$\frac{\int_0^1 PR^n dx}{\int_0^1 PR^{n+1} dx} = \frac{b+n\beta}{a+n\alpha} + \frac{1}{\frac{a+n\alpha}{c+n\gamma}\frac{\int_0^1 PR^{n+1}dx}{\int_0^1 PR^{n+2}dx}} \ . \tag{1.8}$$

Iterating this formula and applying elementary transformations, we get the lemma by Markov's Test. ∎

Euler's brilliant idea is to search P and R as functions satisfying the following identity with *indefinite* integrals

$$(a+n\alpha)\int PR^n dx + R^{n+1}S = (b+n\beta)\int PR^{n+1}dx + (c+n\gamma)\int PR^{n+2}dx \ .$$

If $R^{n+1}S$ vanishes at 0 and 1, then P and R must satisfy the conditions of Lemma 1.4. Euler's formula in differentials looks as follows

$$(a+n\alpha)Pdx + RdS + (n+1)SdR = (b+n\beta)PRdx + (c+n\gamma)PR^2dx \ .$$

Considering it as a polynomial in n, one can replace it with a system

$$\begin{aligned} aPdx + RdS + SdR &= bPRdx + cPR^2dx \ , \\ \alpha Pdx + SdR &= \beta PRdx + \gamma PR^2 dx \ . \end{aligned}$$

Solving both equations in Pdx, we find that

$$Pdx = \frac{RdS + SdR}{bR + cR^2 - a} = \frac{SdR}{\beta R + \gamma R^2 - \alpha} \ . \tag{1.9}$$

It follows from the last equation of (1.9) that

$$\begin{aligned} \frac{dS}{S} &= \frac{(b-\beta)RdR + (c-\gamma)R^2dR - (a-\alpha)dR}{\beta R^2 + \gamma R^3 - \alpha R} \\ &= \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b - \beta a)dR + (\alpha c - \gamma a)RdR}{\alpha(\beta R + \gamma R^2 - \alpha)} \ . \end{aligned} \tag{1.10}$$

2. Stieltjes' Formula [10, Ch 3, §3]. This formula was mentioned without a proof by Stieltjes in his correspondence with d'Hermite. In [10] it is derived from Roger's formulas for Laplace transforms of elliptic functions. Euler's Differential Method provides a very simple proof to Stieltjes' formula:

$$y(s) = \frac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n(n+1)}{s}\right)} = \int_0^\infty \frac{e^{-sx}}{\cosh^2(x)}dx \ . \tag{2.1}$$

We put in Lemma 1.4 and in (1.10)

$$\begin{array}{llll} a=1, & b=s, & c=1, & \alpha b - \beta a = s, \\ \alpha = 1, & \beta = 0, & \gamma = 1, & \alpha c - \gamma a = 0, \end{array}$$

$$\frac{dS}{S} = s\frac{dR}{R^2-1} \Longrightarrow S = \left|\frac{1-R}{1+R}\right|^{s/2} \ .$$

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If $R(x) = x$, then $R^{n+1}S$ vanishes at $x = 0$ and $x = 1$ for $n \geqslant 0$. By (1.9)

$$\int_0^1 P dx = \int_0^1 \left(\frac{1-x}{1+x}\right)^{s/2} \frac{dx}{1-x^2} \overset{x=\frac{1-t}{1+t}}{=} \frac{1}{2}\int_0^1 t^{s/2-1} dt = \frac{1}{s} .$$

Similarly

$$\int_0^1 RP dx = \frac{1}{2}\int_0^1 t^{s/2-1}\left(\frac{1-t}{1+t}\right) dt \overset{t=e^{-x}}{=} \int_0^{+\infty} e^{-sx}\tanh(x)\, dx$$
$$= -\frac{1}{s}\int_0^{+\infty} \tanh(x) de^{-sx} = \frac{1}{s}\int_0^{\infty} \frac{e^{-sx}}{\cosh^2(x)}\, dx ,$$

which proves (2.1).

COROLLARY 2.1. *For $s > 0$*

$$y(s) = \frac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n(n+1)}{s}\right)} = 2s\sum_{k=0}^{\infty} \frac{(-1)^k}{(s+2k)(s+2k+2)} . \tag{2.2}$$

Proof. We have

$$\int_0^1 t^{s/2-1}\left(\frac{1-t}{1+t}\right) dt \int_0^1 \frac{t^{s/2-1} - t^{s/2}}{1+t}\, dt = \tag{2.3}$$
$$= \sum_{k=0}^{\infty}(-1)^k \int_0^1 (t^{s/2-1+k} - t^{s/2+k})\, dt 4\sum_{k=0}^{\infty} \frac{(-1)^k}{(s+2k)(s+2k+2)} . \qquad \square$$

COROLLARY 2.2. *The continued fraction $y(s)$ satisfies*

$$\frac{y(s)}{s} + \frac{y(s+2)}{s+2} = \frac{2}{s(s+2)} .$$

Proof. It follows from (2.2). $\square$

Putting $s = 1$ in (2.2) and summing up Leibniz's series, we obtain another Euler's formula

$$1 + \frac{1}{1 + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n(n+1)}{1}\right)} = \frac{\pi}{2} .$$

By Van Vleck's Theorem (see [8, Theorem 4.29]) the continued fraction in the left-hand side of (2.2) converges uniformly on compact subsets of the right half-plane $\mathbb{C}_+$ to a function with a positive real part in $\mathbb{C}_+$. On the other hand, formula (2.3) shows that the series in (2.2) converges uniformly on compact subsets of $\mathbb{C} \setminus \{-2, -4, -6, \ldots\}$. By the uniqueness theorem for holomorphic functions formula (2.2) extends the continued fraction $y(s)$ in (2.2) to a holomorphic function in $\mathbb{C} \setminus \{-2, -4, -6, \ldots\}$. At point $s = -k$, $k = 2, 4, \ldots$, this function has residue $(-1)^{k-1}4k$ so that formally

$$y(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}4k}{s+2k} .$$

The above series, however does not converge. So the series in (2.2) is its regularization.

By (2.1) and by Van Vleck's Theorem

$$\Re y(it) = \int_0^{+\infty} \frac{\cos(tx)}{\cosh^2(x)}\, dx = p(t) \geqslant 0\ .$$

By Stieltjes' inversion formula

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{p(t)\, dt}{z - it} = y(z)$$

for $\Re z > 0$.

Stieltjes' formula (2.1) shows that $y(s)$ takes its maximal value at $s = 0$:

$$y(0) = \int_0^{\infty} \frac{1}{\cosh^2(x)}\, dx = \tanh(x)\Big|_0^{+\infty} = 1\ .$$

This can also be obtained by Corollary 2.2. It is also clear from (2.1) that $y(s)$ is continuous in the closure of the extended right half plane $\mathbb{C}_+$ and belongs to the Hardy class $H^2(\mathbb{C}_+)$. By Corollary 2.1 $y(s)$ extends to a meromorphic function in $\mathbb{C}$. However, this extension cannot be represented as a quotient of two bounded holomorphic functions, i.e. cannot be a function from Nevanlinna's class, as it is clear from the following theorem.

THEOREM 2.3. *The shifts* $\cosh^{-2}(x + t)$*,* $t \geqslant 0$*, span the whole space* $L^2(0, +\infty)$.

Proof. By Corollary 2.1 the Laplace transform $y(s)$ of $\cosh^{-2}(x)$ extends in $\mathbb{C}_-$ to a meromorphic function with poles at $\{-2, -4, -6, \ldots\}$. The conformal mapping $w(z) = (z+1)/(z-1)$ maps $\mathbb{C}_-$ onto the unit disc $\mathbb{D}$ centered at 0 and transforms $\{-2, -4, -6, \ldots\}$ to

$$z_k = 1 - \frac{2}{2k+1}, \quad k = 1, 2, \ldots\ .$$

Since obviously

$$\sum_{k=1}^{\infty} (1 - |z_k|^2) = +\infty\ ,$$

we see that $\{z_k\}$, and hence $\{-2, -4, -6, \ldots\}$, cannot be zero sets of functions from the Hardy algebra, see [2]. This implies that the extension of $y(s)$ to $\mathbb{C}_-$ does not belong to Nevanlinna's class. On the other hand, if the linear span of the shifts $\cosh^{-2}(x + t)$, $t \geqslant 0$ is not dense in $L^2(0, +\infty)$, then the Laplace transform $y(s)$ must have a pseudo-analytic continuation to a function in Nevanlinna's class (see [16]). This pseudo-analytic continuation coincides with an analytic continuation since $y(s)$ is analytic at the points of the imaginary axis. Since it is not in Nevanlinna's class the linear span of the shifts must be dense. □

It is useful to notice that to the contrary the shifts $e^{-(x+t)} = e^{-t}e^{-x}$ cannot span $L^2(0, +\infty)$ since they all are multiples of e^{-x}. Hence, in spite of the fact that $\cosh^{-2} x$ is so close to $4e^{-2x}$:

$$\frac{1}{\cosh^2(x)} = \frac{4}{(e^x + e^{-x})^2} = 4e^{-2x} + O\left(e^{-4x}\right)\ ,\ x \to +\infty$$

its left shifts span $L^2(0, +\infty)$.

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3. The Second Example. For every $s \geqslant 0$

$$y(s) = e\int_0^1 x^s e^{-x}\,dx = \frac{1}{s+}\frac{1}{s+1+}\frac{2}{s+2+}\frac{3}{s+3+\ldots+}\frac{n}{s+n+\ldots}\;. \tag{3.1}$$

The change of variables $x := e^{-x}$ shows that $y(s)$ is the Laplace transform

$$e\int_0^1 x^s e^{-x}\,dx = e\int_0^{+\infty} e^{-sx}e^{-e^{-x}}e^{-x}\,dx\;. \tag{3.2}$$

We put in Lemma 1.4 and in (1.10)

$$\begin{array}{llll} a=1, & b=s+1, & c=1, & \alpha b-\beta a=s, \\ \alpha=1, & \beta=1, & \gamma=0, & \alpha c-\gamma a=1, \end{array}$$

$$\frac{dS}{S} = (s+1)\frac{dR}{R-1} + dR \Longrightarrow S = (1-R)^{s+1}e^R\;.$$

If $R(x) = 1-x$, then $R^{n+1}S = 0$ at $x=0$ and $x=1$. Thus

$$\int_0^1 P dx = e\int_0^1 x^s e^{-x}\,dx\;;$$
$$\int_0^1 PR dx = -se\int_0^1 x^s e^{-x}\,dx + 1\;,$$

which implies (3.1). Passing to the limit $s \to 0^+$, we obtain Euler's formula

$$e = 2 + \frac{2}{2+}\frac{3}{3+\ldots+}\frac{n}{n+\ldots}\;. \tag{3.3}$$

COROLLARY 3.1. *For $s \geqslant 0$*

$$y(s) = \frac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n}{s+n}\right)} = \sum_{n=1}^{\infty}\frac{1}{(s+1)(s+2)\ldots(s+n)}\;. \tag{3.4}$$

Proof. Integrating by parts, we obtain

$$\begin{aligned} e\int_0^1 x^s e^{-x}\,dx &= \frac{1}{s+1} + \frac{e}{(s+1)}\int_0^1 x^{s+1}e^{-x}\,dx \\ &= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} + \frac{e}{(s+1)(s+2)}\int_0^1 x^{s+2}e^{-x}\,dx = \ldots. \end{aligned}$$

□

It follows from (3.2) that

$$y(s) = e\int_0^{+\infty} e^{-(s+1)x}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}e^{-kx}\,dx e\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{s+k+1} \tag{3.5}$$

is an infinite sum of simple fractions.

Notice that integration by parts in (3.1) shows that $y(s)$ satisfies

$$y(s+1)+1=(s+1)y(s)\ . \tag{3.6}$$

Other analytic properties are similar to those considered in §2. In particular, (3.2) and (3.5) show that left shifts of $e^{-e^{-x}}e^{-x}$ span $L^2(0,+\infty)$. It is easy to see that more generally left shifts of $e^{-e^{-ax}}e^{-x}$ span $L^2(0,+\infty)$ for every $a>0$. Clearly

$$e^{-x}-e^{-e^{-ax}}e^{-x}=O\left(e^{-(1+a)x}\right),\quad x\to+\infty\quad . \tag{3.7}$$

This shows that there are functions approximating the exponential function as indicated in (3.7) with meromorphic Laplace transform in $\mathbb{C}$ and at the same time such that their left shifts span $L^2(0,+\infty)$. The condition (3.7) cannot be weakened since for any continuous function $f(x)$ such that

$$f(x)=O\left(e^{-ax}\right),\quad x\to+\infty$$

for every positive a the Laplace transform of f is an entire function.

4. The Arctangent. This continued fraction was obtained in 1770 by Lambert [11] (see [10, Ch. II, §6]). For every $s>0$

$$\arctan\left(\frac{1}{s}\right)=\frac{1}{s+}\,\frac{1^2}{3s+}\,\frac{2^2}{5s+}\,\frac{3^2}{7s+}\,\frac{4^2}{9s+}\cdots\frac{n^2}{(2n+1)s+}\cdots\ . \tag{4.1}$$

Remark. Since all convergents to the continued fraction (4.1) are odd, the equality (4.1) holds for every real s, $s\neq 0$.

We put in Lemma 1.4 and in (1.10)

$$\begin{array}{llll} a=1, & b=3s, & c=2, & \alpha b-\beta a=s,\\ \alpha=1, & \beta=2s, & \gamma=1, & \alpha c-\gamma a=1, \end{array}$$

$$\frac{dS}{S}=\frac{(R+s)dR}{2sR+R^2-1}\Longrightarrow S=\sqrt{1+s^2-(R+s)^2}\ .$$

If $R(x)=(\sqrt{1+s^2}-s)x$, then $R^{n+1}S$ vanishes at $x=0$ and $x=1$ for every nonnegative n. Next,

$$\int_0^1 PdR=\arcsin\frac{R+s}{\sqrt{1+s^2}}\bigg|_0^1=\frac{\pi}{2}-\arcsin\frac{s}{\sqrt{1+s^2}}\ ;$$

$$\begin{aligned}\int_0^1 PRdR&=-\sqrt{1+s^2-(R+s)^2}-s\arcsin\frac{R+s}{\sqrt{1+s^2}}\bigg|_0^1=\\ &=1-s\left(\frac{\pi}{2}-\arcsin\frac{s}{\sqrt{1+s^2}}\right)\ .\end{aligned}$$

The elementary identity

$$\frac{\pi}{2}-\arcsin\frac{s}{\sqrt{1+s^2}}=\arctan\left(\frac{1}{s}\right)\ ,$$

which can be proved by differentiation, completes the proof of (4.1). Putting $s = 1$ in (4.1), we obtain Euler's formula

$$\frac{\pi}{4} = \frac{1}{1+}\,\frac{1^2}{3+}\,\frac{2^2}{5+}\,\frac{3^2}{7+}\,\frac{4^2}{9+\ldots+}\,\frac{n^2}{2n+1+\ldots}\;. \tag{4.2}$$

Euler's formula

$$\arctan(x) = \frac{i}{2}\ln\left(\frac{i+x}{i-x}\right)$$

shows that

$$\arctan\left(\frac{1}{it}\right) = \frac{i}{2}\ln\left(\frac{t-1}{t+1}\right) = \frac{i}{2}\ln\left|\frac{t-1}{t+1}\right| - \frac{1}{2}\arg\left(\frac{t-1}{t+1}\right)$$

and therefore

$$\arctan\left(\frac{1}{it}\right) = \begin{cases} \frac{\pi}{2} & \text{if } -1 < t < 1\,; \\ 0 & \text{if } 1 < |t|\,. \end{cases}$$

It follows that

$$\frac{1}{2}\int_1^1 \frac{1}{z-it}\,dt = \arctan\left(\frac{1}{z}\right)\ ,\ \Re z > 0\,.$$

5. The Forth Example. Here we establish the following formula

$$\frac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{s+n}\right)} = \phi\sqrt{5}\int_0^1 \frac{x^{s-1+1/\phi}\,dx}{1+\phi+x^{\sqrt{5}}}\,, \tag{5.1}$$

where

$$\phi = \frac{1+\sqrt{5}}{2} = \frac{2}{\sqrt{5}-1} = \phi^2 - 1$$

is the Golden Ratio.

We put in Lemma 1.4 and in (1.10)

$$\begin{array}{llll} a = 1, & b = s+1, & c = 2, & \alpha b - \beta a = s, \\ \alpha = 1, & \beta = 1, & \gamma = 1, & \alpha c - \gamma a = 1, \end{array}$$

$$\frac{dS}{S} = \frac{sdR + RdR}{R^2 + R - 1} = \frac{(R+1/2)dR + (s-1/2)dR}{(R+1/2)^2 - 5/4}\,. \tag{5.2}$$

Integrating (5.2) and using the factorization

$$R^2 + R - 1 = (R+\phi)(R-1/\phi)\,,$$

we obtain

$$S = \frac{\left|R - \frac{1}{\phi}\right|^{(s-\frac{1}{2})\frac{1}{\sqrt{5}}+\frac{1}{2}}}{|R+\phi|^{(s-\frac{1}{2})\frac{1}{\sqrt{5}}-\frac{1}{2}}}\,.$$

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Hence, if $R(x) = x/\phi$, then $R^{n+1}S = 0$ at $x = 0$ and $x = 1$. To simplify the notations we put

$$\delta = \left(s - \frac{1}{2}\right)\frac{1}{\sqrt{5}} - \frac{1}{2} = \frac{s-\phi}{\sqrt{5}} \quad , \quad \Delta = \phi + 1\,.$$

Then

$$\int_0^1 P\,dx = \int_0^1 \left(\frac{1-x}{x+\Delta}\right)^{\delta} \frac{dx}{x+\Delta} \overset{x=\frac{1-t\Delta}{1+t}}{=} \int_0^{1/\Delta} \frac{t^{\delta}}{1+t}dt\,;$$
$$\int_0^1 RP\,dx = \frac{1}{\phi}\int_0^1 x\left(\frac{1-x}{x+\Delta}\right)^{\delta} \frac{dx}{x+\Delta} \overset{x=\frac{1-t\Delta}{1+t}}{=} \frac{1}{\phi}\int_0^{1/\Delta} \frac{t^{\delta}(1-t\Delta)}{(1+t)^2}dt\,.$$

Integration by parts shows that

$$\int_0^{1/\Delta} \frac{t^{\delta}(1-t\Delta)}{(1+t)^2}dt \int_0^{1/\Delta} \frac{t^{\delta}\,dt}{1+t} - (1+\Delta)\int_0^{1/\Delta} \frac{t^{\delta+1}\,dt}{(1+t)^2} =$$
$$= \frac{1}{\Delta^{\delta}} - \{(1+\delta)(1+\Delta) - 1\}\int_0^{1/\Delta} \frac{t^{\delta}\,dt}{(1+t)}\,.$$

Observing that $(1+\delta)(1+\Delta) - 1 = s\phi$, we obtain

$$\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{s+n}\right) = \frac{\int_0^1 RP\,dx}{\int_0^1 P\,dx} = \frac{1}{\phi}\left\{\frac{1}{\Delta^{\delta}\int_0^{1/\Delta}\frac{t^{\delta}\,dt}{1+t}} - s\phi\right\}\,.$$

It follows that

$$\frac{1}{s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{s+n}\right)} = \phi\int_0^1 \frac{t^{\delta}\,dt}{\Delta+t} \overset{t=x^{\sqrt{5}}}{=} \phi\sqrt{5}\int_0^1 \frac{x^{s-1+1/\phi}\,dx}{1+\phi+x^{\sqrt{5}}}\,.$$

COROLLARY 5.1. *For* $s > 0$

$$\frac{1}{s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{s+n}\right)} = \frac{\sqrt{5}}{\phi}\sum_{k=0}^{\infty}\frac{1}{s+1/\phi+k\sqrt{5}}\cdot\frac{(-1)^k}{\phi^{2k}}\,.$$

COROLLARY 5.2.

$$\frac{1}{\phi + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{\phi+n}\right)} = \phi\ln\left(1+\frac{1}{\phi^2}\right)\,. \tag{5.3}$$

Proof. Evaluate the integral in (5.1) at $s = \phi$. $\square$

6. A partial case of Euler's Formula. Let us compare a continued fraction

$$s + \frac{fh}{s} \,{}_{+}\, \frac{(f+r)(h+r)}{s} \,{}_{+}\, \frac{(f+2r)(h+2r)}{s} \,{}_{+\dots}$$

with

$$a\frac{\int_0^1 Pdx}{\int_0^1 PRdx}b + \frac{(a+\alpha)c}{b+\beta} \,{}_{+}\, \frac{(a+2\alpha)(c+\gamma)}{b+2\beta} \,{}_{+}\, \frac{(a+3\alpha)(c+2\gamma)}{b+3\beta} \,{}_{+\dots}\,.$$

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Then it is clear that

$$a = f - r, \quad b = s, \qquad c = h, \quad \alpha = r, \quad \beta = 0, \quad \gamma = r,$$

and the differential equation (1.10) for S takes the form

$$\frac{dS}{S} = \frac{(f-2r)dR}{rR} + \frac{s}{r}\frac{dR}{R^2-1} + \frac{(h-f+r)RdR}{r(R^2-1)} \, .$$

The integral of this differential equation is given by

$$\ln S = \frac{f-2r}{r}\ln R + \frac{s}{2r}\ln\left|\frac{R-1}{R+1}\right| + \frac{h-f+r}{2r}\ln|R^2-1| + C \, .$$

It follows that

$$S = C \cdot R^{\frac{f-2r}{r}} \left|\frac{R-1}{R+1}\right|^{\frac{s}{2r}} |R^2-1|^{\frac{h-f+r}{2r}} \, .$$

If $R = x^r$, then $R^{n+1}S$ vanish at $x = 0$ and $x = 1$ for all nonnegative integer n provided

$$0 < f - r < h + s \, . \tag{6.1}$$

By (1.9) we obtain the formula for P

$$Pdx = C \cdot x^{f-r-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}(1-x^{2r})^{\frac{h-f-r}{2r}}\,dx \, .$$

Here C stands for a constant the value of which is not important for us, since we are interested in the quotient of integrals. By Lemma 1.4

$$s + \mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{(f+nr)(h+nr)}{s}\right) = = (f-r)\frac{\displaystyle\int_0^1 x^{f-r-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}\frac{dx}{(1-x^{2r})^{\frac{f+r-h}{2r}}}}{\displaystyle\int_0^1 x^{f-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}\frac{dx}{(1-x^{2r})^{\frac{f+r-h}{2r}}}} \, . \tag{6.2}$$

If $f = h = g$ and $g > r$, then (6.2) takes the form

$$s + \mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{(g+nr)^2}{s}\right)(g-r)\frac{\displaystyle\int_0^1 x^{g-r-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}\frac{dx}{\sqrt{1-x^{2r}}}}{\displaystyle\int_0^1 x^{g-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}\frac{dx}{\sqrt{1-x^{2r}}}} \, .$$

Assuming now that $g > 0$ and applying the above identity with $g := g + r$, we obtain

$$\mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{(g+nr)^2}{s}\right) = g \cdot \frac{\displaystyle\int_0^1 x^{g+r-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}\frac{dx}{\sqrt{1-x^{2r}}}}{\displaystyle\int_0^1 x^{g-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}\frac{dx}{\sqrt{1-x^{2r}}}} \, . \tag{6.3}$$

7. Laplace transform of hyperbolic secant and Leibnitz's Series.

$$y(s) = \cfrac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\cfrac{n^2}{s}\right)} = \int_0^{+\infty} \frac{e^{-sx}\,dx}{\cosh(x)}\ ,\ s > 0\ . \tag{7.1}$$

This formula easily follows by the substitution $x := e^{-x}$ from the following theorem by Euler.

THEOREM 7.1 ([4], §69). *For* $s > 0$

$$y(s) = \cfrac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\cfrac{n^2}{s}\right)} = 2\int_0^1 \frac{x^s\,dx}{1+x^2}\ . \tag{7.2}$$

Proof. By (6.3) with $g = 1$ and $r = 1$

$$\underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{n^2}{s}\right) = \frac{\displaystyle\int_0^1 x\left(\frac{1-x}{1+x}\right)^{s/2} \frac{dx}{\sqrt{1-x^2}}}{\displaystyle\int_0^1 \left(\frac{1-x}{1+x}\right)^{s/2} \frac{dx}{\sqrt{1-x^2}}}\ .$$

By parts integration followed by the substitution $x := (1-x)/(1+x)$

$$\int_0^1 x\left(\frac{1-x}{1+x}\right)^{s/2} \frac{dx}{\sqrt{1-x^2}} =$$

$$= 1 - s\int_0^1 \left(\frac{1-x}{1+x}\right)^{s/2} \frac{dx}{\sqrt{1-x^2}} 1 - s\int_0^1 \frac{x^{(s-1)/2}\,dx}{1+x}$$

results in (7.2)

$$\cfrac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\cfrac{n^2}{s}\right)} = \int_0^1 \frac{x^{(s-1)/2}\,dx}{1+x} = 2\int_0^1 \frac{x^s\,dx}{1+x^2}\ . \qquad \square \tag{7.3}$$

Since

$$\int_0^1 \frac{x^s\,dx}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k \int_0^1 x^{2k+s}dx \sum_{k=0}^{\infty} \frac{(-1)^k}{s+2k+1}\ ,$$

we obtain a representation of (7.2) as a sum of simple fractions

$$y(s) = \cfrac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\cfrac{n^2}{s}\right)} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{s+2k+1} 4\sum_{k=0}^{\infty} \frac{1}{(s+2k+1)(s+2k+3)}\ . \tag{7.4}$$

By Van Vleck's Theorem the continued fraction (7.4) converges to $y(s)$ in $\mathbb{C}_+$ and to $-y(-s)$ in $\mathbb{C}_-$. The poles of its convergents are located on the imaginary axis. The right-hand part of (7.4) converges uniformly on compact sets of $\mathbb{C}_- \setminus \{-1, -3, -5, \ldots\}$.

COROLLARY 7.2. *The shifts* $\cosh^{-1}(x+t)$, $t \geqslant 0$, *span the whole space* $L^2(0, +\infty)$.

By (7.4) $y(s)$ satisfies

$$y(2s) + y(2s+2) = \frac{2}{2s+1}\ . \tag{7.5}$$

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We apply now (7.4) and (7.5) to Leibnitz's series.

THEOREM 7.3. *For every positive integer s*

$$\pi = 4\sum_{k=0}^{s-1}\frac{(-1)^k}{2k+1} + \frac{2(-1)^s}{2s + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{2s}\right)}\,. \tag{7.6}$$

Proof. Computations with integrals show that

$$2 + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{2}\right) = \frac{2}{4-\pi}\,, \qquad 4 + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{4}\right) = \frac{2}{\pi - \frac{8}{3}}\,;$$
$$6 + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{6}\right) = \frac{2}{\frac{52}{15}-\pi}\,, \qquad 8 + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{8}\right) = \frac{2}{\pi - \frac{105}{304}}\,.$$

Assuming by induction that for an integer s

$$y(2s) = (-1)^s\left\{\frac{\pi}{2} - 2\sum_{k=0}^{s-1}\frac{(-1)^k}{2k+1}\right\}\,,$$

we deduce from (7.5) that

$$y(2s+2)(-1)^{s+1}\left\{\frac{\pi}{2} - 2\sum_{k=0}^{s}\frac{(-1)^k}{2k+1}\right\}\,. \qquad \square$$

Remark. If $s = 50$, then taking the second convergent and adding 50 first terms of Leibnitz's series, we obtain 10 valid places for π. It should also be noticed that by (7.3)

$$\lim_{s\to 0+} s + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{s}\right) = \frac{2}{\pi}\,.$$

8. Some more formulas. In this section using the differential method we establish the following formulas

$$\int_0^{+\infty}\frac{e^{-sx}\,dx}{\cosh^{3/2}(2x)} = \frac{1}{s\,+}\,\frac{2\cdot 3}{s}\,+\,\frac{4\cdot 5}{s}\,+\,\frac{6\cdot 7}{s}\,+\ldots\,, \tag{8.1}$$

$$\int_0^{+\infty}\frac{e^{-sx}\,dx}{\cosh^{3/2}(2x)}\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(s+3)(s+7)\ldots(s+4n+3)}\,, \tag{8.2}$$

$$\int_0^{+\infty}\frac{e^{-sx}\,dx}{\cosh^{3/2}(2x)} = 2\sqrt{2}\sum_{n=0}^{+\infty}(-1)^n\frac{(2n+1)!!}{2^n n!}\,\frac{1}{s+4n+3}\,. \tag{8.3}$$

Notice that for $s = 1$ we obtain from (8.2) that

$$\frac{1}{1\,+}\,\frac{2\cdot 3}{1}\,+\,\frac{4\cdot 5}{1}\,+\,\frac{6\cdot 7}{1}\,+\ldots\sum_{n=0}^{\infty}\frac{(2n+1)!!}{4^{n+1}(n+1)!} = \sqrt{2}-1 \tag{8.4}$$

by the Binomial Formula for $1/\sqrt{1-x}$ evaluated at $x = 1/2$.

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If $f = 4$, $h = 5$, $r = 2$ in (6.2), then

$$s + \frac{4\cdot 5}{s} {}_{+} \frac{6\cdot 7}{s} {}_{+} \frac{8\cdot 9}{s} {}_{+\dots} = 2\frac{\displaystyle\int_0^1 x\left(\frac{1-x^2}{1+x^2}\right)^{s/4}\frac{dx}{(1-x^4)^{1/4}}}{\displaystyle\int_0^1 x^3\left(\frac{1-x^2}{1+x^2}\right)^{s/4}\frac{dx}{(1-x^4)^{1/4}}}\,. \tag{8.5}$$

Next,

$$\sqrt{2}I_1 = \sqrt{2}\int_0^1 x\left(\frac{1-x^2}{1+x^2}\right)^{s/4}\frac{dx}{(1-x^4)^{1/4}} \overset{x=\left(\frac{1-t}{1+t}\right)^{1/2}}{=} \int_0^1 \frac{t^{\frac{s-1}{4}}\,dt}{(1+t)^{3/2}}\,,$$

$$\sqrt{2}I_2 = \sqrt{2}\int_0^1 x^3\left(\frac{1-x^2}{1+x^2}\right)^{s/4}\frac{dx}{(1-x^4)^{1/4}} \overset{x=\left(\frac{1-t}{1+t}\right)^{1/2}}{=} \int_0^1 \frac{t^{\frac{s-1}{4}}(1-t)\,dt}{(1+t)^{5/2}}\,.$$

By Lemma (8.5)

$$z(s) = s + \frac{4\cdot 5}{s} {}_{+} \frac{6\cdot 7}{s} {}_{+\dots} = 2\cdot\frac{\sqrt{2}I_1}{\sqrt{2}I_2}\,.$$

Integration by parts shows that

$$\int_0^1 \frac{t^{\frac{s-1}{4}}(1-t)\,dt}{(1+t)^{5/2}} - \frac{2}{3}\int_0^1 t^{\frac{s-1}{4}}(1-t)d(1+t)^{-3/2} =$$

$$= \frac{s-1}{6}\int_0^1 \frac{t^{\frac{s-5}{4}}\,dt}{(1+t)^{3/2}} - \frac{s+3}{6}\int_0^1 \frac{t^{\frac{s-1}{4}}\,dt}{(1+t)^{3/2}} \overset{t=x^4}{=}$$

$$= \frac{s-1}{6}\int_0^1 \frac{4x^{s-2}\,dx}{(1+x^4)^{3/2}} - \frac{s+3}{6}\int_0^1 \frac{4x^{s+2}\,dx}{(1+x^4)^{3/2}}\,.$$

Therefore

$$s + \frac{2\cdot 3}{z(s)} = \frac{s-1}{2}\cdot\frac{\displaystyle\int_0^1 \frac{x^{s-2}\,dx}{(1+x^4)^{3/2}}}{\displaystyle\int_0^1 \frac{x^{s+2}\,dx}{(1+x^4)^{3/2}}} + \frac{s-1}{2} - 1\,.$$

Applying an obvious identity $x^{s-2}(1+x^4) - x^{s-2} = x^{s+2}$ and integrating by parts, we obtain

$$\int_0^1 \frac{x^{s+2}\,dx}{(1+x^4)^{3/2}} = \int_0^1 \frac{x^{s-2}\,dx}{(1+x^4)^{1/2}} - \int_0^1 \frac{x^{s-2}\,dx}{(1+x^4)^{3/2}}\,;$$

$$U(s) \overset{def}{=} 2\sqrt{2}\int_0^1 \frac{x^{s+2}\,dx}{(1+x^4)^{3/2}} = -1 + \sqrt{2}(s-1)\int_0^1 \frac{x^{s-2}\,dx}{(1+x^4)^{1/2}}\,.$$

It follows that

$$s + \frac{2\cdot 3}{z(s)} = \frac{s-1}{2}\cdot\frac{2\sqrt{2}\left\{\displaystyle\int_0^1 \frac{x^{s-2}\,dx}{(1+x^4)^{1/2}} - \int_0^1 \frac{x^{s+2}\,dx}{(1+x^4)^{3/2}}\right\}}{U(s)} + \frac{s-1}{2} - 1 =$$

$$= \frac{s-1}{2}\cdot\frac{2\sqrt{2}\displaystyle\int_0^1 \frac{x^{s-2}\,dx}{(1+x^4)^{1/2}}}{U(s)} - 1\frac{2\sqrt{2}(s-1)\displaystyle\int_0^1 \frac{x^{s-2}\,dx}{(1+x^4)^{1/2}} - 2U(s)}{2U(s)} = \frac{1}{U(s)}\,.$$

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Now (8.1) follows by

$$U(s)=\int_0^1 x^{s+2}\left(\frac{2}{1+x^4}\right)^{3/2}dx \overset{x:=e^{-x}}{=} \int_0^{+\infty} e^{-sx}\left(\frac{2}{e^{2x}+e^{-2x}}\right)^{3/2}dx\ . \tag{8.6}$$

Formula (8.3) follows from the first equality in (8.6) by the Binomial Theorem. Integration by parts shows by induction that

$$U(s)=\sum_{n=0}^{m}\frac{1\cdot 3\cdot\ldots\cdot(2n+1)}{(s+3)(s+7)\ldots(s+4n+3)}+ \\ +\frac{1\cdot 3\cdot\ldots\cdot(2m+1)\cdot(2m+3)}{(s+3)(s+7)\ldots(s+4m+3)}\int_0^1\left(\frac{2x^4}{1+x^4}\right)^{m+1}\frac{2^{3/2}x^2\,dx}{(1+x)^{3/2}}\ ,$$

which implies (8.2) by passing to the limit in m.

To complete the topic on left-shifts of the negative powers of hyperbolic cotangent we prove the following theorem.

THEOREM 8.1. *For every $p>0$ the shifts* $\cosh^{-p}(x+t)$*, $t\geqslant 0$, span the whole space* $L^2(0,+\infty)$. *Proof.* It is sufficient to prove that the Laplace transform

$$\int_0^{+\infty}\frac{e^{-sx}\,dx}{\cosh^p(x)}=\int_0^1\frac{e^{-sx}\,dx}{\cosh^p(x)}+\int_1^{+\infty}\frac{e^{-sx}\,dx}{\cosh^p(x)}=F(s)+G(s)$$

can be extended analytically to a meromorphic function with poles at arithmetic progression on a negative real semi-axis. Since $F(s)$ is an entire function, it is sufficient to consider $G(s)$. By the Binomial Theorem

$$\frac{e^{-sx}}{\cosh^p(x)}=\frac{2^pe^{-(s+p)x}}{(1+e^{-2x})^p}=2^p\sum_{k=0}^{\infty}(-1)^k\frac{p(p+1)\ldots(p+k-1)}{k!}e^{-(s+p+2k)x}\ .$$

Integrating, we obtain that

$$G(s)=\left(\frac{2}{e}\right)^pe^{-s}\sum_{k=0}^{\infty}\frac{p(p+1)\ldots(p+k-1)}{k!}\,\frac{e^{-2k}}{s+p+2k}\ . \qquad\square$$

9. Three formulas by Ramanujan. In this section we consider three Ramanujan's formulas.

The error function. For every $s>0$

$$e^{s^2/2}\int_s^{+\infty}e^{-x^2/2}\,dx=\frac{1}{s+}\,\frac{1}{s+}\,\frac{2}{s+}\,\frac{3}{s+}\,\frac{4}{s+}\ldots+\frac{n}{s+}\ldots\ , \tag{9.1}$$

see [9, p.8, (1.8)]. In fact this formula was first obtained by Euler in 1754, [5, §29].

Remark. The function

$$\operatorname{erfc}(s)=\frac{2}{\sqrt{\pi}}\int_s^{+\infty}e^{-x^2}\,dx\sqrt{\frac{2}{\pi}}\int_{s\sqrt{2}}^{+\infty}e^{-x^2/2}\,dx$$

is called the *complementary error function*, see [1, p. 196]. The identity (9.1) can be used to get good approximations to erfc(s) for big s.

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To prove (9.1) we put $a = 1$, $b = s$, $c = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$ in Lemma 1.4 and use the point $+\infty$ instead of $x = 1$:

$$\frac{\int_0^{+\infty} PR\,dx}{\int_0^{+\infty} P\,dx} = \frac{1}{s\,+}\,\frac{2}{s\,+}\,\frac{3}{s\,+}\,\frac{4}{s\,+\ldots+}\,\frac{n}{s\,+\ldots}\ .$$

The differential equation (1.10) for S is

$$\frac{dS}{S} = -sdR - RdR\ ,$$

which with $R(x) = x$ shows that $xS(x) = xe^{-sx-x^2/2}$ vanishes at $x = 0$ and $x = +\infty$. It follows that we may put

$$Pdx = e^{-sx-x^2/2}dx, \quad RPdx = xe^{-sx-x^2/2}dx\ .$$

Let us consider

$$\varphi(s) = \int_0^{+\infty} e^{-sx-x^2/2}\,dx = e^{s^2/2}\int_s^{+\infty} e^{-x^2/2}\,dx\ ,$$

implying that φ satisfies

$$\varphi'(s) = s\varphi(s) - 1 = -\int_0^{+\infty} xe^{-sx-x^2/2}dx\ . \tag{9.2}$$

It follows that

$$\frac{\int_0^{+\infty} PR\,dx}{\int_0^{+\infty} P\,dx} = \frac{1 - s\varphi(s)}{\varphi(s)} = \frac{1}{\varphi(s)} - s\ ,$$

which implies (9.1).

Fourier Sine Transform. For every positive t

$$\int_0^{+\infty} \frac{\sin(ts)\,ds}{s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty}\left(\frac{n}{s}\right)} = \frac{\sqrt{\frac{\pi}{2}}}{t + \mathop{\mathbf{K}}\limits_{n=1}^{\infty}\left(\frac{n}{t}\right)}\ , \tag{9.3}$$

see [14, (2), p. 369]. By (9.1) and Fubini's Theorem the left-hand side of (9.3) is

$$\int_0^{+\infty} \sin(ts)\varphi(s)\,ds = \int_0^{+\infty} e^{-x^2/2}\,dx\left\{\int_0^{+\infty} \sin(ts)e^{-sx}\,ds\right\}\ .$$

Double integration by parts shows that

$$\int_0^{+\infty} \sin(ts)e^{-sx}\,ds = \frac{-1}{t}\int_0^{+\infty} e^{-sx}d\cos(ts) =$$
$$= \frac{1}{t} - \frac{x}{t}\int_0^{+\infty} \cos(ts)e^{-sx}\,ds = \frac{1}{t} - \frac{x^2}{t^2}\int_0^{+\infty} \sin(ts)e^{-sx}\,ds\ .$$

It follows that

$$\int_0^{+\infty} \sin(ts)e^{-sx}\,ds = \frac{t}{t^2 + x^2}\ .$$

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Hence

$$\psi(t) \overset{def}{=} \int_0^{+\infty} \sin(ts)\varphi(s)\,ds = \int_0^{+\infty} \frac{1}{1+x^2} e^{-x^2t^2/2}\,dx\,.$$

Differentiating the last integral in t, we obtain that $\psi(t)$ satisfies the differential equation

$$\psi'(t) = t\cdot\psi(t) - \int_0^{+\infty} e^{-x^2}\,dx$$

and therefor ψ must be proportional to φ, satisfying the differential equation (9.2), with the coefficient

$$\int_0^{+\infty} e^{-x^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}\,.$$

A Formula for $\sqrt{\pi e/2}$. Similar arguments prove another Ramanujan's formula

$$1 + \frac{1}{1\cdot 3} + \frac{1}{1\cdot 3\cdot 5} + \frac{1}{1\cdot 3\cdot 5\cdot 7} + \ldots + \frac{1}{1 + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n}{1}\right)} = \sqrt{\frac{\pi e}{2}}\,, \tag{9.4}$$

see [14, (3), p. 370]. Let

$$y(x) = x + \frac{x^3}{1\cdot 3} + \frac{x^5}{1\cdot 3\cdot 5} + \frac{x^7}{1\cdot 3\cdot 5\cdot 7}\cdots\,.$$

Then $y(0) = 0$ and $y'(x) = xy(x) + 1$, which implies that

$$y(x) = e^{x^2/2}\int_0^x e^{-t^2/2}\,dt\,.$$

Then by (9.1)

$$\sum_{n=1}^{+\infty} \frac{x^{2n-1}}{(2n-1)!!} + \frac{1}{x + \underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n}{x}\right)} = e^{x^2/2}\int_0^{+\infty} e^{-t^2/2}\,dt = e^{x^2/2}\sqrt{\frac{\pi}{2}}\,.$$

Putting here $x = 1$, we obtain (9.4).

10. Euler's formula for hyperbolic cotangent. Motivation. In 1737 Euler computed first partial denominators of the regular continued fraction for

$$e = 2,71828182845904\ldots$$

and discovered a remarkable law

$$\boxed{e = 2 + \frac{1}{1} {+} \frac{1}{2} {+} \frac{1}{1} {+} \frac{1}{1} {+} \frac{1}{4} {+} \frac{1}{1} {+} \frac{1}{1} {+} \frac{1}{6} {+} \frac{1}{1} {+} \ldots} \,. \tag{10.1}$$

Formula (10.1) looks not less beautiful than the formulas

$$e = \lim_n \left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\,,$$

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and in addition proves the irrationality of e immediately. After that Euler computed the continued fraction for

$$\sqrt{e} = 1,6487212707\ldots = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{9+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{13+\ldots}\,,$$

which obeys a similar progression law of partial denominators. Next,

$$\sqrt[3]{e} = 1 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{14+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{20+\ldots}\,,$$

confirms this law. To concentrate on the arithmetic progressions in the above formulas Euler removed repeating 1's by elementary transformations:

$$\begin{aligned} a + \frac{1}{m+}\,\frac{1}{n+}\,\frac{1}{w+} \quad a + \frac{nw+1}{mnw+m+w} &= \\ = a + \frac{n}{mn+1} + \frac{nw+1}{mnw+m+w} - \frac{n}{mn+1} &= \\ = a + \frac{n}{mn+1} + \frac{1}{(mn+1)w+m} &= \\ = \frac{1}{mn+1}\left\{(mn+1)a + n + \frac{1}{(m+n)w+m}\right\}&\,, \end{aligned}$$

which show that

$$\begin{aligned} a + \frac{1}{m+}\,\frac{1}{n+}\,\frac{1}{b+}\,\frac{1}{m+}\,\frac{1}{n+}\,\frac{1}{c+\ldots} &= \\ = \frac{1}{mn+1}\left\{(mn+1)a + n + \frac{1}{(mn+1)b+n+m+\ldots}\right\}&\,. \end{aligned} \tag{10.2}$$

This interesting identity, by the way, implies a beautiful formula

$$a + \frac{1}{m+}\,\frac{1}{n+}\,\frac{1}{b+}\,\frac{1}{m+}\,\frac{1}{n+}\,\frac{1}{c+\ldots} - a - \frac{1}{n+}\,\frac{1}{m+}\,\frac{1}{b+}\,\frac{1}{n+}\,\frac{1}{m+}\,\frac{1}{c+\ldots} = \frac{n-m}{nm+1}\,,$$

reminding the addition formula for cotangent. Let us put $m = n = 1$, $a = 2$, $b = 4$, $c = 6$, $d = 8$, etc in (10.2). Then

$$2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+\ldots} = \frac{1}{2}\left\{-1 + 6 + \frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+\ldots}\right\} = \frac{\xi - 1}{2}\,,$$

where ξ denotes the continued fraction related with the progression $6, 10, 14, 18, \ldots$. Consequently by (10.1)

$$e = 2 + \cfrac{1}{1 + \cfrac{2}{\xi - 1}} = 2 + \frac{\xi - 1}{\xi + 1} = 1 + \frac{2}{1 + 1/\xi}\,,$$

which leads us to

$$1 + \frac{1}{\xi} = \frac{2}{e-1} = -1 + \frac{e+1}{e-1}\,,$$

and finally to

$$\coth\left(\frac{1}{2}\right) = \frac{e+1}{e-1} = 2 + \frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\,\frac{1}{18+}\,\frac{1}{22+}\,\frac{1}{26+\ldots}\,.$$

Similarly, Euler obtains

$$\coth\left(\frac{1}{4}\right) = \frac{\sqrt{e}+1}{\sqrt{e}-1} = 4 + \frac{1}{12+}\frac{1}{20+}\frac{1}{28+}\dots$$
$$\coth\left(\frac{1}{6}\right) = \frac{\sqrt[3]{e}+1}{\sqrt[3]{e}-1} = 6 + \frac{1}{18+}\frac{1}{30+}\frac{1}{42+}\dots$$
$$\coth(1) = \frac{e^2+1}{e^2-1} = 1 + \frac{1}{3+}\frac{1}{5+}\frac{1}{7+}\dots .$$

A simple analysis of empirical formulas obtained by Euler shows that can be naturally explained if the following is true

$$q = \coth(p) = \frac{1}{p} + \frac{1}{\frac{3}{p}+}\frac{1}{\frac{5}{p}+}\frac{1}{\frac{7}{p}+}\dots . \tag{10.3}$$

11. Euler's formula for hyperbolic cotangent. A proof. Euler's differential method from §1 applied to (10.3) with $s = 1/p$ corresponds to the choice

$$\begin{array}{lll} a = 1, & b = s, & c = 1, \\ \alpha = 0, & \beta = 2s, & \gamma = 0, \end{array}$$

$$\frac{dS}{S} = -\frac{1}{2}\frac{dR}{R} + \frac{1}{2s}\left(R + \frac{1}{R}\right) \Longrightarrow S = \frac{1}{\sqrt{R}} e^{\frac{1}{2s}\left(R+\frac{1}{R}\right)} .$$

It is clear from the formula for S that under no choice of a positive function R can $R^{n+1}S$ vanish at any point of positive semi-axes. So, the method of §1 does not work. Historically the method of §1 appeared later. But the method Euler applied to $\coth(p)$ also was differential. Taking into account that $q = \coth(p)$ satisfies the differential equation

$$\frac{dq}{dp} = 1 - q^2 \Longleftrightarrow dq + q^2 dp = dp ,$$

Euler finds by induction the differential equation for the reminders q_n.

THEOREM 11.1 (Euler [4, §28]). *For $n = 1, 2, \dots$ we have*

$$q = \coth(p) = \frac{1}{p} + \frac{1}{\frac{3}{p}+}\frac{1}{\frac{5}{p}+}\frac{1}{\frac{7}{p}+}\dots+\frac{1}{\frac{2n-1}{p}} + \frac{1}{\frac{1}{x^{\frac{2n}{2n+1}}y}} , \tag{11.1}$$

where $p = (2n+1)x^{\frac{1}{2n+1}}$ and y satisfies the differential equation

$$\frac{dy}{dx} + y^2 = x^{\frac{-4n}{2n+1}} \tag{11.2}$$

Proof. We put $q_0 = \coth(p)$ and define a sequence of functions $\{q_n\}_{n\geqslant 0}$ by

$$q_n = \frac{2n+1}{p} + \frac{1}{q_{n+1}} . \tag{11.3}$$

Then

$$\frac{dq_n}{dp} = \frac{2n}{p}q_n + 1 - q_n^2 , n = 0, 1, \dots . \tag{11.4}$$

For $n = 0$ this equation coincides with the differential equation for $\coth(p)$. The transaction $n \longrightarrow n+1$ is done by

$$\frac{d}{dp}\left(\frac{2n+1}{p}+\frac{1}{q_{n+1}}\right)\frac{2n}{p}\left(\frac{2n+1}{p}+\frac{1}{q_{n+1}}\right)+1-\left(\frac{2n+1}{p}+\frac{1}{q_{n+1}}\right)^2 .$$

Now the identities $2n+1+2n(2n+1) = (2n+1)^2$ and $2n-2(2n+1) = -2(n+1)$ turn it into (11.4) for $n+1$. Let

$$y(x) = x^{-\frac{2n}{2n+1}} q_n\left((2n+1)x^{\frac{1}{2n+1}}\right) . \tag{11.5}$$

Then

$$\begin{aligned}\frac{dy}{dx} &= -\frac{2n}{2n+1}x^{-\frac{4n+1}{2n+1}}q_n + x^{-\frac{4n}{2n+1}}\frac{dq_n}{dp} = \\ &= -\frac{2n}{2n+1}x^{-\frac{4n+1}{2n+1}}q_n + \frac{2n}{2n+1}x^{-\frac{4n+1}{2n+1}}q_n + x^{-\frac{4n}{2n+1}} - x^{-\frac{4n}{2n+1}}q_n^2 = \\ &= x^{-\frac{4n}{2n+1}} - y^2 .\end{aligned}$$ □

It seems like that the idea to attract Riccati's equation goes back to the first paper by D. Bernoulli published in 1724. In 1724 Jacoppo Riccati (1676–1754) in a paper on the equation

$$dy + y^2 dx = ax^m\, dx ,$$

posed the problem of finding those values of m, for which the equation can be integrated in quadratures. D.Bernoulli, in the same volume of "Acta Eruditorum", announced his solution which was published in 1726 (see footnote on p. 245 of [6]). Since both Euler and Bernoulli worked in St. Petersburg, it is natural to assume that Bernoulli's contribution was known to Euler. In Theorem 11.1 Euler presented the solution in quite an original way and, as result, discovered very interesting relations.

The continued fraction in (10.3) converges since the partial denominators are positive and are members of an arithmetic progression. To be more precise let us put $p_1 = p$, $p_n = p^2$, $n = 2, 3, \ldots$ and $q_n = 2n+1$, $n = 1, 2, \ldots$ in Theorem 1.2. Since

$$\frac{1}{p} + \frac{1}{\frac{3}{p}} + \frac{1}{\frac{5}{p}} + \frac{1}{\frac{7}{p}} + \ldots \approx \frac{1}{p} + \frac{p}{3} + \frac{p^2}{5} + \frac{p^2}{7} + \ldots ,$$

the convergence of (10.3) follows.

It is tempting therefore to conclude that (10.3) may be obtained from Theorem 11.1 by a simple passage to the limit in (11.1). However, this is prevented by the paradox of quadratic equation (1.1). On the other hand Theorem 1.1 guaranties that (10.3) holds provided $q_n > 0$ on $(0, +\infty)$ for every n.

THEOREM 11.2. *Every q_n is positive on $(0, +\infty)$.*

Proof. The idea of the proof is obvious from Fig 11.1.

By (11.4) the derivative of q_n at zeros of q_n is 1, which forces the graph to cross the p-axis in the up direction. Therefore q_n cannot have positive zeros and at the same time behave like hyperbola $q = c/p$, $c > 0$ at a vicinity of 0. This forces q_n to be positive.

By Euler's formula (1.2) for continued fractions $q_n(p)$ is a Möbius transformation of $q_0 = \coth(p)$ with polynomial coefficients in $1/p$ and therefore is meromorphic in $\mathbb{C}$. Observe now that if $u = q_n$ satisfies (11.4), then $v = 1/u$ satisfies

$$\frac{dv}{dp} = -\frac{2n}{p}v + 1 - v^2 . \tag{11.6}$$

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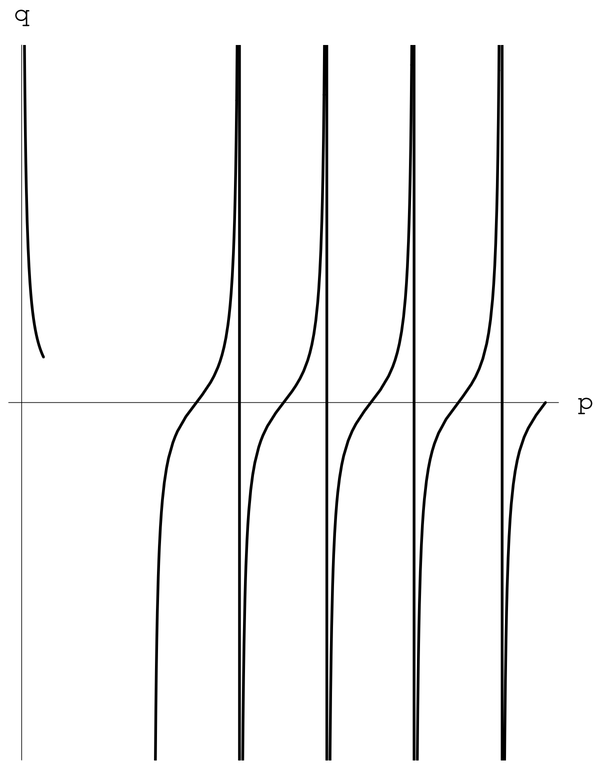

FIG. 11.1.

Therefore poles of u become zeros of v and vice-versa. Thus if either u or v vanishes, then its derivative must be 1. It follows that all nonzero zeros and poles of q_n are simple and

$$q_n(p) = \begin{cases} \dfrac{1}{p-a} + r_n(a,p) & \text{if a is a pole ;} \\ p - a + (p-a)^2 r_n(a,p) & \text{if a is a zero ,} \end{cases} \tag{11.7}$$

where $r_n(a,p)$ is analytic at $p = a$. Similar calculations with (11.4) show that if q_n has a pole at $p = 0$, then

$$q_n = \frac{2n+1}{p} + s_n(p) \ , \tag{11.8}$$

where $s_n(p)$ is holomorphic at 0. In fact $s_n(0) = 0$, which leads to the asymptotic formula

$$q_n(p) = \frac{2n+1}{p} + o(1) \ , p \to 0 \ . \tag{11.9}$$

If $n = 0$, then (11.9) follows from the Taylor series for e^{2p}. Suppose now that $s_{n-1}(0) = 0$, then by (11.3) for $n - 1$:

$$\lim_{p \to 0} q_n(p) = \infty \ ,$$

showing that q_n has a pole at $p = 0$ and therefore implying (11.8). To prove that $s_n(0) = 0$

we substitute (11.8) in (11.4):

$$-\frac{2n+1}{p^2} + s_n' \frac{2n(2n+1)}{p^2} + \frac{2ns_n}{p} + 1 - \frac{(2n+1)^2}{p^2} - \frac{2(2n+1)s_n}{p} - s_n^2 \,.$$

A calculation of the coefficients at $1/p$

$$0 = [2n - 2(2n+1)]s_n(0) = -2(n+1)s_n(0) \,,$$

competes the proof of (11.9).

If poles or zeros exist on $(0, +\infty)$, then there must be the zero or pole of the minimal value. Let it be $p = a$. Then q_n is positive near the left end of $(0, a)$ by (11.9) and q_n is negative near the right end of $(0, a)$ by (11.7). It follows that a continuous function q_n on $(0, a)$ has a zero, which contradicts to our choice of a. If there are no zeros or poles for q_n on $(0, +\infty)$, then q_n is continuous on $(0, +\infty)$ and is positive near 0 by (11.9). If $q_n(a) < 0$ for some $a, a > 0$, then q_n must vanish on $(0, a)$ which contradicts to our assumption. Hence $q_n > 0$ on $(0, +\infty)$. $\square$

Having proved (10.3) we obtain that

$$\coth\left(\frac{1}{2s}\right) = \frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1} 2s + \frac{1}{6s} + \frac{1}{10s} + \frac{1}{14s} + \frac{1}{18s} + \dots \tag{11.10}$$

for $s = 1, 2, \dots$.

Reverting arguments used in the proof of (10.2), we obtain the following formula

$$\begin{aligned} a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \dots &= \\ &= a - n + \frac{mn+1}{m} + \frac{1}{n} + \frac{1}{\frac{b-m-n}{mn+1}} + \frac{1}{m} + \frac{1}{n} + \frac{1}{\frac{c-m-n}{mn+1}} + \frac{1}{m} + \dots \,. \end{aligned} \tag{11.11}$$

In particular, for $m = n = 1$:

$$\begin{aligned} a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \dots &= \\ &= a - 1 + \frac{2}{1} + \frac{1}{1} + \frac{1}{b/2-1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{c/2-1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{d/2-1} + \dots \,, \end{aligned}$$

which implies remarkable Euler's formula

$$\boxed{e^{\frac{1}{s}} = 1 + \frac{1}{s-1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{3s-1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{5s-1} + \frac{1}{1} + \dots \,.} \tag{11.12}$$

Passing to the limit $s \to 1$ in (11.12), we obtain the regular continued fraction (10.1) for e. Formulas (10.1) and (11.12) coupled with Lagrange's Theorem on regular periodic continued fractions (see [13]) show that neither e, nor any its integer root satisfy a quadratic equation with rational coefficients.

12. Conclusion. In [4] Euler only stated Theorem 11.1 and just mentioned that the convergence of the continued fraction implies (10.3). It is not clear weather Euler realized that the paradox of quadratic equations could occur in this case as well. Let us consider this point in more details. In Theorem 11.1 Euler obtains a formula for q_n, which in terms of the independent variable p looks as follows

$$q_n(p) = x^{\frac{2n}{2n+1}} y_n(x) = \left\{\frac{p}{2n+1}\right\}^{2n} y_n\left(\left\{\frac{p}{2n+1}\right\}^{2n+1}\right) \,.$$

Now y_n itself satisfies a differential equation depending on n. Passing to the limit in it, we obtain that $y(x) = \lim_n y_n(x)$ satisfies

$$\frac{dy}{dx} + y^2 = \frac{1}{x^2} \,. \tag{12.1}$$

This differential equation has at least two solutions

$$y_1(x) = \frac{a_1}{x} \quad , \quad y_1(x) = \frac{a_2}{x} \,,$$

where $a_1 = \phi$ (the Golden Ratio) and $a_2 = -1/\phi$ are the solutions to the quadratic equation $a^2 - a - 1 = 0$. The general solution to (12.1) is given by

$$y(x) = \frac{1+\sqrt{5}}{2x} - \frac{\sqrt{5}}{x(1 + Dx^{\sqrt{5}})} \,, D \in \mathbb{R} \,.$$

Putting here $D = 0$ we obtain $y = a_2/x$. To make this formula universal we must allow $D = \infty$. Notice that there is only one solution (corresponding to $D = \infty$), which is positive about $x = 0+$. This is exactly the reason why $q_n(p) > 0$ at least about 0.

Therefore we are exactly in the situation of the paradox of quadratic equations. In any case the asymptotic formula for $q_n(p)$ at $p = 0$ supports the conjecture of the positiveness of y_n at the points

$$\left\{ \frac{p}{2n+1} \right\}^{2n+1} \longrightarrow 0 \,,$$

which are essential for the convergence.

Famous Euler's formulas related exponential and trigonometric functions were discovered later (in 1741) as it follows from the correspondence between Euler and Goldbach. The formula $\cot(p) = i \coth(ip)$ formally leads to the expression

$$\cot(p) = i\coth(ip) = \frac{1}{p} + \frac{i}{3/ip +} \frac{1}{5/ip + \dots} = \frac{1}{p} - \frac{1}{3/p -} \frac{1}{5/p -} \frac{1}{7/p - \dots} \,. \tag{12.2}$$

It is not necessary to use Euler's trigonometric formulas to find this continued fraction, since it can be obtained in a similar way to the continued fraction of the hyperbolic cotangent. One can easily obtain that the remainders u_n for the continued fraction of $\cot(p)$ satisfy the differential equation

$$\frac{du_n}{dp} = \frac{2n}{p} u_n - 1 - u_n^2 \,, n = 0, 1, 2, \dots \tag{12.3}$$

In 1731-37 Euler, seemingly, didn't have in his disposal any tool to control the convergence of the continued fraction for $\cot(p)$. Notice that there is no chance for u_n to keep sign on $(0, +\infty)$, since every u_n has infinitely many poles and zeros on $(0, +\infty)$ as $\cot(p)$ itself has. Possibly this was the reason why Euler didn't include in [4] the development of $\cot(p)$ into a continued fraction. Later Euler returned to this problem in his paper [7], which was published only after his death. This paper also shows that Euler understood the necessity of a justification to the limit in these formulas.

Theorem 11.2 was proved in 1857 by Schlömilch [19]. The fact that q_n are positive also follows from Legendre's proof [18], [13]. Both proofs completely revised Euler's original approach, which resulted in a loss of Euler's clear logic as well as of relationships with integration of Riccati's equations in quadratures. The elementary proof of Theorem 11.2 given here looks very logically related to Euler's ideas. Notice that such computations with series were well-known to Euler.

Euler had these results already in November 1731, almost 45 years before Lagrange announced (July 18, 1776) the method of solution of differential equations with continued fractions [12]. Moreover, in [3] Euler in fact solved in continued fractions Riccati's equation

$$\frac{dq}{dr} + q^2 = nr^{n-2} \, .$$

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