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WEIGHTED APPROXIMATION OF DERIVATIVES ON THE HALF-LINE*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. Weighted polynomial approximation of derivatives on the half line $[0, \infty)$ is considered. The weight function will be of the form $e^{-R(t)}$, a "folded" Freud weight. That is, that $R(x^2) = Q(x)$, where $e^{-Q(x)}$ is a Freud weight on $(-\infty, \infty)$. Linear processes which can be used for approximation of derivatives include interpolation, in particular using node-sets recently developed by J. Szabados.

Key words. Freud weights, derivatives, weighted approximation

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1. Approximation of derivatives with folded Freud weights on $[\mathbf{0}, \infty)$. Let $f \in C^q_R[0, \infty)$, that is, let $f^{(0)}, \ldots, f^{(q)}$ be continuous and $e^{-R(t)} f^{(k)}(t) \to 0$ as $t \to \infty$ for $k = 0, \ldots, q$. The goal is to approximate $f^{(0)}, \ldots, f^{(q)}$ in the same weighted norm, using a linear projection, and using only data about f. The weighted norms will involve *folded* Freud weights on $[0, \infty)$, which are related to Freud weights. The function $e^{-Q(x)}$ is a *Freud weight* on the interval $(-\infty, \infty)$ if the function Q is even, twice continuously differentiable on $(0, \infty)$, with $Q'(x) > 0$ on the same interval, and Q also satisfies there

$$A \le \frac{(xQ'(x))'}{Q'(x)} \le B$$

for two constants $A, B > 1$.

The notation $f \in C^q_Q(-\infty, \infty)$ means that $f^{(0)}, \ldots, f^{(q)}$ are continuous and $e^{-Q(x)} f^{(k)}(x) \to 0$ as $x \to \pm\infty$ for $k = 0, \ldots, q$.

On the interval $[0, \infty)$, the weight function $e^{-R(t)}$ will be called a *folded Freud weight*, if there exists a function Q on $(-\infty, \infty)$ such that $R(x^2) = Q(x)$ and $e^{-Q(x)}$ is a Freud weight. In Balázs and Kilgore [5], the weights $R(t)$ were called *split* Freud weights, but calling them folded Freud weights seems to be better.

Relevant both to Freud weights and to folded Freud weights are the *Freud numbers* q_n which are defined for $n > 0$ as the least positive solution of the equation $q_n Q'(q_n) = n$. From their definition it follows (Mhaskar [8], (4.1.3)) that there exist positive constants c_1, c_2, c_3, c_4 depending on Q, such that

$$c_1 n^{\frac{1}{1+c_2}} < q_n < c_3\, n^{\frac{1}{1+c_4}}. \tag{1.1}$$

Also, we use the notations $E_n(f; e^{-R(t)})$ for a $f \in C_R[0, \infty)$ and $E_n(g; e^{-Q(x)})$ for $g \in C_Q(-\infty, \infty)$ to denote the error in the best weighted approximation.

The following Theorem gives an estimate for the weighted approximation of $f^{(k)}$, $k = 0, \ldots, q$, based upon the approximation of f.

THEOREM 1.1. *Let $f \in C^q_R[0, \infty)$. For any given $k \in \{0, \ldots, q\}$ let P be a polynomial of degree at most $n + k$ such that $t^{-\frac{k}{2}}(f(t) - P(t))$ has a removable singularity at 0 and such that for some M*

$$t^{-\frac{k}{2}} e^{-R(t)} |f(t) - P(t)| \le M \left(\frac{q_n}{n}\right)^k E_n(f^{(k)}; e^{-R(t)}).$$

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Then there is a constant γ_k *such that*

$$e^{-R(t)}|f^{(k)}(t) - P^{(k)}(t)| \le \gamma_k\, M\, E_n(f^{(k)}; e^{-R(t)}).$$

LEMMA 1.2 (cf. Mhaskar [8], Theorem 4.1.1). *There exists a constant* α *such that if* $g \in C^1_Q(-\infty, \infty)$ *then*

$$E_n(g; e^{-Q(x)}) \le \alpha(q_n/n) E_{n-1}(g'; e^{-Q(x)}).$$

LEMMA 1.3 (cf. Mhaskar [8], Theorem 4.1.7). *If* $g \in C^1_Q(-\infty, \infty)$ *and if* P_n *is a polynomial of degree at most* $n \ge 1$ *with*

$$\|e^{-Q(x)}(g(x) - P_n(x))\| \le C\frac{q_n}{n} E_n(g'; e^{-Q(x)}),$$

then there exists a constant β*, such that,*

$$\|e^{-Q(x)}(g'(x) - P_n'(x))\| \le C\beta E_{n-1}(g'; e^{-Q(x)}).$$

Here, we give in detail the proof of Theorem 1.1 for $q = 1$. The result is true for arbitrary q, but the details are quite involved and lengthy.

Proof of Theorem 1.1 for the case $q = 1$. Without loss of generality, the proof may be simplified immediately by assuming that $f(0) = P(0) = 0$. For, if $f(0) \ne 0$ one can replace $f(t)$ by $f(t) - f(0)$ and, as it is already assumed that $f(0) - P(0) = 0$, also replace $P(x)$ by $P(x) - P(0)$. Then, proceeding with the assumption that $f(0) = P(0) = 0$, the transformation $t \leftrightarrow x^2$ induces an isometric isomorphism between $C_R[0, \infty)$ and the even part of $C_Q(-\infty, \infty)$. Furthermore, let $t^{-\frac{1}{2}} f(t) = g(x)$. It follows easily that g is an *odd* function, $g \in C^1_Q(-\infty, \infty)$ and $t^{-\frac{1}{2}} P(t)$ is mapped to an *odd* polynomial $p(x)$. Also,

$$e^{-Q(x)}|g(x) - p(x)| = t^{-\frac{1}{2}} e^{-R(t)}|f(t) - P(t)| \le M\,\frac{q_n}{n} E_n(f'; e^{-R(t)}). \tag{1.2}$$

Furthermore,

$$f'(t) = \frac{1}{2}(g'(x) + \frac{g(x)}{x}) = \frac{1}{2}\left(g'(x) + \frac{f(t)}{t}\right) \tag{1.3}$$

and

$$P'(t) = \frac{1}{2}(p'(x) + \frac{p(x)}{x}) = \frac{1}{2}\left(p'(x) + \frac{P(t)}{t}\right).$$

Combining these observations, we see that

$$f'(t) - P'(t) = \frac{1}{2}\left(g'(x) - p'(x) + \frac{g(x) - p(x)}{x}\right).$$

Now, if $x \ne 0$, there is x_1 between 0 and x such that, using the Mean Value Theorem and the monotonicity of Q, we have

$$e^{-Q(x)}|\frac{g(x) - p(x)}{x}| = e^{-Q(x)}|g'(x_1) - p'(x_1)| \le e^{-Q(x_1)}|g'(x_1) - p'(x_1)|,$$

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and thus

$$e^{-R(t)}|f'(t) - P'(t)| \leq \|e^{-Q(x)}(g'(x) - p'(x))\|.$$

Importantly, from this inequality and from (1.2) we see also that the hypothesis

$$t^{-\frac{1}{2}}e^{-R(t)}|f(t) - P(t)| \leq M\,\frac{q_n}{n}E_n(f'; e^{-R(t)})$$

implies

$$\|e^{-Q(x)}(g(x) - p(x))\| \leq M\,\frac{q_n}{n}E_n(g'; e^{-Q(x)}),$$

and by Lemma 1.3 there exists a constant β such that

$$e^{-Q(x)}|g'(x) - p'(x)| \leq \beta\, M\, E_n(g'; e^{-Q(x)}).$$

Therefore,

$$e^{-R(t)}|f'(t) - P'(t)| \leq \beta\, M\, E_n(g'; e^{-Q(x)}). \tag{1.4}$$

From (1.3) we have

$$g'(x) = 2f'(t) - \frac{f(t)}{t},$$

from which it follows that

$$E_n(g'; e^{-Q(x)}) \leq 2E_n(f'; e^{-R(t)}) + E_n(\frac{f(t)}{t}; e^{-R(t)}). \tag{1.5}$$

Furthermore, it is the case that

$$E_n\left(\frac{f(t)}{t}; e^{-R(t)}\right) \leq E_n(f'; e^{-R(t)}). \tag{1.6}$$

To see this, let $\tilde{P}$ be any polynomial of degree at most $n+1$ such that $\tilde{P}(0) = 0$ and let $t > 0$. Then, using the Mean Value Theorem, there is a t_1 between 0 and t such that

$$e^{-R(t)}\frac{f(t) - \tilde{P}(t)}{t} = e^{-R(t)}(f'(t_1) - \tilde{P}'(t_1)),$$

and, since $R(t_1) < R(t)$, it follows that

$$e^{-Rt}|\frac{f(t) - \tilde{P}(t)}{t}| = e^{-R(t_1)}|f'(t_1) - \tilde{P}'(t_1)|.$$

Since $t > 0$ is arbitrary and since $t = 0$ presents no additional difficulty, it follows that

$$\|e^{-R(t)}\frac{f(t) - \tilde{P}(t)}{t}\| \leq \|e^{-R(t)}(f'(t) - \tilde{P}'(t))\|.$$

As $\tilde{P}(0) = 0$, it is clearly true that *any* polynomial of degree at most n can be written as $\tilde{P}'$, given a suitable choice of $\tilde{P}$, which establishes (1.6).

Combining (1.5) with (1.6) now results in

$$E_n(g'; e^{-Q(x)}) \leq 3E_n(f'; e^{-R(t)}), \tag{1.7}$$

and combining (1.4) with (1.7) gives

$$e^{-R(t)}|f'(t) - P'(t)| \leq 3\beta\, M\, E_n(f'; e^{-R(t)}).$$

which completes the proof of Theorem 1.1 for the case $q = 1$, with $\gamma_1 = 3\beta$. $\square$

2. Approximation of a derivative by interpolation. As a concrete method for choosing an approximating polynomial for our function f, we consider Lagrange interpolation in particular. First we need some definitions:

The set $T_n = \{t_0, \ldots, t_n\}$, where $0 = t_0 < t_1 \ldots < t_n$, will be a set of nodes in $[0, \infty)$. The interpolation operator for $f \in C_R[0, \infty)$ on the nodes T_n is given by

$$L_n(f, t) = \sum_{k=0}^{n} f(t_k)\ell_k(t),$$

in which the functions ℓ_k are the fundamental polynomials, defined for $k = 0, \ldots, n$ by

$$l_k(t) = \prod_{\substack{j = 0 \\ j \neq k}}^{n} \frac{t - tj}{t_k - t_j}, \tag{2.1}$$

It is easily seen that

$$||L_n||_R = ||e^{-R(t)} \sum_{k=0}^{n} e^{R(t_k)} |\ell_k(t)| \, ||.$$

A second interpolation operator based upon the node-set T_n is also useful here. The operator L_n^* is defined only for those $f \in C_R[0, \infty)$ which satisfy $f(0) = f(t_0) = 0$. Its range will consist of the span of its fundamental functions ℓ_k^*, defined for $k = 1, \ldots, n$ by

$$l_k^*(t) = \frac{t^{\frac{1}{2}}}{t_k^{\frac{1}{2}}} \prod_{\substack{j = 1 \\ j \neq k}}^{n} \frac{t - tj}{t_k - t_j}, \tag{2.2}$$

after which

$$L_n^*(f, t) = \sum_{k=1}^{n} f(t_k)\ell_k^*(t).$$

We note that $L_n^*(f, t)$ is *not* a polynomial unless its output is zero, but $t^{\frac{1}{2}} L_n^*(f, t)$ *is* a polynomial.

Also, naturally related to the nodes T_n are the nodes $X_{2n} = \{x_0, x_{\pm 1}, \ldots, x_{\pm n}\}$ satisfying

$$x_0 = 0 \text{ and } x_k = \sqrt{t_k} \text{ and } x_{-k} = -x_k. \tag{2.3}$$

On the nodes X_{2n} there is also a Lagrange interpolation operator P_{2n}. Denoting the kth fundamental polynomial for P_{2n} by λ_k for $k = -n, \ldots, n$, we have

$$\lambda_k(t) = \prod_{\substack{j = -n \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}, \tag{2.4}$$

and for $g \in C_Q(-\infty, \infty)$

$$P_{2n}(g, x) = \sum_{k=-n}^{n} g(x_k)\lambda_k(x),$$

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with

$$\|P_{2n}\|_Q = \|e^{-Q(x)} \sum_{k=-n}^{n} e^{Q(x_k)} |\lambda_k(x)|\,\|.$$

THEOREM 2.1. *Let $e^{-R(t)}$ be a folded Freud weight and $e^{-Q(x)}$ the related Freud weight, and let the node-sets T_n and X_{2n} be related as in (2.3). Then for $f \in C^1_R[0,\infty)$, such that $f(0) = 0$, the following estimates hold:*

$$|e^{-R(t)}(f(t) - L_n(f,t))| \leq (1 + \|L_n\|_R)E_n(f; e^{-R(t)}). \tag{2.5}$$

$$|e^{-R(t)}(f(t) - L_n(f,t))| \leq (1 + \|P_{2n}\|_Q)E_n(f; e^{-R(t)}). \tag{2.6}$$

Also, $t^{-\frac{1}{2}} f(t) \in C_R[0,\infty)$, and

$$|t^{-\frac{1}{2}} e^{-R(t)}(f(t) - L_n(f,t))| \leq (1 + \|L^*_n\|_R)E_n(t^{-\frac{1}{2}} f(t); e^{-R(t)}). \tag{2.7}$$

Furthermore, there is a constant γ such that

$$|t^{-\frac{1}{2}} e^{-R(t)}(f(t) - L_n(f,t))| \leq \gamma \frac{q_n}{n}(1 + \|L^*_n\|_R)E_{n-1}(f'(t); e^{-R(t)}) \tag{2.8}$$

and

$$|t^{-\frac{1}{2}} e^{-R(t)}(f(t) - L_n(f,t))| \leq \gamma \frac{q_n}{n}(1 + \|P_{2n}\|_Q)E_{n-1}(f'; e^{-R(t)}). \tag{2.9}$$

There is a constant C such that for every $f \in C^1_R[0,\infty)$ satisfying $f(0) = 0$

$$\|e^{-R(t)}(f'(t) - (L_n f(t))')\| \leq C(1 + \|L^*_n\|_P)E_n(f'; e^{-R(t)}), \tag{2.10}$$

$$\|e^{-R(t)}(f'(t) - (L_n f(t))')\| \leq C(1 + \|P_{2n}\|_Q)E_n(f'; e^{-R(t)}), \tag{2.11}$$

If $f(0)$ is not assumed to be 0, then all of these results remain valid, as $f(t)$ can then be replaced by $f(t) - f(0)$.

Proof. (2.5) follows easily from the Lebesgue's theorem on bounded linear projection operators.

(2.6) follows by combining (2.5) with the fact that $\|L_n\|_R \leq \|P_{2n}\|_Q$, which is true because $C_R[0,\infty)$ corresponds naturally to the even part of $C_Q(-\infty,\infty)$ and the output of L_n, which operates only on $C_R[0,\infty)$, also naturally corresponds to the output of P_{2n} operating on the even functions in $C_Q(-\infty,\infty)$. More explicitly, if g is defined on $(-\infty,\infty)$ by $g(x) = f(x^2)$, and if $f(0) = 0$, then (2.6) follows from the fact that $L_n(f,t) = L_n(f,x^2) = P_{2n}(g,x)$.

Now, let $f_1(t) = t^{-\frac{1}{2}} f(t)$, with $f_1(0) = 0$. That f_1 is continuous should be clear, as f itself is differentiable and $f(0) = 0$. For the same reasons, f_1 is clearly bounded on $[0,1]$. Also, $|f_1(t)| \leq |f(t)|$ whenever $t \geq 1$. Therefore, $t^{-\frac{1}{2}} f(t) = f_1$ is in $C_R[0,\infty)$. Important in what follows is the identity

$$t^{-\frac{1}{2}} e^{-R(t)}(f(t) - L_n(f,t)) = t^{-\frac{1}{2}} e^{-R(t)}(f(t) - t^{\frac{1}{2}} L^*_n(f_1,t)). \tag{2.12}$$

This is true because $L_n(f,t) = t^{\frac{1}{2}} L^*_n(f_1,t))$, following from

$$f(t_k)\ell_k(t) = t^{\frac{1}{2}} f_1(t_k)\ell^*_k(t), \qquad \text{for } k = 1,\ldots,n,$$

a consequence of (2.1) and (2.2). Therefore, (2.7) follows from Lebesgue's theorem on bounded linear projections. For, L_n^* defines a bounded linear projection whose domain is the set of functions in $C_R[0,\infty)$ which are zero at 0, and by (2.12) the inequality in (2.7) is equivalent to the statement

$$e^{-R(t)}|f_1(t) - L_n^*(f_1,t)| \leq (1 + \|L_n^*\|_R)E_n(f_1(t); e^{-R(t)}).$$

To prove (2.8) and (2.7), we now define a function g_1 by $xg_1(x) = f(x^2)$, noticing that g_1 is *odd*. Since $f(0) = 0$ and $f \in C_R[0,\infty)$, it follows that g_1 is defined and continuous on $(-\infty,\infty)$. In particular, g_1 is defined and continuous at 0, with $g_1(0) = 0$. Indeed, g_1 is the odd extension of $f_1(x^2)$, and it follows that $g_1 \in C_Q(-\infty,\infty)$, with $\|e^{-Q(x)}g_1(x)\| = \|e^{-R(t)}f_1(t)\|$.

(2.8) follows from (2.7). Specifically, we have

$$E_n(t^{-\frac{1}{2}}f(t); e^{-R(t)}) = E_n(g_1; e^{-Q(x)}),$$

and, by using Lemma 1.2 followed by (1.7), we get

$$E_n(t^{-\frac{1}{2}}f(t); e^{-R(t)}) \leq \alpha(q_n/n)E_{n-1}(g_1'; e^{-Q(x)}) \leq 3\alpha(q_n/n)E_{n-1}(f'(t); e^{-R(t)}),$$

and (2.8) follows, with $\gamma = 3\alpha$.

(2.9) follows from (2.8). To see this, recall that $g_1(x)$ is the odd extension of $f_1(x^2)$ and that $g_1 \in C_Q(-\infty,\infty)$. Thus $\|e^{-Q(x)}g_1(x)\| = \|e^{-R(t)}f_1(t)\|$, and $P_{2n}(g_1,x)$ exists. Furthermore, from (2.2) and (2.4) we have for *every* $f \in C_R[0,\infty)$ such that $f(0) = 0$ and for g the odd extension of $f(x^2)$, that $|e^{-R(x^2)}L_n^*(f,x^2)| = |e^{-Q(x)}P_{2n}(g,x)|$ for every x. Therefore, $\|L_n^*\|_R \leq \|P_{2n}\|_Q$. The estimate (2.9) follows.

(2.10) follows directly now from the application of Theorem 1.1 to the inequality (2.8), using $M = \gamma(1 + \|L_n^*\|_R)$ and, also (2.11) follows by the application of Theorem 1.1 to the inequality (2.9), using here $M = \gamma(1 + \|P_{2n}\|_Q)$.

The final statement of the theorem involves preconditioning of a function f which does not satisfy $f(0) = 0$; the statement should be obvious.

This concludes the proof of Theorem 2.1. $\square$

REMARK 1. *The relationship (2.3) which gives the nodes X_n in terms of the nodes T_n is obviously reversible, and it is clearly possible to define operators L_n and L_n^* for use in $C_R[0,\infty)$ based upon an operator P_{2n} defined on $C_Q(-\infty,\infty)$ with better-known properties. The only requirement is that P_{2n} should be based upon nodes which are symmetric with respect to 0. If this construction is carried out, then the three estimates (2.6), (2.9), and (2.11), which are all given in terms of $\|P_{2n}\|_Q$, can be very useful.*

In the next section, we carry out precisely such a construction, based upon a node-set X_n and obtain error estimates and convergence estimates, which use these estimates from Theorem 2.1.

3. A convergence result for approximation of derivatives in $\mathbf{C^1_R[0,\infty)}$.

Here, we give a particular set of nodes T_n for the approximation of derivatives, using the methods of the previous section.

The construction of these nodes of interpolation requires the Mhaskar-Rahmanov-Saff numbers a_n, defined by the equation

$$\frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)\, dt}{\sqrt{1-t^2}} = n \quad \text{for} \quad n \geq 1.$$

It is known (see Mhaskar [8], (6.1.21), p. 131) that $q_n \leq a_n \leq \sqrt{2} q_{2n}$.

THEOREM 3.1. *Let $e^{-R(t)}$ be a folded Freud weight, and let $Q(x) = R(x^2)$. For $n \geq 1$ let the node-set X_{2n} be defined by $x_n = a_{2n-1}$ and $x_{-n} = -a_{2n-1}$, with $x_{-n+1}, \ldots, x_{n-1}$ chosen as the $2n-1$ zeroes of the $(2n-1)$st-degree orthogonal polynomial with respect to the Freud weight $e^{-2Q(x)}$.*

Let the nodes T_n be derived from the nodes X_{2n} by means of (2.3), and let the interpolation operator L_n be defined upon the nodes T_n.

Then, the following estimates hold for interpolation of any function $f \in C^1_R[0, \infty)$, with the constants C and C_1 depending only upon $R(t)$:

$$\|e^{-R(t)}(f(t) - L_n(f,t))\| \leq C \log n E_n(f; e^{-R(t)})$$

and

$$\|e^{-R(t)}(f'(t) - (L_n(f,t))')\| \leq C_1 \log n E_n(f'; e^{-R(t)}).$$

Thus,

$$\|e^{-R(t)}(f(t) - L_n(f,t))\| \to 0 \ \ as \ \ n \to \infty \tag{3.1}$$

and

$$\|e^{-R(t)}(f'(t) - (L_n(f,t))')\| \to 0 \ \ if \ \ \log n E_n(f'; e^{-R(t)}) \to 0 \ \ as \ \ n \to \infty. \tag{3.2}$$

Proof. For the interpolation operator P_{2n} upon the node-set X_{2n} in the Theorem there exists a constant c independent of n such that

$$\|P_{2n}\|_Q \leq c \log n.$$

This estimate is proved in Szabados [9], along with the construction of the node-sets X_{2n}. Using this estimate for $\|P_{2n}\|_Q$, the first two estimates given here follow from Theorem 2.1. In particular, the inequalities (2.6) and (2.11) give estimates directly in terms of $\|P_{2n}\|_Q$.

Now, the estimate (3.1) also follows from (2.9), combined with the observation that

$$\frac{q_n}{n} \log n \to 0 \ \ \text{as} \ \ n \to \infty,$$

which follows from (1.1).

The estimate (3.2) is a straightforward application of the second estimate in the Theorem, already established. $\square$

REMARK 2. *For the operators P_{2n} the lower estimate $c \log n$ for growth rate is also shown to hold in general, in the presence of any Freud weight (Vértesi [10]), and thus the estimates for the growth of $\|P_{2n}\|_Q$ cannot be improved here. For the particular case that the Freud weight is the Hermite weight and the folded weight is the Laguerre weight, the lower estimates for growth rate of both $\|L_n\|_R$ and $\|P_{2n}\|_Q$ which appear in Theorem 3.1 are sharp. For these two weight functions the lower estimate follows from the fact that the weighted operator of interpolation obeys the Bernstein-Erdős conditions (Kilgore [6]), combined with the fact that in any space where the Bernstein-Erdős conditions characterize minimal norm interpolation, the lower estimate $c \log n$ must hold (Kilgore [7]).*

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