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THE CIRCLE THEOREM AND RELATED THEOREMS FOR GAUSS-TYPE QUADRATURE RULES*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. In 1961, P.J. Davis and P. Rabinowitz established a beautiful "circle theorem" for Gauss and Gauss–Lobatto quadrature rules. They showed that, in the case of Jacobi weight functions, the Gaussian weights, suitably normalized and plotted against the Gaussian nodes, lie asymptotically for large orders on the upper half of the unit circle centered at the origin. Here analogous results are proved for rather more general weight functions—essentially those in the Szegö class—, not only for Gauss and Gauss–Lobatto, but also for Gauss–Radau formulae. For much more restricted classes of weight functions, the circle theorem even holds for Gauss–Kronrod rules. In terms of potential theory, the semicircle of the circle theorem can be interpreted as the reciprocal density of the equilibrium measure of the interval $[-1,1]$. Analogous theorems hold for weight functions supported on any compact subset Δ of $(-1,1)$, in which case the (normalized) Gauss points approach the reciprocal density of the equilibrium measure of Δ. Many of the results are illustrated graphically.

Key words. Gauss quadrature formulae, circle theorem, Gauss–Radau, Gauss–Lobatto and Gauss–Kronrod formulae, Christoffel function, potential theory, equilibrium measure

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