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ALTERNATIVE ORTHOGONAL POLYNOMIALS AND QUADRATURES*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. A bidirectional orthogonalization algorithm is applied to construct sequences of polynomials, which are orthogonal over the interval [0, 1] with the weighting function 1. Functional and recurrent relations are derived for the sequences that are the result of inverse orthogonalization procedure. Quadratures, generating by the sequences, are described. An example on approximation of the Cauchy problem is given.

Key words. orthogonal polynomial, recurrence relation, quadrature, initial value problem

AMS subject classifications. 33C45

1. Introduction. Family of the classical orthogonal polynomials originates from a problem on the differential equation of the hypergeometric type, which solution is subjected to certain additional requirements [8]. The polynomials also may be defined by an orthogonalization procedure, if it is applied to the fundamental sequence $\{x^k\}$ in the order of power increase. Generalizing this approach, one can develop the orthogonalization procedure beginning with an arbitrary number of the sequence, both in the direct and inverse order. The bidirectional algorithm of orthogonalization was introduced in [2] for defining orthogonal sequences of exponents. It was also mentioned there that the algorithm may be applied to the fundamental sequence under various orthogonality relations, and, for the polynomials constructed, the inverse algorithm retains the properties of the original sequence as $x \to 0$, if $x \in [0,1]$. Here we describe an example of such alternative sequences and show some applications. The alternative orthogonal polynomials obtained are not solutions of the equation of the hypergeometric type, but they can be expressed in terms of the Jacobi polynomials.

2. n-Sequences. Let n be a fixed whole number, $\boldsymbol{\mathcal{P}}_n$ and $\boldsymbol{P}_n$ are sequences of polynomials

$$\boldsymbol{\mathcal{P}}_n = \{\mathcal{P}_{nk}\}_{k=n}^{0}, \quad \mathcal{P}_{nk} \equiv \mathcal{P}_{nk}(x) = \sum_{l=k}^{n} \tau_{nkl} x^l, \tag{2.1}$$

$$\boldsymbol{P}_n = \{P_{nk}\}_{k=n}^{\infty}, \quad P_{nk} \equiv P_{nk}(x) = \sum_{l=n}^{k} T_{nkl} x^l \tag{2.2}$$

that satisfy the orthogonality relationships

$$\int_0^1 \mathcal{P}_{nk}\mathcal{P}_{nl}dx = \begin{cases} 0, & k \neq l, \\ 1/(k+l+1), & k = l, \end{cases} \qquad k,l = 0,1,...,n, \tag{2.3}$$

$$\int_0^1 P_{nk}P_{nl}dx = \begin{cases} 0, & k \neq l, \\ 1/(k+l+1), & k = l, \end{cases} \qquad k,l = n,n+1,..., \tag{2.4}$$

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and standardizations

$$\operatorname{sign}(\tau_{nkn}) = (-1)^{n-k}, \ \operatorname{sign}(T_{nkk}) = (-1)^{k-n}. \tag{2.5}$$

The coefficients τ_{nkl} and T_{nkl} of the polynomials $\mathcal{P}_{nk}$ and P_{nk} are defined uniquely by requirements (2.3) - (2.5) and the Gram-Schmidt orthogonalization procedure (without normalization), which is realized in the order of decreasing k from n to 0 for sequences (2.1), and in the order of increasing k, originating from $k = n$, for sequences (2.2). The sequences $\boldsymbol{\mathcal{P}}_n$ and $\boldsymbol{P}_n$ have different properties if $x \sim 0$. For fixed n and $x \to 0$ $\quad \mathcal{P}_{nk}(x) \sim x^k, \ k = 0, 1, ...n$, and $P_{nk}(x) \sim x^n, \ k = n, n+1, ...$. The sequence $\boldsymbol{P}_0$ represents the shifted to the interval $[0, 1]$ Legendre polynomials; $\boldsymbol{P}_n, \ n > 0$, is considered as an auxiliary sequence of incomplete polynomials, and the sequence $\boldsymbol{\mathcal{P}}_n$ is introduced here as the *alternative Legendre polynomials* (ALP).

The polynomials $\mathcal{P}_{nk}$ and P_{nk} have properties, which are analogous to the properties of the classical orthogonal polynomials.

Since n is fixed, by verifying property (2.4) and (2.5), the polynomials $P_{nk}(x)$ can be immediately connected to a fixed set of the Jacobi polynomials $P_m^{(\alpha,\beta)}(\xi)$ [9]. Precisely,

$$P_{nk}(x) = x^n P_{k-n}^{(2n,0)}(1-2x). \tag{2.6}$$

This relation can be used directly to describe the properties of P_{nk}, and one of the formulas that follow from (2.6) is the integral representation

$$P_{nk}(x) = \frac{1}{2\pi i} \frac{1}{x^n} \int\limits_C \frac{z^{k+n}(1-z)^{k-n}}{(z-x)^{k-n+1}} dz. \tag{2.7}$$

Here C is a closed curve, which encloses the point $z = x$. Representation (2.7) will be employed below.

The orthogonalization procedure is the only starting point for examining polynomials $\mathcal{P}_{nk}$. Realizing the procedure one can suppose that the explicit definition of the polynomials $\mathcal{P}_{nk}$ is

$$\mathcal{P}_{nk}(x) = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \binom{n+k+1+j}{n-k} x^{k+j}, \ k = 0, 1, ..., n.$$

This yields the Rodrigues' type representation,

$$\mathcal{P}_{nk}(x) = \frac{1}{(n-k)!} \frac{1}{x^{k+1}} \frac{d^{n-k}}{dx^{n-k}} (x^{n+k+1}(1-x)^{n-k}), \ k = 0, 1, ..., n, \tag{2.8}$$

and orthogonality relationships (2.3) are confirmed by applying last formula. It also follows from (2.8) that

$$\int\limits_0^1 \mathcal{P}_{nk}(x) dx = \int\limits_0^1 x^n dx = \frac{1}{n+1}. \tag{2.9}$$

Making use of formula (2.8), the Cauchy integral formula for derivatives of an analytic function and reciprocal substitutions one can obtain the integral representation

$$\mathcal{P}_{nk}(x) = \frac{1}{2\pi i} \frac{1}{x^{n+2}} \int\limits_{C_1} \frac{z^{-(n+k+2)}(1-z)^{n-k}}{(z-x^{-1})^{n-k+1}} dz, \tag{2.10}$$

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where C_1 is a closed curve, which encloses the point $z = x^{-1}$. Employing the theory developed in [8] and representation (2.10) one may complete description of $\mathcal{P}_{nk}$. However, there is more simple way to do it.

Representations (2.7) and (2.10) lead directly to *the reciprocity relation*

$$\mathcal{P}_{nk}(x) = x^{-1} P_{-(n+1),-(k+1)}(x^{-1}), \tag{2.11}$$

which is the result of the bidirectional orthogonalization. Relationship, similar to (2.11), holds also for orthogonal exponential polynomials [2].

Identity (2.11) facilitates description of the ALP, and the results that are shown below can be obtained making use of the auxiliary sequences $P_{nk}(x)$ and relationship (2.11). In particular,

$$\mathcal{P}_{nn} = x^n, \ \mathcal{P}_{n,n-1} = 2nx^{n-1} - (2n+1)x^n,$$

and the following recurrence relations and differentiation formulas hold:

$$a_{nk}\mathcal{P}_{n,k-1} = (b_{nk}x^{-1} - c_{nk})\mathcal{P}_{nk} - d_{nk}\mathcal{P}_{n,k+1} \tag{2.12}$$

$$\alpha_{nk}x(1-x)\mathcal{P}'_{nk} = (\beta_{nk} - \gamma_{nk}x)\mathcal{P}_{nk} - \delta_{nk}x\mathcal{P}_{n,k+1},$$

$$\kappa_{nk}x(x-1)\mathcal{P}'_{nk} = (\lambda_{nk} - \mu_{nk}x)\mathcal{P}_{nk} - \nu_{nk}x\mathcal{P}_{n,k-1},$$

where

$$a_{nk} = (k+1)(n-k+1)(n+k+1), \quad b_{nk} = k(2k+1)(2k+2),$$

$$c_{nk} = (2k+1)((n+1)^2 + k^2 + k), \quad d_{nk} = k(n-k)(n+k+2),$$

$$\alpha_{nk} = 2(k+1), \quad \beta_{nk} = 2k(k+1), \quad \gamma_{nk} = n^2 + k^2 + 2n,$$

$$\delta_{nk} = (n-k)(n+k+2), \quad \kappa_{nk} = 2k, \quad \lambda_{nk} = 2k(k+1),$$

$$\mu_{nk} = (n+1)^2 + k^2, \quad \nu_{nk} = (n-k+1)(n+k+1).$$

The polynomial $x\mathcal{P}_{nk}(x)$ is a solution of the differential equation

$$x^2(1-x)\zeta'' - x^2\zeta' + ((n+1)^2 x - k(k+1))\zeta = 0, \ k = 0, 1, ..., n \tag{2.13}$$

that also follows from the constructions developed. Making use of the substitution $\zeta(x) = x^k u(x)$ one can represent the polynomial solution of equation (2.13) in terms of the hypergeometric function F, and the following relationships hold

$$\mathcal{P}_{nk}(x) = (-1)^{n-k} x^k F(k-n, k+n+2; 1; 1-x),$$

$$\mathcal{P}_{nk}(x) = (-1)^{n-k} x^k P_{n-k}^{(0,2k+1)}(2x-1), \ k = 0, 1, ..., n. \tag{2.14}$$

Thus, the ALP are related to *different* families of the Jacoby polynomials.

Properties of the zeros of the Jacobi polynomials [9] and the last relationship give the following result.

COROLLARY 2.1. *Polynomials $\mathcal{P}_{nk}(x)$ have k multiple zeros $x = 0$ and $n-k$ distinct real zeros in the interval* $[0,1]$.

Recurrence relation (2.12) can be represented in the form that shows polynomial $(2k+1)x^{-1}\mathcal{P}_{nk}(x)$ explicitly; then, regular transformations ([7], for example) lead to the Christoffel-Darboux identity

$$\left(\frac{1}{x}-\frac{1}{t}\right)\sum_{k=1}^{n}(2k+1)\,\mathcal{P}_{nk}(x)\mathcal{P}_{nk}(t) = \frac{n\,(n+2)}{2}\left(\mathcal{P}_{n0}(x)\mathcal{P}_{n1}(t)-\mathcal{P}_{n1}(x)\mathcal{P}_{n0}(t)\right).$$

The identity facilitates calculating coefficients in approximate integration formulas.

3. Quadratures. Let $f(x)$ be a continuous function on the interval $[0,1]$ and $f(0) = 0$.

LEMMA 3.1. *The polynomial of degree n interpolating the function $f(x)$ at $n+1$ distinct points $x_0 = 0$, $x_j \in (0,1]$, $j = 1, 2, ..., n$ can be defined as*

$$p_n(x) = \sum_{j=1}^{n}\ell_j(x)f(x_j), \tag{3.1}$$

where $\ell_j(x)$ are the Lagrange polynomials

$$\ell_j(x) = \frac{x}{x_j}\cdot\frac{\phi_n(x)}{(x-x_j){\phi_n}'(x_j)}, \qquad \phi_n(x) = (x-x_1)...(x-x_n).$$

Proof. The term that includes the factor $\ell_0(x)$ is absent in (3.1), because $f(x_0) = 0$. ∎

Let

$$\int_0^1 f(x)dx \approx \sum_{s=1}^{n} w_s f(x_s) \tag{3.2}$$

be an interpolatory quadrature rule, thus the weighting coefficients can be found by substitution $p_n(x)$ for $f(x)$ and

$$w_s = \int_0^1 \ell_s(x)dx. \tag{3.3}$$

Let indices n and k show the power range $k \le l \le n$ of a polynomial, say $\mathcal{Q}_{nk}(x)$, with the terms x^l.

THEOREM 3.2. *Quadrature rule (3.2),(3.3) is interpolatory iff it is exact for any polynomial $\mathcal{Q}_{n1}(x)$.*

THEOREM 3.3. *Quadrature rule (3.2),(3.3) is exact for any polynomial $\mathcal{Q}_{2n,1}(x)$ iff the polynomial $\phi_n(x)$ is orthogonal to any polynomial $\mathcal{Q}_{n1}(x)$.*

Comparatively to the regular case, the theorems expose the shift in the power range, but it does not introduce peculiarities in their proofs. Proofs for the regular case can be found, for example, in [7].

Being subjected to the requirement of the last theorem, polynomial $\phi_n(x)$ differs polynomial $\mathcal{P}_{n0}(x)$ by a constant factor that results in the definition of the weighting coefficients as follows

$$w_s = \int_0^1 \frac{x}{x_s} \cdot \frac{\mathcal{P}_{n0}(x)}{(x - x_s)\mathcal{P}'_{n0}(x_s)} dx. \tag{3.4}$$

Now, the above described Christoffel-Darboux identity can be involved in evaluating the integral in (3.4) that leads to the following result.

COROLLARY 3.4. *Quadrature (3.2) is exact for any polynomial $\mathcal{Q}_{2n,1}(x)$ iff x_s are the zeros of the polynomial $\mathcal{P}_{n0}(x)$ and the weight factors are*

$$w_s = -\frac{2}{n(n+1)(n+2)} \cdot \frac{\sum_{k=1}^{n} (2k+1)\,\mathcal{P}_{nk}(x_s)}{x_s^2 \mathcal{P}_{n1}(x_s)\mathcal{P}'_{n0}(x_s)}. \tag{3.5}$$

Although alternative Gauss quadrature (3.2),(3.5) is developed for approximate integration of a function $f(x)$ subjected to the requirement $f(0) = 0$, it can be easily extended to the general case.

Let $g(x)$ be a continuous function on the interval $[0, 1]$, and

$$\int_0^1 g(x)dx \approx w_0 g(0) + \sum_{s=1}^{n} w_s g(x_s). \tag{3.6}$$

COROLLARY 3.5. *Quadrature (3.6),(3.5) is exact for any polynomial $\mathcal{Q}_{2n,0}(x)$ iff x_s are the zeros of the polynomial $\mathcal{P}_{n0}(x)$ and*

$$w_0 = 1 - \sum_{s=1}^{n} w_s. \tag{3.7}$$

Proof. Let $f(x) = g(x) - g(0)$, then the quadrature rule (3.2) can be shown in the form (3.6), (3.7). ◻

It may be noticed in (2.14) that the abscissas of the Radau quadrature [7] inside of the interval $[0, 1)$ are the zeros x_s of the equation $\mathcal{P}_{n0}(x) = 0$. Thus, quadrature rule (3.6), (3.5) and (3.7) represents differently the Radau quadrature.

The same abscissas x_s can be employed to generate one more quadrature. Let us approximate the function $g(x)$ by *almost* orthogonal sequence of polynomials $\{1, \mathcal{P}_{nk}\}_{k=n}^{1}$ as follows

$$g(x) \approx a_0 \cdot 1 + \sum_{j=1}^{n} a_j \mathcal{P}_{nj}(x). \tag{3.8}$$

To find coefficients $a_j, j = 0, ..., n$ we examine (3.8) at the abscissas $\{x_s, 1\}_{s=1}^{n}$, $x_s < x_{s+1}$. Making use of the polynomial values $\mathcal{P}_{nj}(1) = (-1)^{n-j}$ one can immediately show that

$$a_0 = -\sum_{j=1}^{n} (-1)^{n-j} a_j + g(1), \tag{3.9}$$

and

$$g(x) \approx \sum_{j=1}^{n} a_j(\mathcal{P}_{nj}(x) - (-1)^{n-j}) + g(1). \tag{3.10}$$

Below we apply the discrete form of integral relationships (2.3)

$$\sum_{s=1}^{n} w_s \mathcal{P}_{nj}(x_s)\mathcal{P}_{nl}(x_s) = \delta_{jl}/(k+l+1) \tag{3.11}$$

and the discrete form of calculation of the first integral in (2.9)

$$\sum_{s=1}^{n} w_s \mathcal{P}_{nl}(x_s) = 1/(n+1) \tag{3.12}$$

that follow from the alternative Gauss quadrature for $k, l = 1, ..., n$; δ_{jl} is the Kronecker delta.

Multiplying (3.10) by $w_s\mathcal{P}_{nl}(x_s)$, adding them, making use of (3.11), (3.12) and inverting the matrix in the right hand side of the equation obtained, one can solve problem (3.8) as follows

$$a_j = (2j+1)\left(\sum_{l=1}^{n} A_{jl} \sum_{s=1}^{n} \mathcal{P}_{nl}(x_s) w_s g(x_s) - (-1)^n g(1)\right), \tag{3.13}$$

$$A_{jl} = (-1)^l \begin{cases} 2l+1, & j \neq l \\ 2l, & j = l, \quad l \text{ odd} \\ 2(l+1), & j = l, \quad l \text{ even.} \end{cases} \tag{3.14}$$

Approximation (3.8), (3.9), (3.13), (3.14) results in a quadrature formula. Indeed, integrating (3.8) and making use of coefficients (3.9), (3.13) one can obtain

$$\int_0^1 g(x)dx \approx \sum_{s=1}^{n} c_s w_s g(x_s) + c_{n+1} g(1), \tag{3.15}$$

$$c_s = \sum_{l=1}^{n} \alpha_l \mathcal{P}_{nl}(x_s), \tag{3.16}$$

$$c_{n+1} = \frac{(-1)^n}{n+1}, \quad \alpha_l = \begin{cases} 2(2l+1)/(n+1), & l \text{ odd} \\ 0, & l \text{ even.} \end{cases} \tag{3.17}$$

Obviously, quadrature rule (3.15) - (3.17) is exact for any polynomial $Q_{n0}(x)$, but it has no nice properties of the Gauss and the Radau quadratures. In particular, for $n = 1$

$$x_1 = 2/3, \quad c_1 \cdot w_1 = 3/2, \quad x_2 = 1, \quad c_2 = -1/2;$$

for $n = 2$

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$$\begin{aligned} x_1 &= (6-\sqrt{6})/10, & c_1 \cdot w_1 &= (12+7\sqrt{6})/36 \approx 0.809623, \\ x_2 &= (6+\sqrt{6})/10, & c_2 \cdot w_2 &= (12-7\sqrt{6})/36 \approx -0.142956, \\ x_3 &= 1, & c_3 &= 1/3, \end{aligned}$$

and thus the weights in (3.15) have mixed signs.

Once one of the quadratures described above has been chosen, the polynomials $\mathcal{P}_{nj}(x)$ provide evaluation of an antiderivative of a continuous function on the interval $[0,1]$. Let us illustrate it for (3.15). Integrating approximation (3.8) and making use of the formula

$$\int_0^x \mathcal{P}_{nj}(t)dt = \frac{x\mathcal{P}_{nj}(x)}{j+1} + \sum_{m=j+1}^{n} (2+1/m)\frac{x\mathcal{P}_{nm}(x)}{m+1}$$

one can represent the result as follows

$$\int_0^x g(t)dt \approx \sum_{s=1}^{n} C_s(x)w_s g(x_s) + C_{n+1}(x)g(1),$$

where the functions $C_s(x)$, $C_{n+1}(x)$ easily can be found by substitution of known coefficients (3.9) and (3.13) in (3.8).

Quadratures (3.6) and (3.15) may be associated with spectral approaches for solving the Cauchy problem. The endpoint abscissas in (3.6) and (3.15) are different, but both quadratures can be applied for constructing algorithms. In this study, we chose (3.15) to develop a discrete analogue of the initial value problem.

4. Approximation of the solution of the initial value problem. The Cauchy problem often is considered as an extension of a problem on integration. This results in high order implicit collocation algorithms based on known quadrature rules ([1], [6], for example).

Still another approach is approximation of the solution by a sequence of functions, i.e. application of spectral methods that usually involve transformations both in the spectral space and in the space of the original variables. Advantage of such an approach is in the opportunity to integrate the Cauchy problem on any subinterval $[t, t+h_p]$, $h_p \le h$ without recomputing its right hand side, if the solution has been approximated on the subinterval. It allows explicit recurrence algorithms and algorithms with translation of the integration interval to be developed.

Recurrence algorithms can be employed to compute a trial vector for approximation of the non-linear initial value problem. In addition, preliminary computations show that the recurrence algorithms itself expose better stability properties than explicit Runge-Kutta methods and may be applied independently.

Algorithms with partial translation might push forward implementation of fully implicit Runge-Kutta methods, making them competitive with multistep methods. Indeed, a translation may involve only a few non-equidistant abscissas.

First fully spectral approach for the Cauchy problem solving that includes the recurrence algorithm was elaborated for arbitrary n in [4], where the sequences $\boldsymbol{\mathcal{E}}_n = \{\mathcal{E}_{nk}(t)\}_{k=n}^{1}$ of alternatively constructed orthogonal polynomials of exponents [2] were exploited as the basis functions. (One may mention that the system of functions $\mathcal{E}_{n0} \cup \boldsymbol{\mathcal{E}}_n$ generates the Gauss-type quadratures for exponents on the semi-axis [3].)

We develop here an algorithm of discretization of the initial value problem analogous to that one in [4]. The sequences $\mathcal{P}_{nk}$ are applied for the solution approximation. A preliminary result, the procedure for $n = 1$, is revealed. It shows that well known low order methods [5] are the components of the spectral approach.

Let us consider the initial value problem

$$y'(x) = f(x, y(x)), \quad y(t) = y_0 \tag{4.1}$$

for the function y on the interval $x \in [t, t+h]$, $h > 0$, where the function f is known, and it has the properties that are necessary for the following constructions. Given vector y_0 and the functions y and f have dimension N. Substitutions

$$x = t + h\tau, \quad \xi(\tau) = y(t + h\tau), \quad \zeta(\tau,\xi(\tau)) = hf(t + h\tau, y(t + h\tau)) \tag{4.2}$$

reduce the original problem to the following

$$\xi'(\tau) = \zeta(\tau,\xi(\tau)), \quad \xi(0) = y_0, \quad \tau \in [0,1]\,. \tag{4.3}$$

Approximation $\xi_n(\tau) \approx \xi(\tau)$ to problem (4.3) in the spectral form

$$\xi_n(\tau) = y_0 + \zeta_0\tau + \sum_{j=1}^{n} \mathfrak{a}_{nj}(\mathcal{S}_{nj}(\beta_n, \tau) \; - (\,-\,1)^{n-j}\tau), \tag{4.4}$$

$$\mathcal{S}_{nj}\,(\beta_n, \tau) = \int_0^\tau \mathcal{P}_{nj}\,(\beta_n\,(1 - \eta))\, d\eta, \quad \beta_n = 1 \tag{4.5}$$

satisfies the initial conditions in (4.3) and the initial conditions for the derivative of the unknown variable ξ. Here $\zeta_0 = \zeta(0, \xi(0))$, and $\mathfrak{a}_{nj}$ are the unknown vector coefficients with dimension N.

To find the coefficients $\mathfrak{a}_{nj}$, one can substitute approximation (4.4) to the differential equation in (4.3) and examine its discrete form

$$\xi'_n(\lambda_{ns}) = \zeta(\lambda_{ns}, \xi_n(\lambda_{ns}))$$

at the abscissas $\{\lambda_{ns}\}_{s=1}^{n}$, $\lambda_{ns} < \lambda_{n,s+1}$, which are the zeros of the polynomial $\mathcal{P}_{n0}(1 - x)$. Making use of spectral representations (4.4), (4.5) and the results of the previous section one can obtain n functional equations for the vectors $\mathfrak{a}_{nj}$ as follows

$$\mathfrak{a}_{nj} = (2j + 1)\left(\sum_{l=1}^{n} A_{jl} \sum_{s=1}^{n} \mathcal{P}_{nl}(1 - \lambda_{ns})w_{n-s+1}\zeta_{ns} - (-1)^n\zeta_0\right), \tag{4.6}$$

$$\zeta_{ns} \equiv \zeta(\lambda_{ns},\; \xi_n(\lambda_{ns})).$$

Equation (4.6) together with (4.4) at $\tau = \lambda_{ns}$ are the spectral form of initial value problem (4.1); the form allows to return to the space of original variables.

Substitution of (4.6) in (4.4) results in approximation

$$\xi_n(\tau) = y_0 + \Omega_{n0}(\tau)\zeta_0 + \sum_{s=1}^{n} \Omega_{ns}(\tau)w_{n-s+1}\zeta_{ns}, \tag{4.7}$$

where the functions $\Omega_{n0}(\tau)$ and $\Omega_{ns}(\tau)$ are

$$\Omega_{n0}(\tau) = (-1)^n \left((n + 1)\tau - \sum_{j=1}^{n} (2j + 1)\mathcal{S}_{nj}(\beta_n, \tau)\right),$$

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$$\Omega_{ns}(\tau) = \sum_{l=1}^{n} \mathcal{P}_{nl}(1-\lambda_{ns})\rho_{nl}(\tau),$$

$$\rho_{nl}(\tau) = (n+1)(-1)^{l+1}(2l+1)\tau + \sum_{j=1}^{n}(2j+1)A_{jl}\mathcal{S}_{nj}(\beta_n,\tau).$$

Vector ξ_n in (4.7) can be calculated at any $\tau \in [0,1]$, if values of ζ_{ns} are known.

Equating (4.7) at $\tau = \{\lambda_{ns}, 1\}_{s=1}^{n}$ and making use of substitutions (4.2) we finally obtain the discrete analogue of the Cauchy problem (4.1) that represents an implicit Runge-Kutta method as follows

$$x_p = t + \varkappa_p h, \quad y_p = y(x_p),$$

$$y_p = y_0 + \omega_{p0} h f(t,y_0) + h\sum_{s=1}^{n}\omega_{ps} f(x_s,y_s), \quad p = 1,2,...n, \tag{4.8}$$

$$y(t+h) \approx y_0 + \sigma_0 h f(t,y_0) + h\sum_{s=1}^{n}\sigma_s f(x_s,y_s). \tag{4.9}$$

We dropped index n in (4.8), (4.9) introducing coefficients

$$\varkappa_p = \lambda_{np}, \quad \omega_{p0} = \Omega_{n0}(\lambda_{np}), \quad \omega_{ps} = \Omega_{ns}(\lambda_{np}), \quad \sigma_0 = c_{n+1}, \quad \sigma_s = c_{n-s+1}w_{n-s+1}.$$

Solving equations (4.8) is a difficult problem, and trial values for the vectors y_p are necessary. It follows from (2.14) that the zeros λ_{ns} of $\mathcal{P}_{nk}(1-x)$ thicken near $x = 0$ when increasing n. This indicates that for fixed n computation can originate in low degree approximation and then can be extended from point to point by drawing in the polynomials with successively increasing degree. Following [4], one can develop an explicit recurrence algorithm for evaluating trial values on the given interval.

Here we complete the description of algorithm (4.2) - (4.9) by considering the case $n = 1$ that can be shown as follows

$$y\left(t+\varkappa_1 h\right) = y\left(t\right) + \omega_{10} h f\left(t, y\left(t\right)\right) + \omega_{11} h f\left(t+\varkappa_1 h, y\left(t+\varkappa_1 h\right)\right), \tag{4.10}$$

$$\varkappa_1 = 1/3, \quad \omega_{10} = 1/6, \quad \omega_{11} = 1/6,$$

$$y\left(t+h\right) \approx y\left(t\right) + \sigma_0 h f\left(t, y\left(t\right)\right) + \sigma_1 h f\left(t+\varkappa_1 h, y\left(t+\varkappa_1 h\right)\right), \tag{4.11}$$

$$\sigma_0 = -1/2, \quad \sigma_1 = 3/2.$$

The trapezoidal rule (4.10) [5] with $h^* = \varkappa_1 h$ coupled with completion (4.11) has the order of $O(h^3)$ and requires one calculation of the right hand side of the initial value problem and solving one nonlinear equation.

Procedure (4.10), (4.11) is not A-stable; however, its core (4.10) possesses this stability property. It also may be the case for greater n, when the last abscissa is much closer to the end of the integration interval, and procedures without completion might be better for solving stiff problems.

It appears that the recurrence algorithm

$$y^E\left(t+\varkappa_1 h\right)=y\left(t\right)+\varkappa_1 hf\left(t,y\left(t\right)\right), \tag{4.12}$$

$$y^H\left(t+\varkappa_1 h\right)=y\left(t\right)+\omega_{10}hf\left(t,y\left(t\right)\right)+\omega_{11}hf\left(t+\varkappa_1 h,y^E\left(t+\varkappa_1 h\right)\right), \tag{4.13}$$

$$y\left(t+h\right)\approx y\left(t\right)+\sigma_0 hf\left(t,y\left(t\right)\right)+\sigma_1 hf\left(t+\varkappa_1 h,y^H\left(t+\varkappa_1 h\right)\right) \tag{4.14}$$

represents explicit Runge-Kutta method of the order of $O(h^3)$, and its part (4.12), (4.13) is the Heun method with $h^*=\varkappa_1 h$ [5]. Together with completion (4.14), it has better characteristics of stability and monotonicity comparatively to the standard Runge-Kutta method of the order of $O(h^5)$. The vector y^H may be employed as an initial guess for solving equation (4.10).

In general, the recurrence algorithms are not the Runge-Kutta methods, because they are not subjected to the requirement of elimination of low order terms in asymptotic error estimates. However, special collocation of abscissas may result in reducing coefficients in the estimates when increasing n. Convergence of the recurrence algorithms for chosen n and small step of size h may be proven following theorems described in [5].

In addition to the procedures without completion mentioned above, there is another way to consider stiff problems. Approximation ξ_n in (4.4),(4.5) can be set in the form

$$\xi_n(\tau)=y_0+\zeta_1\tau+\sum_{j=1}^{n}\mathfrak{a}_{nj}\mathcal{S}_{nj}(1,\tau), \tag{4.15}$$

where $\zeta_1=\zeta(1,\xi(1))$. This leads to a spectral method based exactly on the Radau abscissas. It is known that for $n=2$ the implicit Runge-Kutta procedure corresponding to (4.15) is both A- and L-stable (3-stage algorithm Radau IIA, or Ehle method, [1, 6]).

Abscissas for approximation (4.15) differ from those ones in Runge-Kutta method (4.8), (4.9) only at the endpoints of the interval $[0,1]$, and the explicit version of (4.8), (4.9) may be applied to compute trial values for $\xi_n(\tau)$ in (4.15) on a shifted interval of integration.

5. Conclusions. The alternative orthogonal polynomials keep distinctively attributes of the classical orthogonal polynomials. In particular, the algorithm of inverse orthogonalization of the fundamental sequence results in redistribution of the zeros. The polynomials may be applied to different problems on approximation.

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