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ETNA
Kent State University
etna@mcs.kent.edu

THE CIRCLE THEOREM AND RELATED THEOREMS FOR GAUSS-TYPE QUADRATURE RULES*

WALTER GAUTSCHI†

Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. In 1961, P.J. Davis and P. Rabinowitz established a beautiful "circle theorem" for Gauss and Gauss–Lobatto quadrature rules. They showed that, in the case of Jacobi weight functions, the Gaussian weights, suitably normalized and plotted against the Gaussian nodes, lie asymptotically for large orders on the upper half of the unit circle centered at the origin. Here analogous results are proved for rather more general weight functions—essentially those in the Szegö class—, not only for Gauss and Gauss–Lobatto, but also for Gauss–Radau formulae. For much more restricted classes of weight functions, the circle theorem even holds for Gauss–Kronrod rules. In terms of potential theory, the semicircle of the circle theorem can be interpreted as the reciprocal density of the equilibrium measure of the interval $[-1, 1]$. Analogous theorems hold for weight functions supported on any compact subset Δ of $(-1, 1)$, in which case the (normalized) Gauss points approach the reciprocal density of the equilibrium measure of Δ. Many of the results are illustrated graphically.

Key words. Gauss quadrature formulae, circle theorem, Gauss–Radau, Gauss–Lobatto and Gauss–Kronrod formulae, Christoffel function, potential theory, equilibrium measure

AMS subject classifications. 65D32, 42C05

1. Introduction.

One of the gems in the theory of Gaussian quadrature relates to the distribution of the Gaussian weights. In fact, asymptotically for large orders, the weights, when suitably normalized and plotted against the Gaussian nodes, come to lie on a half circle drawn over the support interval of the weight function under consideration. This geometric view of Gauss quadrature rules was first taken by Davis and Rabinowitz [2, §II], who established the asymptotic property described—a "circle theorem", as they called it—in the case of Jacobi weight functions $w(t) = (1-t)^\alpha(1+t)^\beta$, $\alpha > -1$, $\beta > -1$, not only for the Gauss formula, but also for the Gauss-Lobatto formula. For the Gauss-Radau formula, they only conjectured it "with meager numerical evidence at hand". It should be mentioned, however, that the underlying asymptotic formula (see eqn (2.2) below) has previously been obtained by Erdös and Turán [4, Theorem IX], and even earlier by Akhiezer [1, p. 81, footnote 9], for weight functions $w(t)$ on $[-1, 1]$ such that $w(t)\sqrt{1-t^2}$ is continuous and $w(t)\sqrt{1-t^2} \geq m > 0$ on $[-1, 1]$. This answers, in part, one of the questions raised in [2, last paragraph of §IV] regarding weight functions other than those of Jacobi admitting a circle theorem. In §§2–4 we show that the circle theorem, not only for Gaussian quadrature rules, but also for Gauss–Radau and Gauss–Lobatto rules, holds essentially for all weight functions in the Szegö class, i.e., weight functions w on $[-1, 1]$ for which

$$\frac{\ln w(t)}{\sqrt{1-t^2}} \in L_1(-1, 1). \tag{1.1}$$

We say "essentially", since an additional, mild condition, viz.

$$1/w(t) \in L_1(\Delta),$$

must also be satisfied, where Δ is any compact subinterval of $(-1, 1)$. In §5, we show, moreover, that circle theorems, under suitable assumptions, hold also for Gauss–Kronrod

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†Department of Computer Sciences, Purdue University, West Lafayette, Indiana 47907-2066 (wxg@cs.purdue.edu).

ETNA
Kent State University
etna@mcs.kent.edu

formulae. In §6 we give a potential-theoretic interpretation of the circle theorem, namely that the semicircle in question is the reciprocal density of the equilibrium measure of the interval $[-1, 1]$. This is true in more general situations, where the support of the given weight function is any compact subset Δ of $(-1, 1)$, in which case the (normalized) Gauss points come to lie on the reciprocal density of the equilibrium measure of Δ. This is illustrated in the case of Δ being the union of two disjoint symmetric subintervals of $[-1, 1]$. The equation of the limiting curve can be written down in this case and answers in the affirmative another question raised in [2, last sentence of §IV].

2. Gaussian quadrature. We write the Gaussian quadrature formula for the weight function w in the form

$$\int_{-1}^{1} f(t)w(t)\mathrm{d}t = \sum_{\nu=1}^{n} \lambda_\nu^G f(\tau_\nu^G) + R_n^G(f), \tag{2.1}$$

where τ_ν^G are the *Gaussian nodes* and λ_ν^G the *Gaussian weights*; cf., e.g., [7, §1.4.2]. (Their dependence on n is suppressed in our notation.) The remainder satisfies

$$R_n^G(p) = 0 \quad \text{for any } p \in \mathbb{P}_{2n-1},$$

where $\mathbb{P}_{2n-1}$ is the class of polynomials of degree $\le 2n-1$. Without loss of generality we have assumed that the support of the weight function w is the interval $[-1, 1]$. The circle theorem can then be formulated as follows.

THEOREM 2.1. *(Circle theorem) Let w be a weight function in the Szegö class (cf. §1, (1.1)) satisfying $1/w(t) \in L_1(\Delta)$ for any compact interval $\Delta \subset (-1, 1)$. Then*

$$\frac{n\lambda_\nu^G}{\pi w(\tau_\nu^G)} \sim \sqrt{1-(\tau_\nu^G)^2} \quad \text{as } n \to \infty, \tag{2.2}$$

for all nodes τ_ν^G (and corresponding weights) that lie in Δ. (The relation $a_n \sim b_n$ here means that $\lim_{n\to\infty} a_n/b_n = 1$.)

As mentioned in §1, this was shown to be true by Davis and Rabinowitz [2] in the case of the Jacobi weight function $w(t) = (1-t)^\alpha(1+t)^\beta$ on $[-1, 1]$, $\alpha > -1$, $\beta > -1$. We illustrate the theorem in Fig. 2.1 by plotting all quantities on the left of (2.2) for $\alpha, \beta = -0.75 : 0.25 : 1.0, 1.5 : 0.5 : 3.0$, $\beta \ge \alpha$, and for $n = 20 : 5 : 40$ in the plot on the left, and for $n = 60 : 5 : 80$ in the plot on the right.

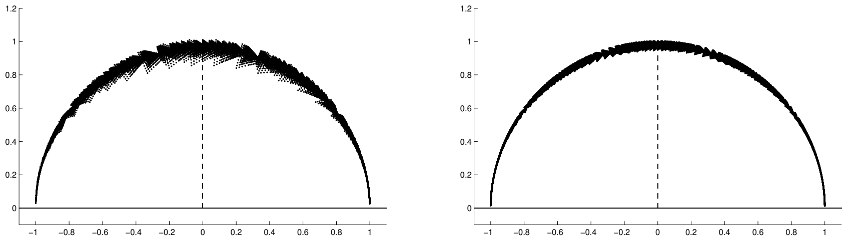

FIG. 2.1. *The circle theorem for Jacobi weight functions*

The circle theorem for the more general weight function indicated in Theorem 2.1 has been around implicitly for some time. Indeed, it is contained in an important asymptotic result for Christoffel functions $\lambda_n(t; w)$ due to Nevai [12, Theorem 34]. According to this result, one has

$$\frac{n\lambda_n(t;w)}{\pi w(t)} \sim \sqrt{1-t^2} \quad \text{as } n \to \infty, \tag{2.3}$$

uniformly for $t \in \Delta$. Recalling that $\lambda_n(\tau_\nu^G; w) = \lambda_\nu^G$ (cf. [5, Theorem 3.2 and last paragraph of §I.3]) yields Theorem 2.1.

COROLLARY TO THEOREM 2.1. *If* $w(t) = (1-t^2)^{-1/2}$ *on* $(-1, 1)$, *then*

$$\frac{n\lambda_\nu^G}{\pi w(\tau_\nu^G)} = \sqrt{1-(\tau_\nu^G)^2}, \quad \nu = 1, 2, \dots, n.$$

Proof. This follows from the well-known fact that $\lambda_\nu^G = \pi/n$, $\nu = 1, 2, \dots, n$, in this case. □

REMARK 2.2. *Theorem 2.1 in a weaker form (pointwise convergence almost everywhere) holds also when* w *is locally in Szegö's class, i.e.,* w *has support* $[-1, 1]$ *and satisfies*

$$\int_\Delta \ln w(t) \mathrm{d}t > -\infty, \tag{2.4}$$

where Δ *is an open subinterval of* $[-1, 1]$. *Then (2.2) holds for almost all* $\tau_\nu \in \Delta$ *([11, Theorem 8]).*

EXAMPLE 1. The Pollaczek weight function $w(t; a, b)$ on $[-1, 1]$, $a \geq |b|$ (cf. [14]).

The weight function is given explicitly by (*ibid.*, eqn (3), multiplied by 2)

$$w(t; a, b) = \frac{2\exp(\omega \cos^{-1}(t))}{1 + \exp(\omega\pi)}, \quad |t| \leq 1, \tag{2.5}$$

where $\omega = \omega(t) = (at + b)(1 - t^2)^{-1/2}$. It is not in Szegö's class, but is so locally. The recurrence coefficients are known explicitly (*ibid.*, eqn (14)),

$$\begin{aligned} \alpha_k &= \frac{-b}{2k + a + 1}, \quad k \geq 0, \\ \beta_0 &= \frac{2}{a+1}, \quad \beta_k = \frac{k^2}{(2k+a)^2 - 1}, \quad k \geq 1. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), it is straightforward to compute the ratios $n\lambda_\nu^G/\pi w(\tau_\nu^G; a, b)$. Their behavior, when $n = 380 : 5 : 400$, is shown in Fig. 2.2 for $a = b = 0$ on the left, and for $a = 4$, $b = 1$ on the right. The circle theorem obviously holds when $a = b = 0$ (i.e., $w = 1$), but also, as expected from the above remark, with possible isolated exceptions, for other values of a and b.

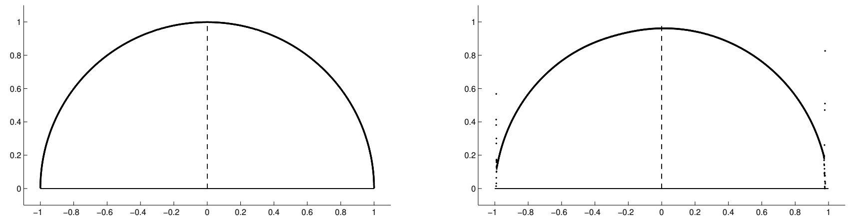

FIG. 2.2. *The circle theorem for Pollaczek weight functions*

3. Gauss–Radau formula. Our analysis of the Gauss–Radau formula (and also the Gauss–Lobatto formula in §4) seeks to conclude from the validity of the circle theorem for the Gauss formula (2.1) the same for the corresponding Gauss–Radau formula,

$$\int_{-1}^{1} f(t)w(t)\mathrm{d}t = \lambda_0^R f(-1) + \sum_{\nu=1}^{n} \lambda_\nu^R f(\tau_\nu^R) + R_n^R(f), \tag{3.1}$$

where $R_n^R(\mathbb{P}_{2n}) = 0$. (Here, as in (2.1), the nodes and weights depend on n.)

THEOREM 3.1. *Let the weight function w satisfy the conditions of Theorem 2.1. Then not only the Gaussian quadrature rule (2.1) for w, but also the Gauss–Radau rule (3.1) for w admits a circle theorem.*

Proof. It is known that τ_ν^R are the zeros of $\pi_n(\,\cdot\,; w_{-1})$, the polynomial of degree n orthogonal with respect to the weight function $w_{-1}(t) = (t+1)w(t)$ (cf. [7, §1.4.2, p. 25]). Let

$$\ell_\nu^*(t) = \prod_{\mu \neq \nu} \frac{t - \tau_\mu^R}{\tau_\nu^R - \tau_\mu^R}, \quad \nu = 1, 2, \ldots, n, \tag{3.2}$$

be the elementary Lagrange interpolation polynomials for the nodes $\tau_1^R, \tau_2^R, \ldots, \tau_n^R$. Since the Gauss–Radau formula is interpolatory, there holds

$$\begin{aligned} \lambda_\nu^R &= \int_{-1}^{1} \frac{(t+1)\pi_n(t; w_{-1})}{(\tau_\nu^R + 1)(t - \tau_\nu^R)\pi_n'(\tau_\nu^R; w_{-1})}\, w(t)\mathrm{d}t \\ &= \int_{-1}^{1} \frac{(t+1)\ell_\nu^*(t)}{\tau_\nu^R + 1}\, w(t)\mathrm{d}t. \end{aligned} \tag{3.3}$$

If λ_ν^* are the n Gaussian weights for the weight function w_{-1}, we have, again by the interpolatory nature of the Gaussian quadrature formula, and by (3.3),

$$\lambda_\nu^* = \int_{-1}^{1} \ell_\nu^*(t)(t+1)w(t)\mathrm{d}t = (\tau_\nu^R + 1)\lambda_\nu^R.$$

By assumption, the Gauss formula for the weight function w, and hence also the one for the weight function w_{-1} (which satisfies the same conditions as those imposed on w) admits a circle theorem. Therefore,

$$\frac{n\lambda_\nu^R}{\pi w(\tau_\nu^R)} = \frac{n\lambda_\nu^*}{\pi(\tau_\nu^R + 1)w(\tau_\nu^R)} = \frac{n\lambda_\nu^*}{\pi w_{-1}(\tau_\nu^R)} \sim \sqrt{1 - (\tau_\nu^R)^2}, \quad n \to \infty.$$

□

EXAMPLE 2. The logarithmic weight function $w(t) = t^\alpha \ln(1/t)$ on $[0, 1]$, $\alpha > -1$. Here, Gauss–Radau quadrature is over the interval $[0, 1]$,

$$\int_0^1 f(t)t^\alpha \ln(1/t)\mathrm{d}t = \lambda_0 f(0) + \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + R_n(f).$$

A linear transformation of variables, mapping $[0, 1]$ onto $[-1, 1]$, yields the Gauss–Radau quadrature formula over $[-1, 1]$, to which Theorem 3.1 is applicable. The circle theorem, therefore, by a simple computation, now assumes the form

$$\frac{n\lambda_\nu}{\pi\tau_\nu^\alpha \ln(1/\tau_\nu)} \sim \sqrt{(\tfrac{1}{2})^2 - (\tau_\nu - \tfrac{1}{2})^2}, \quad n \to \infty.$$

This is illustrated in Fig. 3.1, on the left for $n = 20 : 5 : 40$, on the right for $n = 60 : 5 : 80$, and $\alpha = -0.75 : 0.25 : 1.0, 1.5 : 0.5 : 3$ in both cases.

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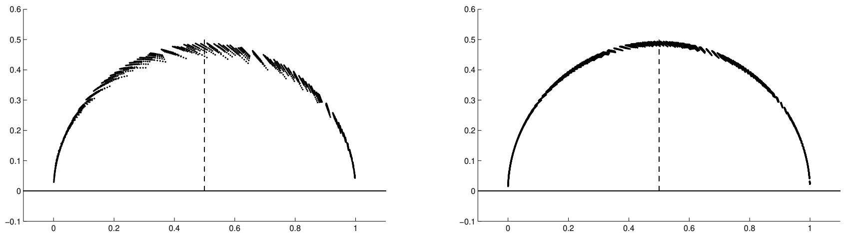

FIG. 3.1. *The circle theorem for Gauss–Radau quadrature*

4. Gauss–Lobatto formula. The argumentation, in this case, is quite similar to the one in §3 for Gauss–Radau formulae. We recall that the Gauss–Lobatto formula for the weight function w is

$$\int_{-1}^{1} f(t)w(t)\mathrm{d}t = \lambda_0^L f(-1) + \sum_{\nu=1}^{n} \lambda_\nu^L f(\tau_\nu^L) + \lambda_{n+1}^L f(1) + R_n^L(f), \tag{4.1}$$

where $R_n^L(\mathbb{P}_{2n+1}) = 0$ and τ_ν^L are the zeros of $\pi_n(\,\cdot\,; w_{\pm 1})$, the polynomial of degree n orthogonal with respect to the weight function $w_{\pm 1}(t) = (1-t^2)w(t)$ (cf. [7, §1.4.2, p. 26]).

THEOREM 4.1. *Let the weight function w satisfy the conditions of Theorem 2.1. Then the Gauss–Lobatto rule (4.1) for w admits a circle theorem.*

Proof. In analogy to (3.2), we define

$$\ell_\nu^*(t) = \prod_{\mu \neq \nu} \frac{t - \tau_\mu^L}{\tau_\nu^L - \tau_\mu^L}\,, \quad \nu = 1, 2, \ldots, n,$$

and denote by λ_ν^* the n Gaussian weights for the weight function $w_{\pm 1}$. Then we have

$$\lambda_\nu^L = \int_{-1}^{1} \frac{(1-t^2)\ell_\nu^*(t)}{1-(\tau_\nu^L)^2}\, w(t)\mathrm{d}t,$$

while, on the other hand,

$$\lambda_\nu^* = \int_{-1}^{1} \ell_\nu^*(t)(1-t^2)w(t)\mathrm{d}t = (1-(\tau_\nu^L)^2)\lambda_\nu^L.$$

Consequently,

$$\frac{n\lambda_\nu^L}{\pi w(\tau_\nu^L)} = \frac{n\lambda_\nu^*}{\pi(1-(\tau_\nu^L)^2)w(\tau_\nu^L)} = \frac{n\lambda_\nu^*}{\pi w_{\pm 1}(\tau_\nu^L)} \sim \sqrt{1-(\tau_\nu^L)^2}, \quad n \to \infty,$$

by Theorem 2.1 and the fact that $w_{\pm 1}$ satisfies the same conditions as those imposed on w. ☐

5. Gauss–Kronrod formula. While the quadrature rules discussed so far are products of the 19th century, the rules to be considered now are brainchilds of the 20th century ([10]). The idea[1] is to expand the Gaussian n-point quadrature formula (2.1) into a $(2n+1)$-point formula by inserting $n+1$ additional nodes and redefining all weights in such a manner as to achieve maximum degree of exactness. It turns out, as one expects, that this optimal degree

[1] In a germinal form, the idea can already be found in work of Skutsch [16]; see [8].

of exactness is $3n + 1$; it comes at an expenditure of only $n + 1$ new function evaluations, but at the expense of possibly having to confront complex-valued nodes and weights.

The quadrature formula described, called *Gauss–Kronrod formula*, thus has the form

$$\int_{-1}^{1} f(t)w(t)\mathrm{d}t = \sum_{\nu=1}^{n} \lambda_\nu^K f(\tau_\nu^G) + \sum_{\mu=1}^{n+1} \lambda_\mu^{*K} f(\tau_\mu^K) + R_n^K(f), \tag{5.1}$$

where τ_ν^G are the Gaussian nodes for the weight function w and

$$R_n^K(p) = 0 \quad \text{for all } p \in \mathbb{P}_{3n+1}. \tag{5.2}$$

The formula (5.1) is uniquely determined by the requirement (5.2); indeed (cf. [7, §3.1.2]), the inserted nodes τ_μ^K—the *Kronrod nodes*—must be the zeros of the polynomial π_{n+1}^K of degree $n+1$ orthogonal to all lower-degree polynomials with respect to the "weight function" $\pi_n(t)w(t)$, where π_n is the orthogonal polynomial of degree n relative to the weight function w,

$$\int_{-1}^{1} \pi_{n+1}^K(t)p(t)\pi_n(t)w(t)\mathrm{d}t = 0, \quad \text{all } p \in \mathbb{P}_n. \tag{5.3}$$

The weights in (5.1) are then determined "by interpolation".

Interestingly, in the simplest case $w(t) = 1$, the polynomial π_{n+1}^K has already been considered by Stieltjes in 1894, though not in the context of quadrature. It is nowadays, for arbitrary w, called the *Stieltjes polynomial* for the weight function w.

Orthogonality in the sense (5.3) is problematic for two reasons: the "weight function" $w_n^K = \pi_n w$ is oscillatory and sign-varying on the interval $[-1, 1]$, and it depends on n. The zeros of π_{n+1}^K, therefore, are not necessarily contained in $(-1, 1)$, or even real, although in special cases they are. A circle theorem for Gauss–Kronrod formulae is therefore meaningful only if all Kronrod nodes are real, distinct, contained in $(-1, 1)$, and different from any Gaussian node. If that is the case, and moreover, w is a weight function of the type considered in Theorem 2.1, there is a chance that a circle theorem will hold. The best we can prove is the following theorem.

THEOREM 5.1. *Assume that the Gauss–Kronrod formula (5.1) exists with τ_μ^K distinct nodes in $(-1, 1)$ and $\tau_\mu^K \neq \tau_\nu^G$ for all μ and ν. Assume, moreover, that*

(i) the Gauss quadrature formula for the weight function w admits a circle theorem;

(ii) the $(n+1)$-point Gaussian quadrature formula for $w^K(t) = \pi_n(t)w(t)$, with Gaussian weights λ_μ^*, admits a circle theorem in the sense

$$\frac{2n\lambda_\mu^*}{\pi w^K(\tau_\mu^K)} \sim \sqrt{1 - (\tau_\mu^K)^2} \quad \text{as } n \to \infty$$

for all μ such that $\tau_\mu^K \in \Delta$, where Δ is any compact subinterval of $(-1, 1)$;

(iii) $\lambda_\nu^K \sim \frac{1}{2}\lambda_\nu^G$ as $n \to \infty$ for all ν such that $\tau_\nu^G \in \Delta$.

Then the Gauss–Kronrod formula (5.1) admits a circle theorem in the sense

$$\frac{2n\lambda_\nu^K}{\pi w(\tau_\nu^G)} \sim \sqrt{1 - (\tau_\nu^G)^2}, \quad \frac{2n\lambda_\mu^{*K}}{\pi w(\tau_\mu^K)} \sim \sqrt{1 - (\tau_\mu^K)^2}, \quad n \to \infty, \tag{5.4}$$

for all ν, μ as defined in assumptions (ii) and (iii).

Proof. The first relation in (5.4) is an easy consequence of assumptions (i) and (iii):

$$\frac{2n\lambda_\nu^K}{\pi w(\tau_\nu^G)} \sim \frac{n\lambda_\nu^G}{\pi w(\tau_\nu^G)} \sim \sqrt{1-(\tau_\nu^G)^2}, \quad n\to\infty.$$

To prove the second relation in (5.4), we first note that the $n+1$ Gaussian nodes for $w^K = \pi_n w$ are precisely the Kronrod nodes τ_μ^K. By assumption (ii),

$$\frac{2n\lambda_\mu^*}{\pi\pi_n(\tau_\mu^K)w(\tau_\mu^K)} \sim \sqrt{1-(\tau_\mu^K)^2}, \quad n\to\infty. \tag{5.5}$$

Since the Gauss formula for w^K is certainly interpolatory, we have

$$\lambda_\mu^* = \int_{-1}^1 \ell_\mu^*(t) w_K(t)\mathrm{d}t = \int_{-1}^1 \ell_\mu^*(t)\pi_n(t)w(t)\mathrm{d}t$$

with

$$\ell_\mu^*(t) = \prod_{\kappa\neq\mu} \frac{t-\tau_\kappa^K}{\tau_\mu^K-\tau_\kappa^K}, \quad \mu = 1,2,\ldots,n+1,$$

denoting the elementary Lagrange interpolation polynomials for the nodes $\tau_1^K, \tau_2^K, \ldots, \tau_{n+1}^K$. On the other hand, by the interpolatory nature of (5.1), we have similarly

$$\lambda_\mu^{*K} = \int_{-1}^1 \frac{\pi_n(t)}{\pi_n(\tau_\mu^K)}\,\ell_\mu^*(t)w(t)\mathrm{d}t = \frac{1}{\pi_n(\tau_\mu^K)}\,\lambda_\mu^*. \tag{5.6}$$

By (5.5) and (5.6), therefore,

$$\frac{2n\lambda_\mu^{*K}}{\pi w(\tau_\mu^K)} \sim \frac{2n\lambda_\mu^*}{\pi\pi_n(\tau_\mu^K)w(\tau_\mu^K)} \sim \sqrt{1-(\tau_\mu^K)^2}, \quad n\to\infty.$$

□

EXAMPLE 3. Jacobi weight function $w(t) = (1-t)^\alpha(1+t)^\beta$, $\alpha,\beta\in[0,\frac{5}{2})$.

For these weight functions, (5.4) has been proved by Peherstorfer and Petras [13, Theorem 2], from which assumptions (ii) and (iii) can be recovered by "inverse implication". Assumption (i), of course, is satisfied for these weight functions by virtue of Theorem 2.1.

The circle theorem, in this case, is illustrated in Fig. 5.1, on the left for $n = 20:5:40$, on the right for $n = 60:5:80$, with $\alpha,\beta = 0:0.4:2$, $\beta\geq\alpha$, in both cases.

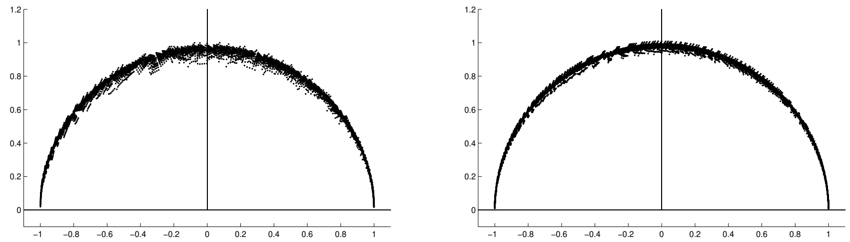

FIG. 5.1. *The circle theorem for Gauss–Kronrod quadrature*

We remark that asymptotic results of Ehrich [3, Corollary 3] imply the circle theorem also for negative values of $\alpha=\beta>-\frac{1}{2}$.

6. Potential-theoretic interpretation and extension of the circle theorem. There is a deep connection between Christoffel functions (and hence Gaussian weights) and equilibrium measures in potential theory. For the necessary potential-theoretic concepts, see [15]. Thus, for example, the density of the equilibrium measure $\omega_{[-1,1]}$ of the interval $[-1,1]$ is $\omega'_{[-1,1]}(t) = 1/(\pi\sqrt{1-t^2})$, showing that (2.3) can be interpreted by saying that as $n \to \infty$ the ratio $n\lambda_n(t;w)/w(t)$ converges to the reciprocal of the density of the equilibrium measure of $[-1,1]$. Here we consider a weight function w that is compactly supported on a (regular) set $E \subset \mathbb{R}$ and $\Delta \subset E$ an interval on which w satisfies the Szegö condition (2.4). Then, for almost all ν,

$$\frac{n\lambda_\nu^G}{w(t)} \sim \frac{1}{\omega'_E}, \quad n \to \infty, \tag{6.1}$$

where ω'_E is the density of the equilibrium measure of E (cf. [17, Theorem 1]).

EXAMPLE 4. A weight function supported on two intervals,

$$w(t) = \begin{cases} |t|^\gamma (t^2-\xi^2)^p (1-t^2)^q, & t \in [-1,-\xi] \cup [\xi,1], \\ 0 \quad \text{elsewhere}, \end{cases}$$

where $0 < \xi < 1$, $p > -1$, $q > -1$ and $\gamma \in \mathbb{R}$.

The recursion coefficients for the weight function w are explicitly known if $\gamma = \pm 1$ and $p = q = \pm 1/2$ (see [6, §5]). The quantities $n\lambda_\nu^G/(\pi w(\tau_\nu^G))$ in these cases are therefore easily computable; plotting them for $\xi = \frac{1}{2}$, and $n = 60 : 5 : 80$, yields the graph in Fig. 6.1.

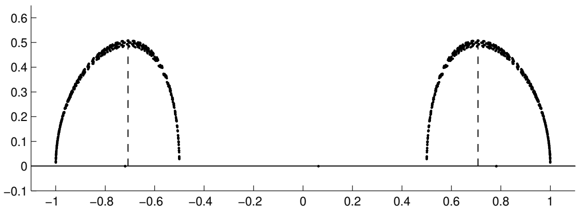

FIG. 6.1. *Analogue of the circle theorem for the weight function of Example 4*

The limiting curve for general ξ must be related to the reciprocal density $\omega'_{[-1,-\xi]\cup[\xi,1]}$ of the two support intervals. We can find its equation by using the known fact [6, §6] that for $\gamma = 1$ and $p = q = -1/2$, when n is even, the Gauss weights λ_ν^G are all equal to π/n. Consequently, for these n, and $\tau_\nu^G \in [\xi, 1]$,

$$\frac{n\lambda_\nu^G}{\pi w(\tau_\nu^G)} = \frac{1}{w(\tau_\nu^G)} = |\tau_\nu^G|^{-1}[(\tau_\nu^G)^2 - \xi^2]^{1/2}[1-(\tau_\nu^G)^2]^{1/2}, \tag{6.2}$$

so that the right branch of the limiting curve, and by symmetry the curve itself, has the equation $y = \varphi(t)$, where

$$\varphi(t) = |t|^{-1}(t^2-\xi^2)^{1/2}(1-t^2)^{1/2}.$$

The extrema of φ are attained at $t_0 = \pm\sqrt{\xi}$ and have the value $\varphi_0 = 1 - \xi$. For $\xi = \frac{1}{2}$, therefore, $t_0 = \pm\sqrt{1/2} = \pm 0.7071\ldots$, $\varphi_0 = \frac{1}{2}$.

We conclude from (6.1) and (6.2) that

$$\omega'_{[-1,-\xi]\cup[\xi,1]}(t) = \pi^{-1}|t|(t^2-\xi^2)^{-1/2}(1-t^2)^{-1/2}, \quad t \in [-1,-\xi]\cup[\xi,1]. \tag{6.3}$$

Actually, the equilibrium measure is known for any set E whose support consists of several intervals and is an inverse polynomial image of $[-1,1]$,

$$E = \mathcal{T}_N^{-1}([-1,1]),$$

where $\mathcal{T}_N$ is a polynomial of degree N. Then indeed [9, p. 577],

$$\omega'_E(t) = \frac{|\mathcal{T}'_N(t)|}{N\pi\sqrt{1-\mathcal{T}_N^2(t)}}, \quad t \in E. \tag{6.4}$$

In the case at hand, $E = [-1,-\xi]\cup[\xi,1]$, $0<\xi<1$, we have

$$\mathcal{T}_2(t) = \frac{2t^2-\xi^2-1}{1-\xi^2},$$

and (6.4) becomes (6.3).

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