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BERNSTEIN'S WEIGHTED APPROXIMATION ON $\mathbb{R}$ STILL HAS PROBLEMS*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Let $W:\mathbb{R}\to(0,1]$ be continuous. Bernstein's approximation problem, posed in 1924, dealt with approximation by polynomials in the norm

$$\|f\|_W := \|fW\|_{L_\infty(\mathbb{R})}.$$

The qualitative form of this problem was solved by Achieser, Mergelyan, and Pollard, in the 1950's. Quantitative forms of the problem were actively investigated starting from the 1960's. We survey old and recent aspects of this topic. One recent finding is that there are weights for which the polynomials are dense, but which do not admit a Jackson-Favard inequality. In fact the weight $W(x)=\exp(-|x|)$ exhibits this peculiarity. Moreover, not all L_p spaces are the same when degree of approximation is considered. We also pose some open problems.

Key words. weighted approximation, polynomial approximation, Jackson-Bernstein theorems

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1. Introduction. Suppose we wish to approximate by polynomials on the whole real line, obtaining analogues of Weierstrass' Theorem. Then we have to deal with the unboundedness of polynomials on unbounded intervals. To cope with this difficulty, that distinguished approximator S. N. Bernstein multiplied by a weight, considering weighted polynomials such as

$$P(x)\exp\left(-x^2\right), \qquad x\in\mathbb{R},$$

where P is a polynomial, or more generally,

$$P(x)\,W(x).$$

Here W decays sufficiently fast at $\pm\infty$ to counteract the growth of every polynomial.

The most intriguing question is what can be approximated, and in what sense. This problem is known as *Bernstein's approximation problem*, after it was posed by Bernstein in 1924. A more precise statement is as follows: let $W:\mathbb{R}\to(0,1]$ be continuous. When is it true that for every continuous $f:\mathbb{R}\to\mathbb{R}$ with

$$\lim_{|x|\to\infty}(fW)(x)=0,$$

there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty}\|(f-P_n)W\|_{L_\infty(\mathbb{R})}=0?$$

We say then that the polynomials are *dense*. The restriction that fW has limit 0 at $\pm\infty$ is essential: if $x^kW(x)$ is bounded on the real line for every non-negative k, then $x^kW(x)$ has limit 0 at $\pm\infty$ for every such k, and so the same is true of every weighted polynomial PW. So we could not hope to approximate in the uniform norm, any function f for which fW does not have limit 0 at $\pm\infty$. The version of Bernstein's problem considered here is not the

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most general form: in some versions, W is not assumed to be continuous, or defined on all of $\mathbb{R}$, allowing (for example), a weight defined on a countable set of points.

Bernstein's approximation problem was solved independently by Achieser, Mergelyan, and Pollard, in the 1950's. Their solutions involve regularization of the weight. For example [10, p. 153] Mergelyan showed that there is a positive answer to Bernstein's problem iff

$$\int_{-\infty}^{\infty} \frac{\log \Omega(t)}{1+t^2} \, dt = \infty,$$

where Mergelyan's regularization of W is

$$\Omega(z) = \sup \left\{ |P(z)| : P \text{ a polynomial and } \sup_{t \in \mathbb{R}} \frac{|P(t) W(t)|}{\sqrt{1+t^2}} \leq 1 \right\}.$$

In another formulation, there is a positive answer iff

$$\Omega(z) = \infty$$

for at least one non-real z (and then $\Omega(z) = \infty$ for all non-real z).

Akhiezer [10, p. 158] used instead the regularization

$$W_*(z) = \sup \left\{ |P(z)| : P \text{ a polynomial with } \|PW\|_{L_\infty(\mathbb{R})} \leq 1 \right\}.$$

He showed that the polynomials are dense iff

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2} \, dt = \infty.$$

Finally, Pollard [10, p. 164] showed that the polynomials are dense essentially iff

$$\sup \left\{ \int_{-\infty}^{\infty} \frac{\log |P(x)|}{1+x^2} dx : P \text{ a polynomial with } \|PW\|_{L_\infty(\mathbb{R})} \leq 1 \right\} = \infty.$$

Of course, these are not very transparent criteria. When the weight is in some sense regular, simplifications are possible. If W is even, and $\ln 1/W(e^x)$ is even and convex, a simpler necessary and sufficient condition for density of the polynomials is [10, p. 170]

$$\int_0^{\infty} \frac{\ln 1/W(x)}{1+x^2} \, dx = \infty.$$

In particular, for

$$W_\alpha(x) = \exp\left(-|x|^\alpha\right),$$

the polynomials are dense iff $\alpha \geq 1$. As regards necessary conditions, Hall showed that

$$\int_{-\infty}^{\infty} \frac{\log W(t)}{1+t^2} dt = \infty$$

is necessary for density. When density fails, only a limited class of entire functions can be approximated. A comprehensive treatment of this topic is given in Koosis' book [10]. A concise elegant exposition appears in [9, p. 28 ff.].

In the 1950's the search began for a quantitative form of Bernstein's Theorem. Bernstein and Jackson had provided quantitative forms of Weierstrass' Theorem before the first World

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War, and it is natural to look for analogues in the weighted setting. Let us first recall the classical unweighted case. Jackson and Bernstein independently proved that

$$E_n[f]_\infty := \inf_{\deg(P)\le n} \|f-P\|_{L_\infty[-1,1]} \le \frac{C}{n}\|f'\|_{L_\infty[-1,1]}, \tag{1.1}$$

with C independent of f and n, and the inf being over (algebraic) polynomials of degree at most n. The rate is best possible amongst absolutely continuous functions f on $[-1,1]$ whose derivative is bounded. More generally, if f has a bounded kth derivative, then the rate is $O\left(\frac{1}{n^k}\right)$. In addition, Jackson obtained general results involving moduli of continuity: for example, if f is continuous, and its modulus of continuity is

$$\omega(f;\delta) = \sup\{|f(x)-f(y)| : x,y\in[-1,1] \text{ and } |x-y|\le\delta\},$$

then

$$E_n[f]_\infty \le C\omega\left(f;\frac{1}{n}\right),$$

where C is independent of f and n.

Bernstein also obtained remarkable converse theorems, which show that the rate (or degree) of approximation is determined by the smoothness of f. These are best stated for trigonometric polynomial approximation: let $0<\alpha<1$. Bernstein showed that the error of approximation of a $2\pi-$periodic function g on $[0,2\pi]$ by trigonometric polynomials of degree at most n decays with rate $n^{-\alpha}$ iff g satisfies a Lipschitz condition of order α. For non-integer $\alpha>1$, the error decays with rate $O(n^{-\alpha})$ iff the $[\alpha]$th derivative of f satisfies a Lipschitz condition of order $\{\alpha\}$. (Here $[\alpha],\{\alpha\}$ respectively denote the integer and fractional parts of α). Bernstein never resolved the exact smoothness required for a rate of decay of n^{-1}; that was solved much later in 1945 by A. Zygmund (the father of the Chicago school of harmonic analysis, and author of the classic "Trigonometric Series" [24]). Zygmund used a second order modulus of continuity.

For approximation by algebraic polynomials, converse theorems are more complicated, as better approximation is possible near the endpoints of the interval of approximation. Only in the 1980's were complete characterizations obtained, with the aid of the Ditzian-Totik modulus of continuity [6]. An earlier alternative approach is that of Brudnyi-Dzadyk-Timan [3]. We shall discuss only the Ditzian-Totik approach, since that has been adopted in weighted polynomial approximation. Define the symmetric differences

$$\begin{aligned} \Delta_h f(x) &= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right); \\ \Delta_h^2 f(x) &= \Delta_h(\Delta_h f(x)); \\ &\vdots \\ \Delta_h^k f(x) &= \Delta_h\left(\Delta_h^{k-1} f(x)\right) \end{aligned}$$

so that

$$\Delta_h^k f(x) = \sum_{i=0}^{k}\binom{k}{i}(-1)^i f\left(x+\frac{kh}{2}-ih\right).$$

If any of the arguments of f lies outside the interval of approximation — $[-1,1]$ in this setting — we adopt the convention that the difference is 0. The rth order Ditzian-Totik modulus of

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continuity in L_p is

$$\omega_{\varphi}^{r}(f;h)_p = \sup_{0<h\leq t} \|\Delta^{r}_{h\sqrt{1-x^2}} f(x)\|_{L_p[-1,1]}.$$

Note the factor

$$\varphi(x) = \sqrt{1-x^2}$$

multiplying the increment h. This forces a smaller increment near the endpoints ± 1 of $[-1,1]$, reflecting the possibility of better approximation rates there.

For $1 \leq p \leq \infty$, Ditzian and Totik [6, p. 79] proved the estimate

$$E_n[f]_p := \inf_{\deg(P)\leq n} \|f-P\|_{L_p[-1,1]} \leq C\omega_{\varphi}^{r}\left(f;\frac{1}{n}\right)_p,$$

with C independent of f and n. This implies the Jackson (or Jackson-Favard) estimate [3, p. 260]

$$E_n[f]_p \leq Cn^{-r}\|\varphi^r f^{(r)}\|_{L_p[-1,1]},$$

$n \geq r$, provided $f^{(r-1)}$ is absolutely continuous, and the norm on the right-hand side is finite. Moreover, they showed that if $0<\alpha<r$, then [3, p. 265]

$$E_n[f]_p = O\left(n^{-\alpha}\right), \qquad n\to\infty, \tag{1.2}$$

iff

$$\omega_{\varphi}^{r}(f;h)_p = O(h^{\alpha}), \qquad h\to 0+.$$

For example, if (1.2) holds with $\alpha = 3\frac{1}{2}$, this implies that f has 3 continuous derivatives inside $(-1,1)$ and f''' satisfies a Lipschitz condition of order 1/2 in each compact subinterval of $(-1,1)$.

This equivalence is easily deduced from the Jackson inequality above, and the general converse inequality [6, Theorem 7.2.4, p. 83]

$$\omega_{\varphi}^{r}(f;t)_p \leq Mt^r \sum_{0<n<\frac{1}{t}} (n+1)^{r-1} E_n[f]_p.$$

The constant M depends on r, but is independent of f and t.

For weights on the whole real line, the first attempts at general Jackson theorems seem due to Dzrbasjan. In the 1960's and 1970's, Freud and Nevai made major strides in this topic [22]. Let us review some of the fundamental features discovered by Freud, in the case of the weight $W_\alpha(x) = \exp(-|x|^{\alpha})$, $\alpha > 1$. A little calculus shows that the weighted monomial $x^n W_\alpha(x)$ attains its maximum modulus on the real line at

$$q_n = (n/\alpha)^{1/\alpha}.$$

Thereafter it decays quickly to zero. With this in mind, Freud and Nevai proved that there are constants C_1 and C_2 such that for all polynomials P_n of degree at most n,

$$\|P_n W_\alpha\|_{L_p(\mathbb{R})} \leq C_1 \|P_n W_\alpha\|_{L_p[-C_1 q_n, C_1 q_n]}. \tag{1.3}$$

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The constants C_1 and C_2 can be taken independent of n, P_n and even the L_p parameter $p \in [1, \infty]$. Outside the interval $[-C_1 q_n, C_1 q_n]$, $P_n W_\alpha$ decays quickly to zero. This meant that one cannot hope to approximate fW by $P_n W$ outside $[-C_1 q_n, C_1 q_n]$. So either a "tail term" $\|f W_\alpha\|_{L_p[|x| \geq C_1 q_n]}$ must appear in the error estimate, or be handled some other way. Inequalities of the form (1.3) are called *restricted range inequalities*, or *infinite-finite range inequalities*.

The sharp form of these was found later by Mhaskar and Saff, using potential theory [19], [21], [23]. For example, they showed that if $\alpha > 0$, and

$$a_n = \left\{ 2^{\alpha-2} \frac{\Gamma(\alpha/2)^2}{\Gamma(\alpha)} n \right\}^{1/\alpha},$$

then for not identically zero polynomials P_n of degree at most n,

$$\|P_n W_\alpha\|_{L_\infty(\mathbb{R})} = \|P_n W_\alpha\|_{L_\infty[-a_n, a_n]};$$
$$\|P_n W_\alpha\|_{L_\infty(\mathbb{R} \setminus [-a_n, a_n])} < \|P_n W_\alpha\|_{L_\infty[-a_n, a_n]}.$$

Moreover, a_n is asymptotically the "smallest" such number. There are various L_p analogues of these; obviously one can no longer have equality of the norm over the real line and that over a finite interval. One form, valid for all $p > 0$, is [14, Thm. 4.1, p. 95], [19, Thm. 6.2.4, p. 142]

$$\|P_n W_\alpha\|_{L_p(\mathbb{R})} \leq 2\|P_n W_\alpha\|_{L_\infty[-a_{n+2/p}, a_{n+2/p}]};$$
$$\|P_n W_\alpha\|_{L_\infty(\mathbb{R} \setminus [-a_{n+2/p}, a_{n+2/p}])} < \|P_n W_\alpha\|_{L_\infty[-a_{n+2/p}, a_{n+2/p}]}.$$

If instead one fixes $\varepsilon > 0$ and takes the "tail" over $\mathbb{R} \setminus [-a_n(1+\varepsilon), a_n(1+\varepsilon)]$, one obtains for some C independent of n and P_n, [19, Thm. 6.2.4, p. 142]

$$\|P_n W_\alpha\|_{L_\infty(\mathbb{R} \setminus [-a_n(1+\varepsilon), a_n(1+\varepsilon)])} \leq e^{-Cn} \|P_n W_\alpha\|_{L_\infty[-a_{n+2/p}, a_{n+2/p}]}.$$

The number a_n is called the *Mhaskar-Rakhmanov-Saff* number for W_α. It plays an important descriptive role in asymptotics of orthogonal polynomials for the weights W_α. It may be defined for very general weights $W = \exp(-Q)$ as the positive root of the formula

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}\, dt.$$

In the case of W_α, Freud's number q_n and a_n differ by a multiplicative constant, and we may confine ourselves to q_n. However, especially when Q grows faster than any polynomial, there need not be such a simple relation. In the remainder of this paper we use only q_n.

The next task is to determine what happens on $[-C_1 q_n, C_1 q_n]$. Now if we had to approximate in the unweighted setting on this interval, a scale change in the Jackson-Bernstein estimate (1.1) gives

$$\inf_{\deg(P) \leq n} \|f - P\|_{L_\infty[-C_1 q_n, C_1 q_n]} \leq \frac{C C_1 q_n}{n} \|f'\|_{L_\infty[-C_1 q_n, C_1 q_n]}.$$

Remarkably, the same is true when we insert the weight W_α in both norms:

$$\text{(1.4)} \qquad \inf_{\deg(P) \leq n} \|(f - P) W_\alpha\|_{L_\infty[-C_1 q_n, C_1 q_n]} \leq \frac{C_3 q_n}{n} \|f' W_\alpha\|_{L_\infty[-C_1 q_n, C_1 q_n]}.$$

Very roughly, this works for the following reason: it seems that if C_1 is small enough, we can approximate $1/W_\alpha$ on $[-C_1 q_n, C_1 q_n]$ by a polynomial $R_{n/2}$ of degree $\leq n/2$, and then use the remaining part $n/2$ degree polynomial in P to approximate fW_α itself on $[-C_1 q_n, C_1 q_n]$. In real terms, this approach works only for a small class of weights. Nevertheless, it at least indicated the form that general results should take. To obtain an estimate over the whole real line, Freud then proved a "tail inequality," such as

$$\|fW_\alpha\|_{L_p[|x|\geq C_1 q_n]} \leq \frac{C_4 q_n}{n} \|f' W_\alpha\|_{L_p(\mathbb{R})}, \tag{1.5}$$

with C_4 independent of f and n. Combining (1.4), (1.5), and that suitable weighted polynomials are tiny outside $[-C_1 q_n, C_1 q_n]$ yielded an estimate of the form

$$E_n [f; W_\alpha]_p := \inf_{\deg(P)\leq n} \|(f-P)\, W_\alpha\|_{L_p(\mathbb{R})} \leq \frac{C_5 q_n}{n} \|f' W_\alpha\|_{L_p(\mathbb{R})}, \tag{1.6}$$

with C_5 independent of f and n.

While this might illustrate some of the ideas, we emphasize the technical details underlying proper proofs of this Jackson (or Jackson-Favard) inequality are formidable. Freud and Nevai developed an original theory of orthogonal polynomials for the weights W_α^2 partly to use in this approximation theory. In this short paper, we shall not present all the technical details. We note that Freud proved (1.6) for W_α for $\alpha \geq 2$. The technical estimates required to extend this to the case $1 < \alpha < 2$ were provided by the author and Eli Levin [13]. What about $\alpha \leq 1$? Well, recall that the polynomials are only dense if $\alpha \geq 1$, so there is no point in considering $\alpha < 1$. But $\alpha = 1$ is still worth consideration, and we shall discuss that below.

One consequence of (1.6) is an estimate of the rate of weighted polynomial approximation of f in terms of that of f'. Indeed if P_n is any polynomial of degree $\leq n-1$, then

$$E_n [f; W_\alpha]_p = E_n [f - P_n; W_\alpha]_p \leq \frac{C_5 q_n}{n} \|(f-P_n)'\, W_\alpha\|_{L_p(\mathbb{R})},$$

and since P_n' may be any polynomial of degree $\leq n-1$, we obtain

$$E_n [f; W_\alpha]_p \leq \frac{C_5 q_n}{n} E_{n-1} [f'; W_\alpha]_p\,, \tag{1.7}$$

which can be iterated. The inequality (1.7) (and sometime even (1.6)) is called a Favard or Jackson-Favard inequality.

Freud also obtained estimates involving moduli of continuity. Here one cannot avoid the tail term, and has to build it directly into the modulus. Partly for this reason, there are many forms of the modulus, and more than one way to decide which interval is the principal interval, and over what interval we take the tail. We shall follow essentially the modulus used by Ditzian and Totik [6], Ditzian and the author [4], and Mhaskar [19].

The first order modulus for the weight W_α has the form

$$\begin{aligned}\omega_{1,p}(f, W_\alpha, t) = \sup_{0<h\leq t} &\|W_\alpha (\Delta_h f)\|_{L_p[-h^{\frac{1}{1-\alpha}}, h^{\frac{1}{1-\alpha}}]} \\ &+ \inf_{c\in\mathbb{R}} \|(f-c)\, W_\alpha\|_{L_p\left(\mathbb{R}\setminus[-t^{\frac{1}{1-\alpha}}, t^{\frac{1}{1-\alpha}}]\right)}.\end{aligned}$$

Why the inf over the constant c in the tail term? It ensures that if f is constant, then the modulus vanishes identically, as one expects from a first order modulus. Why the strange interval $[-h^{\frac{1}{1-\alpha}}, h^{\frac{1}{1-\alpha}}]$? It ensures that when we substitute

$$h = \frac{q_n}{n} = \alpha^{-1/\alpha} n^{-1+1/\alpha},$$

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then

$$[-h^{\frac{1}{1-\alpha}}, h^{\frac{1}{1-\alpha}}] = [-Cq_n, Cq_n],$$

for an appropriate constant C (independent of n). More generally if $r \geq 1$, the rth order modulus is

$$\begin{aligned} \omega_{r,p}(f, W_\alpha, t) = & \sup_{0<h\leq t} \|W_\alpha (\Delta_h^r f)\|_{L_p[-h^{\frac{1}{1-\alpha}}, h^{\frac{1}{1-\alpha}}]} \\ & + \inf_{\deg(P)\leq r-1} \|(f-P) W_\alpha\|_{L_p\left(\mathbb{R}\setminus[-t^{\frac{1}{1-\alpha}}, t^{\frac{1}{1-\alpha}}]\right)}. \end{aligned} \tag{1.8}$$

Again the inf in the tail term ensures that if f is a polynomial of degree $\leq r-1$, then the modulus of continuity vanishes identically, as is expected from an rth order modulus. The Jackson theorem takes the form

$$E_n [f; W_\alpha]_p \leq C\omega_{r,p}(f, W_\alpha, n^{-1+\frac{1}{\alpha}}). \tag{1.9}$$

This is valid for $1 \leq p \leq \infty$, and the constant C is independent of f and n (but depends on p and W_α).

One can consider more general weights than W_α of course. Almost invariably the weight considered has the form $W = \exp(-Q)$, and the rate of growth of Q has a major impact on the form of the modulus. Let us suppose for example, that Q is of polynomial growth at ∞, the so-called Freud case. The most general class of such weights for which a Jackson theorem is known is the following:

DEFINITION 1.1 (Freud Weights). *Let $W = \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, Q' exists and is positive in $(0, \infty)$. Moreover, assume that $xQ'(x)$ is strictly increasing, with right limit 0 at 0, and for some $\lambda, A, B > 1$, $C > 0$,*

$$A \leq \frac{Q'(\lambda x)}{Q'(x)} \leq B, \qquad x \geq C.$$

Then we write $W \in \mathcal{F}$.

For such W, we take q_n to be the positive root of the equation

$$n = q_n Q'(q_n).$$

Again, this is the point where $x^n W(x)$ assumes its maximum modulus on the real line. To replace the function $t^{\frac{1}{1-\alpha}}$, we can use the function

$$\sigma(t) := \inf\left\{q_n : \frac{q_n}{n} \leq t\right\}, \qquad t > 0.$$

The modulus of continuity becomes

$$\begin{aligned} \omega_{r,p}(f, W, t) = & \sup_{0<h\leq t} \|W (\Delta_h^r f)\|_{L_p[-\sigma(h),\sigma(h)]} \\ & + \inf_{\deg(P)\leq r-1} \|(f-P) W\|_{L_p(\mathbb{R}\setminus[-\sigma(t),\sigma(t)])}. \end{aligned}$$

The Jackson theorem is the obvious analogue of (1.9) [4, Theorem 1.2, p. 102]:

$$E_n [f; W]_p \leq C\omega_{r,p}(f, W, \frac{q_n}{n}). \tag{1.10}$$

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Moreover, if W satisfies a mild additional condition on Q'', or admits an appropriate Markov-Bernstein inequality, and $\alpha < r$, then there is the equivalence [4, p. 105]

$$E_n\left[f;W\right]_p = O\left(\left(\frac{q_n}{n}\right)^{\alpha}\right), n \to \infty$$
$$\iff \omega_{r,p}(f,W,t) = O\left(t^{\alpha}\right), \qquad t \to 0+.$$

This equivalence is an easy consequence of the Jackson inequality (1.10) and the converse inequality [4, Cor. 1.6, p. 105]

$$\omega_{r,p}(f,W,t) \le C\left(\frac{q_n}{n}\right)^r \sum_{-1\le j\le \log_2 n}\left(\frac{2^j}{q_{2^j}}\right)^r E_{2^j}\left[f;W\right]_p.$$

Here $E_{2^{-1}} := E_0$ and C is independent of f and n. One of the important tools in establishing this is $K-$functionals and the concept of realization. This is a topic on its own. In the setting of weighted polynomial approximation, it has been explored by Freud and Mhaskar, and later Ditzian and Totik, Damelin and the the author. See [1], [2], [4], [19], [20] for references.

In the (technical) proof of the Jackson theorem (1.10), the function f is first approximated by a piecewise polynomial (or spline). Then special polynomials that approximate characteristic functions, and Whitney's theorem on local polynomial approximation are used to turn the spline approximation into a polynomial approximation. For the case where Q is of faster than polynomial growth, the modulus of continuity becomes more complicated, as again there are endpoint effects, close to $\pm Cq_n$. We refer the reader to [2]. There are also analogous developments for exponential weights on $[-1,1]$ [15].

In recent years, there has been less focus on this type of weighted approximation. Instead much of the focus has been on *Saff's Polynomial Approximation Problem*, which involves varying weights, rather than a fixed weight. Thus one might seek to approximate by weighted polynomials of the form $P_n(x)\,W(x)^n$ or $P_n(x)\,W(a_n x)$, where a_n is the Mhaskar-Rakhmanov-Saff number for Q defined above. Saff's approximation problem and its circle of ideas has applications in asymptotics of orthogonal and extremal polynomials, mathematical physics, random matrices ... — see for example [19], [14], [23].

Recall that we left discussion of $W_1(x) = \exp(-|x|)$ till later. Curiously it is issues close to that weight that have arisen most recently — and have served to renew at least the author's interest in classical weighted approximation. While investigating asymptotics of Sobolev orthogonal polynomials, the question arose of which weights admit some form of the Jackson inequality (1.6). Curiously, these inequalities enable one to relate asymptotic behavior of derivatives of Sobolev orthogonal polynomials to classical orthogonal polynomials [8].

This forced the author to revisit some very old results of Freud. In 1978, Freud, Giroux and Rahman [7, p. 360] proved that

$$E_n[f;W_1]_1 = \inf_{\deg(P)\le n} \|\,(f-P)\,W_1\|_{L_1(\mathbb{R})}$$
$$\le C\left[\omega\left(f,\frac{1}{\log n}\right) + \int_{|x|\ge\sqrt{n}} |fW_1|(x)\,dx\right],$$

where

$$\omega(f,\varepsilon) = \sup_{|h|\le\varepsilon}\int_{-\infty}^{\infty} |(fW_1)(x+h)-(fW_1)(x)|\,dx + \varepsilon\int_{-\infty}^{\infty}|fW_1|.$$

Here C is independent of f and n, and $\sqrt{n}$ could be replaced by $n^{1-\delta}$ for any fixed $\delta \in (0,1)$. Ditzian, the author, Nevai and Totik [5] later extended this result to a characterization in L_1. The technique used by Freud, Giroux and Rahman was essentially an L_1 technique, using the relation between one-sided weighted approximation, Gauss quadratures, and Christoffel functions. Only recently has it been possible to establish the analogous results in L_p, $p > 1$ [16]. The author modified the spline method from [4]. As the peaking polynomials used there do not work for W_1, they were replaced by the reproducing kernel for orthogonal polynomials for W_1^2, and in the proofs, the author needed bounds for these orthogonal polynomials, implied by recent work of Kriecherbauer and McLaughlin [12].

If we examine the modulus used in (1.8) for $W_\alpha, \alpha > 1$, we see that the interval $[-h^{\frac{1}{1-\alpha}}, h^{\frac{1}{1-\alpha}}]$ is no longer meaningful for $\alpha = 1$. It turns out to be replaced by $\left[-\exp\left(\frac{1-\varepsilon}{h}\right),\right.$ $\left.\exp\left(\frac{1-\varepsilon}{h}\right)\right]$, for some fixed $\varepsilon \in (0,1)$. The modulus becomes

$$\begin{aligned}\omega_{r,p}(f, W_1, t) = &\sup_{0<h\leq t} \|W_1 (\Delta_h^r f)\|_{L_p\left[-\exp\left(\frac{1-\varepsilon}{t}\right),\exp\left(\frac{1-\varepsilon}{t}\right)\right]} \\ &+ \inf_{\deg(P)\leq r-1} \|(f-P) W_1\|_{L_p\left(\mathbb{R}\setminus\left[-\exp\left(\frac{1-\varepsilon}{t}\right)+1, -\exp\left(\frac{1-\varepsilon}{t}\right)+1\right]\right)}.\end{aligned}$$

The author proved [16] that for $0 < p \leq \infty$, and $n \geq C_3$,

$$E_n [f; W_1]_p \leq C_1 \omega_{r,p} \left(f, W, \frac{1}{\log(C_2 n)}\right). \tag{1.11}$$

Here C_1, C_2, C_3 are independent of f and n.

While this may be a technical achievement, it is scarcely surprising, given that Freud, Giroux and Rahman already had the rate $O\left(\frac{1}{\log n}\right)$. What is perhaps more interesting is that the rate $n^{-1+1/\alpha}$ for $W_\alpha, \alpha > 1$, becomes $\frac{1}{\log n}$ as $\alpha \to 1+$. This suggests that we ought to obtain an analogue of (1.6) of the form

$$E_n [f; W_1]_p \leq \frac{C}{\log n} \|f' W_1\|_{L_p(\mathbb{R})}.$$

Remarkably enough this is false, and there is no Jackson-Favard inequality for W_1, not even if we replace $\frac{1}{\log n}$ by a sequence decreasing arbitrarily slowly to 0. More generally we answered in [17] the question: which weights admit a Jackson type theorem, of the form (1.6), with $\{q_n/n\}_{n=1}^\infty$ replaced by some sequence $\{\eta_n\}_{n=1}^\infty$ with limit 0? We proved:

THEOREM 1.2. *Let $W : \mathbb{R} \to (0, \infty)$ be continuous. The following are equivalent:*
(a) There exists a sequence $\{\eta_n\}_{n=1}^\infty$ of positive numbers with limit 0 and with the following property. For each $1 \leq p \leq \infty$, and for all absolutely continuous f with $\|f'W\|_{L_p(\mathbb{R})}$ finite, we have

$$\inf_{\deg(P)\leq n} \|(f-P) W\|_{L_p(\mathbb{R})} \leq \eta_n \|f'W\|_{L_p(\mathbb{R})}, n \geq 1. \tag{1.12}$$

(b) Both

$$\lim_{x\to\infty} W(x) \int_0^x W^{-1} = 0 \tag{1.13}$$

and

$$\lim_{x\to\infty} W(x)^{-1} \int_x^\infty W = 0, \tag{1.14}$$

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with analogous limits as $x \to -\infty$. Two fairly direct corollaries of this are:

COROLLARY 1.3. *Let* $W : \mathbb{R} \to (0, \infty)$ *be continuous, with* $W = e^{-Q}$*, where* $Q(x)$ *is differentiable for large* $|x|$*, and*

$$\lim_{x \to \infty} Q'(x) = \infty \text{ and } \lim_{x \to -\infty} Q'(x) = -\infty.$$

Then there exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ *of positive numbers with limit* 0 *such that for each* $1 \leq p \leq \infty$*, and for all absolutely continuous* f *with* $\|f'W\|_{L_p(\mathbb{R})}$ *finite, we have (1.12).*

COROLLARY 1.4. *Let* $W : \mathbb{R} \to (0, \infty)$ *be continuous, with* $W = e^{-Q}$*, where* $Q(x)$ *is differentiable for large* $|x|$*, and* $Q'(x)$ *is bounded for large* $|x|$*. Then for both* $p = 1$ *and* $p = \infty$*, there does not exist a sequence* $\{\eta_n\}_{n=1}^{\infty}$ *of positive numbers with limit* 0 *satisfying (1.12) for all absolutely continuous* f *with* $\|f'W\|_{L_p(\mathbb{R})}$ *finite.*

In particular for W_1, there is no Jackson-Favard inequality, since both (1.13) and (1.14) are false. Thus there is a real difference between density of weighted polynomials, and weighted Jackson-Favard theorems. It is possible to have the former without the latter.

Essentially (1.13) is necessary and sufficient for an L_∞ Jackson theorem, and (1.14) is necessary and sufficient for an L_1 Jackson theorem. An obvious question is the independence of these conditions (1.13) and (1.14). Does either imply the other? In fact they are independent. Moreover, there are weights satisfying one but not the other, and also admitting an L_1 Jackson theorem but not an L_∞ Jackson theorem (or conversely). This is a highly unusual occurrence in weighted approximation - in fact the first occurrence of this phenomenon known to this author. Density of polynomials, and the degree of approximation is almost invariably the same for any L_p space (suitably weighted of course). Koosis [10, pp. 210–211] makes a lengthy remark about the latter. We proved:

THEOREM 1.5. *(a) There exists continuous* $W : \mathbb{R} \to (0, \infty)$ *with*

$$1 \leq W(x) / \exp\left(-x^2\right) \leq 2(1 + |x|), \qquad x \in \mathbb{R},$$

admitting an L_∞ *Jackson theorem, but not an* L_1 *Jackson theorem. That is, for* $p = \infty$*, there exist* $\{\eta_n\}_{n=1}^{\infty}$ *with limit* 0 *at* ∞ *satisfying (1.12), but there does not exist such a sequence for* $p = 1$*.*

(b) There exists continuous $W : \mathbb{R} \to (0, \infty)$ *with*

$$1 \geq W(x) / \exp\left(-x^2\right) \geq 2/(1 + |x|), \qquad x \in \mathbb{R},$$

admitting an L_1 *Jackson theorem, but not an* L_∞ *Jackson theorem. That is, for* $p = 1$*, there exist* $\{\eta_n\}_{n=1}^{\infty}$ *with limit* 0 *at* ∞ *satisfying (1.12), but there does not exist such a sequence for* $p = \infty$*.*

We note that the weights in this result are equal to the Hermite weight $W_2(x) = \exp\left(-x^2\right)$ "most" of the time, with spikes upwards or downwards in small intervals. The weights we construct are not decreasing in $[0, \infty)$, though they can be made infinitely differentiable. We expect that with more work one can construct decreasing W in $[0, \infty)$ still satisfying these conclusions.

A key ingredient in the above theorem is an estimate for tails:

THEOREM 1.6. *Assume that* $W : \mathbb{R} \to (0, \infty)$*is continuous.*

(a) Assume W *satisfies (1.13) and (1.14), with analogous limits at* $-\infty$*. Then there exists a decreasing positive function* $\eta : [0, \infty) \to (0, \infty)$ *with limit* 0 *at* ∞ *such that for* $1 \leq p \leq \infty$ *and* $\lambda \geq 0$*,*

$$\|fW\|_{L_p(\mathbb{R} \setminus [-\lambda, \lambda])} \leq \eta(\lambda) \|f'W\|_{L_p(\mathbb{R})} \tag{1.15}$$

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for all absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ *for which* $f(0) = 0$ *and the right-hand side is finite.*
(b) Conversely assume that (1.15) holds for $p = 1$ *and for* $p = \infty$*, for large enough* λ*. Then the limits (1.13) and (1.14) in Theorem 1.2 are valid, with analogous limits at* $-\infty$*.*

The above results deal with L_p for all $1 \le p \le \infty$. What happens if we focus on a single L_p space? We recently proved [18]:

THEOREM 1.7. *Let* $W : \mathbb{R} \to (0, \infty)$ *be continuous and let* $1 \le p \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. The following are equivalent:*
(a) There exists a sequence $\{\eta_n\}_{n=1}^{\infty}$ *of positive numbers with limit* 0 *and with the following property. For all absolutely continuous* f *with* $\|f'W\|_{L_p(\mathbb{R})}$ *finite, we have*

$$\inf_{\deg(P) \le n} \|(f - P)W\|_{L_p(\mathbb{R})} \le \eta_n \|f'W\|_{L_p(\mathbb{R})}, \qquad n \ge 1.$$

(b)

$$\lim_{x \to \infty} \left\|W^{-1}\right\|_{L_q[0,x]} \|W\|_{L_p[x,\infty)} = 0, \tag{1.16}$$

with analogous limits as $x \to -\infty$*.*

As a consequence one can construct weights that admit a Jackson theorem in L_p but not in $L_{p'}$ for any $1 \le p, p' \le \infty$ with $p \ne p'$. Finally, we note that weights close to W_1 are worthwhile candidates for investigating Jackson theorems involving moduli of continuity. To be explicit, we pose:

PROBLEM 1. Find the analogue of (1.11) for the weight

$$W(x) = \exp\left(-|x|\left(\log\left(2 + x^2\right)\right)^a\right), \qquad a > -1.$$

For $a \le -1$, Bernstein's polynomial approximation problem does not have a positive solution, and so there cannot be an analogue of (1.11). Even for $a > -1$, there can be no Jackson-Favard inequality (1.12), since these weights violate both (1.13) and (1.14).

One shortcoming of (1.12) is that no information is given regarding the rate of decay of $\{\eta_n\}_{n=1}^{\infty}$. One could recast it in the form of a Jackson-Favard inequality, and iterate to obtain

$$\begin{aligned}\inf_{\deg(P) \le n} \|(f - P)W\|_{L_p(\mathbb{R})} &\le \eta_n \inf_{\deg(P) \le n-1} \|(f - P)'W\|_{L_p(\mathbb{R})} \le \cdots \\ &\le \eta_n \eta_{n-1} \cdots \eta_{n-k+1} \inf_{\deg(P) \le n-k} \left\|(f - P)^{(k)} W\right\|_{L_p(\mathbb{R})},\end{aligned}$$

provided the right-hand side is meaningful. However, this does not really help without information on the size of η_n. Accordingly, we pose:

PROBLEM 2. What is the best choice of η_n in (1.12), for a given W satisfying (1.13) and (1.14)?

While our proof of (1.12) gives no information, we know that for fairly general Freud weights, the correct η_n is q_n/n.

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