Electronic Transactions on Numerical Analysis.
Volume 25, pp. 101-114, 2006.

ISSN 1068-9613.

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QUADRATURE-FREE QUASI-INTERPOLATION ON THE SPHERE*

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. We construct certain quasi-interpolatory operators for approximation of functions on the sphere, using tensor product scattered data satisfying certain symmetry conditions. Our operators are constructed without using any quadrature formulas. We use instead a special class of orthonormal bivariate trigonometric polynomials. These polynomials are functions on the sphere, and are constructed in a numerically stable way, based on the data locations. The quasi-interpolatory operators give near best approximation to every continuous function.

We demonstrate our constructions numerically with several benchmark functions using randomly generated data locations.

Key words. function approximation on the sphere, scattered data, quasi-interpolation, Jacobi matrices

AMS subject classifications. 42A15, 65D32, 33C55

1. Introduction.

The problem of approximation of functions on the sphere arises in almost all applications involving modelling of data collected on the surface of the earth. A simple, classical technique for approximation is to interpolate the data. However, in many practical applications, interpolation is not a suitable method for approximation. For example, if the data has noise, one may not wish to require that the approximation should reproduce this noise as well. Also, in the case when the number of sites at which the data is collected is too large, it may be neither feasible nor necessary to obtain interpolation from a subspace that is commensurately large. We note also that the sequence of interpolants may not always converge, and hence, the accuracy of approximation may not be good in spite of the effort needed to compute the interpolation operator.

The idea behind quasi-interpolation is to consider a sequence $\{V_n\}$ of finite dimensional subspaces of the space $C(K)$ of all continuous functions on a compact set K. One then uses the data to construct an operator $\mathcal{T}_n(f)$ taking values in V_n, where the dimension of V_n is less than the number of data points. Instead of requiring interpolation, one requires that the operator norms of $\mathcal{T}_n$ be uniformly bounded in n, and that $\mathcal{T}_n(P) = P$ for all $P \in V_{\alpha n}$ for some constant $\alpha \in (0,1)$. Denoting by $\|\circ\|$ the supremum norm for $C(K)$, and by B the bound on the operator norms of $\mathcal{T}_n$, these conditions imply that for every $f \in C(K)$ and $P \in V_{\alpha n}$,

$$\|f - \mathcal{T}_n(f)\| = \|(f-P) - \mathcal{T}_n(f-P)\| \le (1+B)\|f-P\|.$$

Taking infimum over $P \in V_{\alpha n}$, this implies that

$$\|f - \mathcal{T}_n(f)\| \le (1+B)\,\mathrm{dist}\,(f, V_{\alpha n}) := (1+B)\inf_{P\in V_{\alpha n}}\|f-P\|.$$

Thus, if the union of the spaces V_n is dense in $C(K)$, the quasi-interpolatory operators $\mathcal{T}_n(f)$ *always* converge to f for every $f \in C(K)$, and moreover, at a near optimal rate of approximation.

*Received May 2, 2005. Accepted for publication January 28, 2006. Recommended by I. Sloan.

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Such quasi-interpolatory operators are used routinely in the context of spline functions [2, 3, 4]. In the case of approximation of periodic functions, S. N. Bernstein [1] constructed quasi-interpolatory trigonometric polynomial operators, based on equidistant data. In the case of approximation on the sphere, there is no natural, coordinate–free analogue of "equidistant data." Several substitutes have been investigated by Saff and his collaborators, among others (cf., for example, [10, 11] and citations therein). However, in all the applications where the data is generated experimentally, for example, by a satellite, one does not have any control on choosing the locations of the sites where the data is collected. Such data is called *scattered data*.

In [14, 17, 16, 13, 20, 9], we have studied quasi-interpolatory operators using scattered data. These operators yield spherical polynomials, and are guaranteed to yield a near best approximation to every continuous function from polynomials of a commensurate degree. These results have been applied to obtain several theorems concerning approximation by neural networks, construction of polynomial and zonal function frames for an analysis of data, solution of pseudo-differential equations, and representation of functions using finitely many bits. Our construction requires the use of quadrature formulas that are exact for polynomials of higher and higher degrees. While the existence of these quadrature formulas has been proved in [15], and their use illustrated by a few numerical examples, a numerically stable construction is not yet available in the case when the data set is large.

In [5], interpolation from certain spaces of bivariate trigonometric polynomials that are functions on the sphere was used in the numerical solution of certain integral equations on the sphere. In [6], we have given matrix–free constructions for interpolation by such spherical trigonometric polynomials at judiciously chosen sites. In this paper, we explore the construction of quasi-interpolatory operators for these spaces, without using any quadrature formulas. Instead, we use the data locations to construct orthogonal bases for spaces of bivariate trigonometric polynomials which are functions on the sphere.

In Section 2, we describe the construction of our operators with scattered data on the unit circle, and use these operators to construct corresponding operators based on tensor product scattered data on the sphere. The numerical aspects of these constructions and applications to the approximation of certain benchmark functions using randomly generated data points are described in Section 3. The proofs of the results in Section 2 are given in Section 4.

We would like to thank the two referees for their valuable comments on the first submitted draft of this paper.

2. Quasi-interpolation.

2.1. Quasi-interpolation on the unit circle. The starting point of our investigation is a finite set of points $\mathcal{S} \subset [0, 2\pi)$, such that $\theta \in \mathcal{S}$ implies that $(2\pi - \theta) \in \mathcal{S}$. As usual, we identify points which are equal modulo 2π. We assume that there exists an integer $N \geq 4$ such that each subinterval of $[0, \pi]$ with length π/N contains at least one point in $\mathcal{S}$. We now choose N distinct points $\{\theta_j\}_{j=1}^N$ such that $\theta_j \in [(j-1)\pi/N, j\pi/N] \cap \mathcal{S}$, $j = 1, \cdots, N$, and observe that $\theta_{N+j} := 2\pi - \theta_{N-j+1} \in \mathcal{S}$. Our constructions will depend only on the points $\{\theta_j\}$. Therefore, we may now assume that $\mathcal{S} = \{\theta_j\}_{j=1}^{2N}$, where the point π, if present, is listed twice. We note in this connection that the points not selected above may be used for such purposes as noise reduction, numerical verification, etc.

For $x \geq 0$, we denote the class of all algebraic polynomials of degree at most x by Π_x, and the class of all trigonometric polynomials of order at most x by $\mathbb{H}_x$. Let $\{t_k \in \Pi_k\}_{k=0}^{N-1}$

and $\{u_k \in \Pi_k\}_{k=0}^{N-3}$ be polynomials such that

$$\frac{\pi}{N}\sum_{j=1}^{N} t_k(\cos\theta_j)t_\ell(\cos\theta_j) = \frac{\pi}{N}\sum_{j=1}^{N} u_k(\cos\theta_j)u_\ell(\cos\theta_j)\sin^2\theta_j = \begin{cases} 1, & \text{if } k=\ell, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $h : \mathbb{R} \to [0,\infty)$ be an even function, nonincreasing on $[0,\infty)$, that can be expressed as an indefinite integral of a function with bounded variation on $\mathbb{R}$. We assume further that $h(t) = 1$ if $0 \le t \le 1/2$ and $h(t) = 0$ if $t \ge 1$. For every integer $n \ge 0$, we write $h_{k,n} := h(k/(n+1))$, $k = 0, \pm1, \pm2, \cdots$.

If f is a continuous, 2π–periodic function, we write $\|f\|_\infty = \max_{\theta\in[0,2\pi]}|f(\theta)|$, and for integer $k \ge 0$,

$$a_k(f) := \frac{\pi}{2N}\sum_{j=1}^{2N} f(\theta_j)t_k(\cos\theta_j), \qquad b_k(f) = \frac{\pi}{2N}\sum_{j=1}^{2N} f(\theta_j)\sin\theta_j u_k(\cos\theta_j).$$

For integer n, $0 \le n \le N-3$, we define the quasi-interpolatory operator of degree at most n, based on the $2N$ data points by

$$T_{n,N}(f,\psi) = \sum_{k=0}^{n} h_{k,n}a_k(f)t_k(\cos\psi) + \sum_{k=0}^{n-1} h_{k+1,n}b_k(f)u_k(\cos\psi)\sin\psi. \tag{2.2}$$

We remark that Szabados [21] has shown that the operator $f \mapsto \sum_{k=0}^{n} h_{k,n}a_k(f)t_k(\cos\circ)$ is quasi-interpolatory for approximation of functions on $[-1,1]$ by algebraic polynomials, in the case when $N = 3n$, $\theta_j = j\pi/N$, and h is chosen to be

$$h(t) := \begin{cases} 1, & \text{if } 0 \le t \le 1/2, \\ 2(1-t), & \text{if } 1/2 < t \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

We will prove the following theorem in Section 4. Throughout this paper, the symbol c denotes a generic, absolute constant.

THEOREM 2.1. *Let $N \ge 4$, $0 \le n \le N-3$ be integers. Then $T_{n,N}(P) = P$ for $P \in \mathbb{H}_{(n+1)/2}$. Let f be a continuous 2π–periodic function. We have $T_{n,N}(f) \in \mathbb{H}_n$,*

$$\|T_{n,N}(f)\|_\infty \le c(1+\sqrt{n^2/N})\|f\|_\infty, \tag{2.4}$$

and hence,

$$\|f - T_{n,N}(f)\|_\infty \le c(1+\sqrt{n^2/N})\ \mathrm{dist}\,(f, \mathbb{H}_{(n+1)/2}). \tag{2.5}$$

Thus, if $N \ge cn^2$, then the operators $T_{n,N}$ are quasi-interpolatory operators (with uniformly bounded norms) in the sense of Section 1. Numerical experiments in Section 3 indicate that $N = n + 10$ may be sufficient to obtain good results.

2.2. Quasi–interpolation on the sphere. In this section, we use the operators developed above to construct corresponding operators on the sphere

$$\mathbb{S}^2 := \{(x,y,z)^T \in \mathbb{R}^3 \ :\ x^2+y^2+z^2 = 1\}.$$

We assume the standard parametrization

$$\hat{\mathbf{x}} = \boldsymbol{p}(\theta,\phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}, \qquad \hat{\mathbf{x}} \in \mathbb{S}^2.$$

Following the ideas in the previous subsection, we assume that the set $\mathcal{C}$ of data locations on the sphere is of the form $\{\boldsymbol{p}(\theta_j,\phi_k)\}$, where $\theta_j \in [(j-1)\pi/N, j\pi/N]$, $j = 1,\cdots,N$, are distinct points, and for $k = 1,\cdots,M$, $\phi_k \in [(k-1)\pi/M, k\pi/M]$ are distinct points, and $\phi_{M+k} = 2\pi - \phi_{M-k+1}$. We write $\theta_{N+j} := 2\pi - \theta_{N-j+1}$, $j = 1,\cdots,N$, and observe that $\boldsymbol{p}(\theta_{N+j},\phi_k) = \boldsymbol{p}(\theta_{N-j+1},\phi_k+\pi)$, $j = 1,\cdots,N$, $k = 1,\cdots,2M$. We assume further that the values of the target function are available at $\boldsymbol{p}(\theta_j,\phi_k)$, $j = 1,\cdots,2N$, $k = 1,\cdots,2M$.

Let C^* denote the class of all continuous functions on $\mathbb{R}^2$ that are 2π-periodic in each of their variables, equipped with the uniform norm $\|\circ\|^*_\infty$. For real $x, y \ge 0$, we denote the class of all bivariate trigonometric polynomials of order at most x in the first variable and at most y in the second variable by $\mathbb{H}_{x,y}$. For $g \in C^*$, we define the operator $U_{n,m}(g) := U_{n,m,N,M}(g)$ by applying the operator $T_{n,N}$, defined in (2.2), to g in the variable θ and the operator $T_{m,M}$ to g in the variable ϕ. We have from (2.5) that

$$\|g - U_{n,m}(g)\|^*_\infty \le c(1+\sqrt{n^2/N})(1+\sqrt{m^2/M})\ \mathrm{dist}\,(g, \mathbb{H}_{(n+1)/2,(m+1)/2}). \tag{2.6}$$

Our quasi-interpolatory operator on the sphere is obtained by modifying the operator $U_{n,m}$ to ensure that if f is a function on the sphere, then the resulting trigonometric polynomial is also a function on the sphere. Towards this goal, we review the connection between bivariate periodic functions and functions on the sphere.

The space of all continuous functions on $\mathbb{S}^2$, equipped with the uniform norm $\|\circ\|^S_\infty$, will be denoted by $C(\mathbb{S}^2)$. For $f \in C(\mathbb{S}^2)$, let $f^*(\theta,\phi) := f(\boldsymbol{p}(\theta,\phi))$. Since $\boldsymbol{p}(-\theta,\phi+\pi) = \boldsymbol{p}(\theta,\phi)$, $\theta,\phi \in \mathbb{R}$, and $\boldsymbol{p}(0,\phi)$, $\boldsymbol{p}(\pi,\phi)$ are independent of ϕ, it is clear that $f \in C(\mathbb{S}^2)$ if and only if $f^* \in C^*$, and satisfies the following symmetry conditions:

$$f^*(-\theta,\phi+\pi) = f^*(\theta,\phi), \qquad \theta,\phi \in \mathbb{R},$$

and

$$f^*(0,\phi),\ f^*(\pi,\phi) \text{ are independent of } \phi.$$

We will denote by C° the subspace of C^* comprising of functions satisfying the above two conditions. If $\tilde{f} \in C^\circ$, there exists a unique $f \in C(\mathbb{S}^2)$ such that $\tilde{f} = f^*$. We will write $f = (\tilde{f})^\circ$. It is clear that $\|f^*\|^*_\infty = \|f\|^S_\infty$. The class of all P for which $P^* \in \mathbb{H}_{n,m}$ will be denoted by $\mathcal{X}_{n,m}$. Following the proof of [6, Theorem 2.1], it is straightforward to obtain the following characterization for $\mathcal{X}^*_{n,m} := C^\circ \cap \mathbb{H}_{n,m}$:

PROPOSITION 2.1. *For integer $n, m \ge 0$, $T \in \mathcal{X}^*_{n,m}$ if and only if*

$$\begin{aligned} T(\theta,\phi) &= S_0(\cos\theta) + \sin^2\theta \sum_{\substack{|\ell|\le m,\ \ell\ne 0 \\ \ell \text{ even}}} Q_\ell(\cos\theta)\exp(i\ell\phi) \\ &\quad + \sin\theta \sum_{\substack{|\ell|\le m \\ \ell \text{ odd}}} R_\ell(\cos\theta)\exp(i\ell\phi) \\ &= L(\cos\theta) + \sin^2\theta \sum_{\substack{|\ell|\le m \\ \ell \text{ even}}} Q_\ell(\cos\theta)\exp(i\ell\phi) \\ &\quad + \sin\theta \sum_{\substack{|\ell|\le m \\ \ell \text{ odd}}} R_\ell(\cos\theta)\exp(i\ell\phi), \end{aligned}$$

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where $S_0 \in \Pi_n$, $L \in \Pi_1$, *and for* $|\ell| \le m$, $Q_\ell \in \Pi_{n-2}$, $R_\ell \in \Pi_{n-1}$.

Let $f \in C(\mathbb{S}^2)$. As in [6], we have

$$\begin{aligned}&(1/2)[U_{n,m}(f^*,\theta,\phi) + U_{n,m}(f^*,-\theta,\phi+\pi)]\\ &= \sum_{\substack{|\ell|\le m\\ \ell \text{ even}}} S_\ell(\cos\theta)e^{i\ell\phi} + \sin\theta \sum_{\substack{|\ell|\le m\\ \ell \text{ odd}}} R_\ell(\cos\theta)e^{i\ell\phi},\end{aligned}$$

where $S_\ell \in \Pi_n$, $R_\ell \in \Pi_{n-1}$. We set

$$\widetilde{S}_\ell(x) = S_\ell(x) - S_\ell(1)(1+x)/2 - S_\ell(-1)(1-x)/2, \qquad 1 \le |\ell| \le m,\ \ell \text{ even},$$

and

$$\begin{aligned}\widetilde{U_{n,m}}(f^*,\theta,\phi) = S_0(\cos\theta) &+ \sum_{\substack{1\le|\ell|\le m\\ \ell \text{ even}}} \widetilde{S}_\ell(\cos\theta)e^{i\ell\phi}\\ &+ \sin\theta \sum_{\substack{|\ell|\le m\\ \ell \text{ odd}}} R_\ell(\cos\theta)e^{i\ell\phi}.\end{aligned}$$

Then $\widetilde{U_{n,m}}(f^*)^\circ \in \mathcal{X}_{n,m}$, and it is easy to check (cf. [6]) that

$$\|f - \widetilde{U_{n,m}}(f^*)^\circ\|_\infty^S \le c\|f^* - U_{n,m}(f^*)\|_\infty^*. \tag{2.7}$$

Hence, (2.6) implies that

$$\|f - \widetilde{U_{n,m}}(f^*)^\circ\|_\infty^S \le c(1+\sqrt{n^2/N})(1+\sqrt{m^2/M}) \operatorname{dist}(f^*, \mathbb{H}_{(n+1)/2,(m+1)/2}).$$

We summarize the properties of the quasi-interpolatory operator

$$\mathcal{V}_{n,m}(f) := \mathcal{V}_{n,m,N,M}(f) := \widetilde{U_{n,m}}(f^*)^\circ$$

on the sphere in the following theorem.

THEOREM 2.2. *Let* $N, M \ge 4$, $0 \le n \le N-3$, $0 \le m \le M-3$ *be integers, Then* $\mathcal{V}_{n,m}(P) = P$ *for* $P \in \mathcal{X}_{(n+1)/2,(m+1)/2}$. *Let* $f \in C(\mathbb{S}^2)$. *We have* $\mathcal{V}_{n,m}(f) \in \mathcal{X}_{n,m}$,

$$\|\mathcal{V}_{n,m}(f)\|_\infty^S \le c(1+\sqrt{n^2/N})(1+\sqrt{m^2/M})\|f\|_\infty^S,$$

and hence,

$$\|f - \mathcal{V}_{n,m}(f)\|_\infty^S \le c(1+\sqrt{n^2/N})(1+\sqrt{m^2/M}) \operatorname{dist}(f, \mathcal{X}_{(n+1)/2,(m+1)/2}).$$

Finally, we remark that arguments similar to those in the proof of [6, Lemma 4.5] show that the degrees of approximation of f from $\mathcal{X}_{n,m}$ and that of f^* by bivariate trigonometric polynomials are related by

$$\operatorname{dist}(f^*, \mathbb{H}_{n,m}) \le \operatorname{dist}(f, \mathcal{X}_{n,m}) \le c \operatorname{dist}(f^*, \mathbb{H}_{n-1,m-1}).$$

3. Numerical aspects. In this section, we demonstrate the approximation properties of our quasi-interpolatory operator $\mathcal{V}_{n,m}$ numerically, using a few benchmark functions. Since a critical component of our constructions is the computation of orthogonal polynomials with respect to discrete inner products, we first review in Subsection 3.1 the algorithms we use for this purpose. The approximation of the benchmark functions is discussed in Subsection 3.2.

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3.1. Computation of orthogonal polynomials. The material in this section is based primarily on [7]. Given a discrete inner product with weights w_j and points $x_j, j = 1, \cdots, N$, the collection $\{p_k : k = 0, \cdots, N\}$ of monic polynomials that are orthogonal with respect to the inner product satisfies the three-term recurrence relation

$$p_{k+1}(x) = (x - \alpha_k)p_k(x) - \beta_k^2 p_{k-1}(x), \qquad k = 0, \cdots, N-1,$$

where $p_0(x) = 1$, $\alpha_0 = (\sum_{j=1}^N w_j x_j)/(\sum_{j=1}^N w_j)$, $\beta_0 = 0$. The $2N-1$ coefficients determine the $N \times N$ real symmetric tridiagonal Jacobi matrix

$$J = \begin{bmatrix} \alpha_0 & \beta_1 & & & & \\ \beta_1 & \alpha_1 & \beta_2 & & & \\ & \beta_2 & \alpha_2 & \beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta_{N-2} & \alpha_{N-2} & \beta_{N-1} \\ & & & & \beta_{N-1} & \alpha_{N-1} \end{bmatrix}.$$

A numerically stable construction of J is equivalent to that of the orthogonal polynomials.

Given a real symmetric $(N+1) \times (N+1)$ matrix A, the Lanczos algorithm can be used to compute the *unique* matrices $\mathcal{T}$ and Q, satisfying $Q^T A Q = \mathcal{T}$, where $\mathcal{T}$ is a tridiagonal matrix (with non–zero sub– and super–diagonal elements) and Q is an orthogonal matrix with first column $\mathbf{e}_1 = [1, 0, \cdots, 0]^T \in \mathbb{R}^{N+1}$.

Using the given weights and nodes of the discrete inner product, if we choose

$$A = \begin{bmatrix} 1 & \sqrt{w_1} & \sqrt{w_2} & \cdots & \sqrt{w_N} \\ \sqrt{w_1} & x_1 & 0 & \cdots & 0 \\ \sqrt{w_2} & 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \sqrt{w_N} & 0 & 0 & \cdots & x_N \end{bmatrix},$$

then for $i, j = 1, \cdots, N$, $\mathcal{T}(i+1, j+1) = J(i,j)$ [7, p. 154], and, theoretically, the unique matrices Q and $\mathcal{T}$ can be computed using the Lanczos algorithm.

However, the Lanczos algorithm is numerically unstable, even in the modified Gram-Schmidt form [8]. We use instead the Rutishauser-Kahan-Pal-Walker (RKPW) algorithm, described in [8, p. 328] using a sequence of Givens rotations. The RKPW algorithm requires at most $6N^2$ operations, and many numerical experiments in [8] demonstrate its stability. Our numerical experiments modeling some random data on the sphere further demonstrate the stability of RKPW algorithm.

3.2. Approximation of benchmark functions. We demonstrate the quality of our quasi-interpolatory operator using *random data locations* on the sphere. We first choose N random latitudinal angles $\theta_j \in ((j-1)\pi/N, j\pi/N)$, $j = 1, \cdots, N$.

Next, we fix $0 < n \leq N$, and for $j = 1, \cdots, N$, let $x_j^\theta = \cos(\theta_j)$, $w_{j,1}^\theta = \pi/N$, and $w_{j,2}^\theta = (\pi/N)\sin^2\theta_j$. Using the RKPW algorithm, we construct $n+1$ algebraic polynomials $t_k^\theta \in \Pi_k$ (resp. $u_k^\theta \in \Pi_k$), $k = 0, \cdots, n$, orthonormal with respect to the inner product defined by the weights $w_{j,1}^\theta$ (resp. $w_{j,2}^\theta$) with x_j^θ, $j = 1, \cdots, N$.

Following the procedure for the latitudinal case, we choose M random partial longitudinal data locations $\phi_j \in [0, \pi]$, $j = 1, \cdots, M$, and for $0 < m \leq M$, we use the RKPW algorithm to construct $\{t_k^\phi : k = 0, \cdots, m\}$ and $\{u_k^\phi : k = 0, \cdots, m\}$ that are orthonormal

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with respect to its associated inner product with points $x_j^\phi = \cos(\phi_j)$, and respective weights $w_{j,1}^\phi = \pi/M$ and $w_{j,2}^\phi = (\pi/M)\sin^2\phi_j$, for $j = 1, \cdots, M$.

We augment the $N \times M$ random data locations, by defining $\theta_{N+j} = 2\pi - \theta_{N-j+1}$, for $j = 1, \cdots, N$, and $\phi_{M+k} = 2\pi - \phi_{M-k+1}$ and $k = 1, \cdots, M$. We note that the random generation of the data implies that the entire procedure to compute the quasi-interpolant, including the computation of the orthogonal polynomials, has to be repeated afresh each time the procedure is called.

In this work, we demonstrate properties of our quadrature-free quasi-interpolatory operator on the sphere using data values arising from (i) spherical polynomials of degree $n/2$; (ii) a smooth function with steep boundary layer near the north pole; (iii) a continuous function that is not differentiable; (iv) a locally supported continuous function with Hölder exponent $3/4$; and (v) a weather model cosine cap test function [22] that is once (but not twice) continuously differentiable on the unit sphere. More precisely, using the random data locations $\hat{\mathbf{x}}_{j,k} = \boldsymbol{p}(\theta_j, \phi_k)$, $j = 1, \cdots, N$, $k = 1, \cdots, M$ on the sphere, we obtain five distinct sets of data values $f(\boldsymbol{p}(\theta_j, \phi_k))$, where for $\hat{\mathbf{x}} = (x_1, x_2, x_3) \in \mathbb{S}^2$, the functions are defined by

$$f_1^n(\hat{\mathbf{x}}) = x_1^{n/4-1} x_2^{n/4} x_3, \qquad f_2(\hat{\mathbf{x}}) = \frac{1}{101 - 100x_3}, \qquad f_3(\hat{\mathbf{x}}) = \frac{1}{|x_1| + |x_2| + |x_3|},$$

$$f_4(\hat{\mathbf{x}}) = \{\max(x_1 - 0.9, 0)\}^{3/4} + \{\max(x_3 - 0.9, 0)\}^{3/4},$$

and

$$f_5(\hat{\mathbf{x}}) = \begin{cases} \cos^2\left(\frac{3\pi}{2}\operatorname{dist}(\hat{\mathbf{x}}, \boldsymbol{p}(\pi/4, 5\pi/4)\right), & \text{if } \operatorname{dist}(\hat{\mathbf{x}}, \boldsymbol{p}(\pi/4, 5\pi/4) < 1/3, \\ 0, & \text{if } \operatorname{dist}(\hat{\mathbf{x}}, \boldsymbol{p}(\pi/4, 5\pi/4) \geq 1/3, \end{cases}$$

where the $\operatorname{dist}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \cos^{-1}(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}})$ is the geodesic distance between two points $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{S}^2$. We note that in the case when n is an integer divisible by 4, $f_1^n \in \mathcal{X}_{n/2,n/2-1}$.

For our numerical experiments, we chose $M = N$, $m = n$, with $n = N - 10$. We computed approximations of the corresponding periodic functions in $\mathbb{H}_{n,n}$, using the data points and the quasi-interpolation operators $U_{n,m,N,M} = U_n$, with h given by (2.3). For each $i = 1, \cdots, 5$, the global uniform norm errors $\operatorname{Err}(f_i) := \|f_i^* - U_n(f_i^*)\|_\infty^*$ were estimated by taking the maximum of errors over $19,000$ points on the sphere. We recall from (2.7) that $\|f_i - \mathcal{V}_{n,n}(f_i)\|_\infty^S \leq c\|f_i^* - U_n(f_i^*)\|_\infty^*$, where $f_i^*(\theta, \phi) := f_i(\boldsymbol{p}(\theta, \phi))$.

The results in Table 3.1 clearly demonstrate that our operators yield a good reconstruction of these functions with various smoothness properties from their semi–random data information. In the case of approximation of the functions f_1^n, the second column of Table 3.1 demonstrates the reproduction of spherical trigonometric polynomials of degree at most $n/2$ in each of its variables. For the non–smooth functions, we also estimated local uniform norm errors $\operatorname{Err}(f_i)^{loc} := \|f_i^* - U_n(f_i^*)\|_\infty^{loc}$ by taking the maximum of errors over $19,000$ points on the sphere inside the local support (or smooth) part of the function. The results in Table 3.2 demonstrate that our operators yield better reconstructions in smoother parts of the functions.

4. Proof of Theorem 2.1. A major part of the proof of Theorem 2.1 can be carried out in a very abstract setting. The following Theorem 4.1 estimates the difference between the norms of the operators based on certain kernels constructed from orthogonal systems with respect to different measures. Let (Ω, μ) be a finite measure space, $\{f_k\}_{k=0}^\infty \subseteq L^1(\mu; \Omega) \cap L^\infty(\mu; \Omega)$ be an orthonormal set of functions:

$$\int_\Omega f_k f_j d\mu = \delta_{k,j}, \qquad k, j = 0, 1, \cdots.$$

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TABLE 3.1
Global error in approximation of f_i^* by $U_N(f_i^*)$, $i = 1, \cdots, 5$.

n	Err(f_1^n)	Err(f_2)	Err(f_3)	Err(f_4)	Err(f_5)
16	6.9389e-17	1.7390e-01	9.7269e-02	2.9174e-02	1.0066e-01
32	4.6621e-18	3.0945e-02	3.9318e-02	2.5249e-02	1.2674e-02
64	6.2968e-14	2.3181e-03	2.5230e-02	1.0341e-02	3.2474e-03
128	2.7311e-13	9.3681e-06	1.4340e-02	6.9765e-03	1.0170e-03
256	4.0804e-21	1.1054e-09	8.0303e-03	3.8792e-03	2.5965e-04
512	1.4811e-40	7.1924e-12	2.7159e-03	2.4981e-03	3.7007e-05

TABLE 3.2
Local error in approximation of non–smooth f_i^* by $U_N(f_i^*)$, $i = 3, 4, 5$.

n	Err(f_3)^{loc}	Err(f_4)^{loc}	Err(f_5)^{loc}
16	5.0579e-03	4.5752e-03	6.6953e-02
32	1.1327e-03	4.4026e-04	4.0600e-03
64	2.4900e-04	1.5809e-04	5.3541e-04
128	1.7856e-05	6.8432e-05	3.0011e-05
256	4.4250e-06	7.6379e-06	2.4601e-06
512	4.2287e-07	2.0442e-06	3.6010e-07

For integer $n \ge 0$, let $V_n = \text{span}\{f_0, \cdots, f_n\}$. In this section, let $n \ge 1$ be a fixed integer. Let $0 < \epsilon \le 1/2$ and ν be a positive measure on Ω such that

$$\left| \|P\|_{\mu;2}^2 - \|P\|_{\nu;2}^2 \right| \le \epsilon \|P\|_{\mu;2}^2, \qquad P \in V_n. \tag{4.1}$$

Let $\{\tilde{f}_k\}_{k=0}^n$ be an orthonormal set with respect to ν, such that for each integer $m = 0, \cdots, n$, $V_m = \text{span}\{f_0, \cdots, f_m\} = \text{span}\{\tilde{f}_0, \cdots, \tilde{f}_m\}$.

THEOREM 4.1. *Let $\{h_k\}$ be a nonincreasing sequence of nonnegative numbers with $h_k = 0$ if $k \ge n+1$. For $x \in \Omega$,*

$$\left| \int_\Omega \left| \sum_{k=0}^n h_k f_k(x) f_k(t) \right| d\mu(t) - \int_\Omega \left| \sum_{k=0}^n h_k \tilde{f}_k(x) \tilde{f}_k(t) \right| d\mu(t) \right|$$
$$\le (6\mu(\Omega) h_0 \epsilon)^{1/2} \left\{ \sum_{m=0}^n h_m f_m(x)^2 \right\}^{1/2}.$$

REMARK 1. *In the case when Ω is a real interval, $\int_\Omega |t|^n d\mu(t) < \infty$ for all integer $n \ge 0$, and μ has infinitely many points of increase, we may choose f_k to be the orthonormalized polynomial of degree k on Ω with respect to μ. During the 1970's G. Freud initiated a theory of strong $(C, 1)$ summability of orthogonal polynomial expansions, based only the estimates on $\sum_{m=0}^n f_m(x)^2$ (cf. [12, Chapter 3]). In light of Lemma 4.2 below, this approach has an advantage that if one knows the strong $(C, 1)$ summability of orthogonal polynomial expansions with respect to one measure μ, one can immediately conclude the same about orthogonal polynomial expansions with respect to another measure ν that is equivalent to μ in the sense that $\|P\|_{\mu;2} \sim \|P\|_{\nu;2}$ for all polynomials. The same ideas were used in [18] to construct quasi-interpolatory operators using generalized Jacobi polynomials. To the best of our knowledge, there is no such general theory governing the higher order Cesàro means for orthogonal polynomial expansions. Nevertheless, Theorem 4.1 provides a way to obtain the*

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summability of orthogonal expansions with respect to one measure from that with respect to another measure.

The proof of this theorem relies upon the following extremal property, analogous to that for orthogonal polynomials [12, Theorem 1.3.2].

LEMMA 4.2. *Let* $0 \le m \le n$ *be an integer. For* $y \in \Omega$,

$$\max_{P\in V_m} \frac{|P(y)|^2}{\|P\|^2_{\mu;2}} = \sum_{k=0}^{m} f_k^2(y), \quad \max_{P\in V_m} \frac{|P(y)|^2}{\|P\|^2_{\nu;2}} = \sum_{k=0}^{m} \tilde{f}_k^2(y). \tag{4.2}$$

Moreover, if (4.1) *holds for some* ϵ, $0 < \epsilon < 1$, *then*

$$\left|\sum_{k=0}^{m} f_k^2(y) - \sum_{k=0}^{m} \tilde{f}_k^2(y)\right| \le \frac{\epsilon}{1-\epsilon}\sum_{k=0}^{m} f_k^2(y). \tag{4.3}$$

Proof. Let $P(y) = \sum_{k=0}^m c_k f_k(y)$. Using the Schwarz inequality followed by the Parseval identity, we deduce that

$$|P(y)|^2 \le \left\{\sum_{k=0}^{m} c_k^2\right\}\left\{\sum_{k=0}^{m} f_k^2(y)\right\} = \|P\|^2_{\mu;2}\sum_{k=0}^{m} f_k^2(y).$$

The equality is attained for $P = \sum_{k=0}^m f_k(y) f_k$. This proves the first equation in (4.2). The second equation is proved in the same way. In view of (4.1),

$$\frac{1}{\|P\|^2_{\mu;2}} \le \frac{1+\epsilon}{\|P\|^2_{\nu;2}} \le \frac{1+\epsilon}{(1-\epsilon)\|P\|^2_{\mu;2}}, \qquad P \in V_m \setminus \{0\}.$$

Therefore, (4.2) implies that

$$\sum_{k=0}^{m} f_k^2(y) \le (1+\epsilon)\sum_{k=0}^{m} \tilde{f}_k^2(y) \le \frac{1+\epsilon}{1-\epsilon}\sum_{k=0}^{m} f_k^2(y). \tag{4.4}$$

Therefore,

$$\sum_{k=0}^{m} f_k^2(y) - \sum_{k=0}^{m} \tilde{f}_k^2(y) \le \epsilon\sum_{k=0}^{m} \tilde{f}_k^2(y) \le \frac{\epsilon}{1-\epsilon}\sum_{k=0}^{m} f_k^2(y),$$

and

$$\sum_{k=0}^{m} \tilde{f}_k^2(y) - \sum_{k=0}^{m} f_k^2(y) \le \frac{\epsilon}{1-\epsilon}\sum_{k=0}^{m} f_k^2(y).$$

This proves (4.3). ☐

Proof of Theorem 4.1. In this proof only, let, for $m = 0, \cdots, n$, $x, t \in \Omega$,

$$K_m(x,t) = \sum_{k=0}^{m} f_k(x) f_k(t), \quad \tilde{K}_m(x,t) = \sum_{k=0}^{m} \tilde{f}_k(x)\tilde{f}_k(t).$$

We observe that

$$\int_\Omega K_m(x,t)P(t)d\mu(t) = \int_\Omega \tilde{K}_m(x,t)P(t)d\nu(t) = P(x), \qquad P \in V_m.$$

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Since for each $x \in \Omega$, $K_m(x,\circ),\ \tilde{K}_m(x,\circ) \in V_m$, it follows that

$$\int_\Omega K_m(x,t)^2 d\mu(t) = K_m(x,x),\ \int_\Omega \tilde{K}_m(x,t)^2 d\nu(t) = \tilde{K}_m(x,x),$$

and

$$\int_\Omega K_m(x,t)\tilde{K}_m(x,t) d\mu(t) = \tilde{K}_m(x,x).$$

Therefore, using (4.1), we obtain

$$\begin{aligned}
&\int_\Omega \left(K_m(x,t) - \tilde{K}_m(x,t)\right)^2 d\mu(t) \\
&= \int_\Omega K_m(x,t)^2 d\mu(t) - 2\int_\Omega K_m(x,t)\tilde{K}_m(x,t)d\mu(t) + \int_\Omega \tilde{K}_m(x,t)^2 d\mu(t) \\
&\le K_m(x,x) - 2\tilde{K}_m(x,x) + \frac{1}{1-\epsilon}\int_\Omega \tilde{K}_m(x,t)^2 d\nu(t) \\
&= K_m(x,x) - \tilde{K}_m(x,x) + \frac{\epsilon}{1-\epsilon}\tilde{K}_m(x,x). \qquad (4.5)
\end{aligned}$$

In view of (4.3), $K_m(x,x) - \tilde{K}_m(x,x) \le \epsilon K_m(x,x)/(1-\epsilon)$, and in view of (4.4), $\tilde{K}_m(x,x) \le K_m(x,x)/(1-\epsilon)$. Thus, (4.5) implies that

$$\int_\Omega \left(K_m(x,t) - \tilde{K}_m(x,t)\right)^2 d\mu(t) \le \frac{\epsilon}{1-\epsilon}\left[1 + \frac{1}{1-\epsilon}\right] K_m(x,x).$$

Since $0 < \epsilon \le 1/2$, $1/(1-\epsilon) \le 2$, and we obtain

$$\int_\Omega \left(K_m(x,t) - \tilde{K}_m(x,t)\right)^2 d\mu(t) \le 6\epsilon K_m(x,x).$$

Therefore, Hölder's inequality implies that

$$\begin{aligned}
&\int_\Omega |K_m(x,t) - \tilde{K}_m(x,t)| d\mu(t) \\
&\qquad \le \{6\mu(\Omega)\epsilon K_m(x,x)\}^{1/2}, \qquad x \in \Omega,\ m = 0, \cdots, n. \qquad (4.6)
\end{aligned}$$

Now, we write $K_{-1}(x,t) := \tilde{K}_{-1}(x,t) := 0$ and $g_m := h_m - h_{m+1}$, and recall that $h_{n+1} = 0$. Then

$$\begin{aligned}
\sum_{m=0}^n h_m f_m(x) f_m(t) &= \sum_{m=0}^n h_m K_m(x,t) - \sum_{m=0}^n h_m K_{m-1}(x,t) \\
&= \sum_{m=0}^n h_m K_m(x,t) - \sum_{m=0}^n h_{m+1} K_m(x,t) = \sum_{m=0}^n g_m K_m(x,t),
\end{aligned}$$

and similarly, $\sum_{m=0}^n h_m \tilde{f}_k(x)\tilde{f}_m(t) = \sum_{m=0}^n g_m \tilde{K}_m(x,t)$. Since h_k is a nonincreasing sequence, each $g_m \ge 0$. Therefore, (4.6) implies that

$$\left| \int_\Omega \left| \sum_{k=0}^n h_k f_k(x) f_k(t) \right| d\mu(t) - \int_\Omega \left| \sum_{k=0}^n h_k \tilde{f}_k(x) \tilde{f}_k(t) \right| d\mu(t) \right|$$

$$
\begin{aligned}
&\le \int_\Omega \left| \sum_{m=0}^{n} h_m f_m(x) f_m(t) - \sum_{m=0}^{n} h_m \tilde{f}_k(x) \tilde{f}_m(t) \right| d\mu(t) \\
&= \int_\Omega \left| \sum_{m=0}^{n} g_m K_m(x,t) - \sum_{m=0}^{n} g_m \tilde{K}_m(x,t) \right| d\mu(t) \\
&\le \sum_{m=0}^{n} g_m \int_\Omega |K_m(x,t) - \tilde{K}_m(x,t)| d\mu(t) \\
&\le \sum_{m=0}^{n} g_m \left\{ 6\mu(\Omega)\epsilon K_m(x,x) \right\}^{1/2} \\
&\le (6\mu(\Omega)\epsilon)^{1/2} \left(\sum_{m=0}^{n} g_m \right)^{1/2} \left\{ \sum_{m=0}^{n} g_m K_m(x,x) \right\}^{1/2} \\
&= (6\mu(\Omega) h_0 \epsilon)^{1/2} \left\{ \sum_{m=0}^{n} h_m f_m(x)^2 \right\}^{1/2}.
\end{aligned}
$$

This completes the proof. □

In light of Theorem 4.1, we may prove Theorem 2.1 by first showing an estimate of the form (4.1), where μ is the Lebesgue measure on $[0, 2\pi)$, and ν is a discrete measure. Towards this end, we first prove the following lemma (cf. [19, Lemma 3.1]).

LEMMA 4.3. *Let $M \ge 1$ be a positive integer, and for $j = 1, \cdots, M$, $\psi_j \in [2(j-1)\pi/M, 2j\pi/M]$. If F is any absolutely continuous, 2π–periodic function, then*

$$
\left| \int_0^{2\pi} |F(t)| dt - \frac{2\pi}{M} \sum_{j=1}^{M} |F(\psi_j)| \right| \le \frac{2\pi}{M} \int_0^{2\pi} |F'(u)| du. \tag{4.7}
$$

In particular, if $T \in \mathbb{H}_n$, then

$$
\left| \int_0^{2\pi} |T(t)| dt - \frac{2\pi}{M} \sum_{j=1}^{M} |T(\psi_j)| \right| \le \frac{2\pi n}{M} \int_0^{2\pi} |T(t)| dt. \tag{4.8}
$$

and

$$
\left| \int_0^{2\pi} |T(t)|^2 dt - \frac{2\pi}{M} \sum_{j=1}^{M} |T(\psi_j)|^2 \right| \le \frac{4\pi n}{M} \int_0^{2\pi} |T(t)|^2 dt. \tag{4.9}
$$

Proof. Let $I_j = [2(j-1)\pi/M, 2j\pi/M]$. For $j = 1, \cdots, M$,

$$
\begin{aligned}
\left| \int_{I_j} |F(t)| dt - \frac{2\pi}{M} |F(\psi_j)| \right| &\le \int_{I_j} |F(t) - F(\psi_j)| dt \\
&\le \int_{I_j} \int_{I_j} |F'(u)| du\, dt = \frac{2\pi}{M} \int_{I_j} |F'(u)| du.
\end{aligned}
$$

Therefore,

$$
\left| \int_0^{2\pi} |F(t)| dt - \frac{2\pi}{M} \sum_{j=1}^{M} |F(\psi_j)| \right| \le \sum_{j=1}^{M} \left| \int_{I_j} |F(t)| dt - \frac{2\pi}{M} |F(\psi_j)| \right|
$$

$$\leq \frac{2\pi}{M}\int_0^{2\pi}|F'(u)|du.$$

This proves (4.7). If $T \in \mathbb{H}_n$, the Bernstein inequality may be used to arrive at the estimates (4.8). The estimates (4.9) are obtained by using (4.8) with $|T(\circ)|^2 \in \mathbb{H}_{2n}$ in place of T. ∎

Our next lemma follows from [19, Proposition 2.1(b)].

LEMMA 4.4. *Let $h : \mathbb{R} \to [0,\infty)$ be an even function, nonincreasing on $[0,\infty)$ that can be expressed as an indefinite integral of a function with bounded variation on $\mathbb{R}$. We assume further that $h(t) = 1$ if $0 \le t \le 1/2$ and $h(t) = 0$ if $t \ge 1$. For integer $n \ge 1$,*

$$\int_0^{2\pi}\left|\frac{1}{2} + \sum_{k=1}^{n} h(k/(n+1))\cos k\theta\right| d\theta \le c.$$

Proof of Theorem 2.1. Let ν be the measure that associates the mass π/N with each of the points θ_j, $j = 1,\cdots,2N$. Let $n \le N-3$ be an integer, $\mathbb{H}_n^e$ (respectively $\mathbb{H}_n^o$) be the class of all even (respectively, odd) trigonometric polynomials of order at most n. Then $\{t_k(\cos\theta)\}_{k=0}^n$ (respectively, $\{\sin\theta u_k(\cos\theta)\}_{k=0}^{n-1}$) is an orthonormal basis for $\mathbb{H}_n^e$ (respectively, $\mathbb{H}_n^o$) with respect to $d\nu$. Taking μ to be the Lebesgue measure on $[0,2\pi]$, (4.9) with $M = 2N$ shows that (4.1) holds for both $\mathbb{H}_n^e$ and $\mathbb{H}_n^o$ with $\epsilon = 2\pi n/N$. Next, we take

$$f_0^e = \sqrt{1/(2\pi)}, \qquad f_k^e(\theta) = \pi^{-1/2}\cos k\theta,$$
$$f_k^o(\theta) = \pi^{-1/2}\sin(k+1)\theta, \qquad k = 0,\cdots,n. \tag{4.10}$$

Since $h_{k,n} \le 1$ for all k, it is clear that

$$\sum_{k=0}^{n} h_{k,n} f_k^e(\psi)^2 \le cn, \qquad \sum_{k=0}^{n} h_{k+1,n} f_k^o(\psi)^2 \le cn.$$

Hence (4.10), (2.1). and Theorem 4.1 imply that

$$\begin{aligned}
&\int_0^{2\pi}\left|\sum_{k=0}^{n} h_{k,n} t_k(\cos\psi)t_k(\cos\theta)\right| d\mu(\theta)\\
&\le \int_0^{2\pi}\left|\sum_{k=0}^{n} h_{k,n} f_k^e(\psi) f_k^e(\theta)\right| d\mu(\theta) + c\sqrt{n^2/N},
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&\int_0^{2\pi}\left|\sum_{k=0}^{n-1} h_{k+1,n}\sin\theta u_k(\cos\theta)\sin\psi u_k(\cos\psi)\right| d\mu(\theta)\\
&\le \int_0^{2\pi}\left|\sum_{k=0}^{n-1} h_{k+1,n} f_k^o(\psi) f_k^o(\theta)\right| d\mu(\theta) + c\sqrt{n^2/N}.
\end{aligned} \tag{4.12}$$

Since

$$\sum_{k=0}^{n} h_{k,n} f_k^e(\psi) f_k^e(\theta) = (1/2\pi)\sum_{k=0}^{n} h_{k,n}\left[\cos k(\psi-\theta) + \cos k(\psi+\theta)\right],$$
$$\sum_{k=0}^{n-1} h_{k+1,n} f_k^o(\psi) f_k^o(\theta) = (1/2\pi)\sum_{k=0}^{n} h_{k,n}\left[\cos k(\psi-\theta) - \cos k(\psi+\theta)\right],$$

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we deduce from Lemma 4.4, (4.11), and (4.12) that

$$\int_0^{2\pi} \left| \sum_{k=0}^{n} h_{k,n} t_k(\cos\psi) t_k(\cos\theta) \right| d\mu(\theta) \le c(1+\sqrt{n^2/N}),$$

$$\int_0^{2\pi} \left| \sum_{k=0}^{n-1} h_{k+1,n} \sin\theta u_k(\cos\theta) \sin\psi u_k(\cos\psi) \right| d\mu(\theta) \le c(1+\sqrt{n^2/N}).$$

Further, in view of (4.8) and the fact that $n \le N$, we have

$$\int_0^{2\pi} \left| \sum_{k=0}^{n} h_{k,n} t_k(\cos\psi) t_k(\cos\theta) \right| d\nu(\theta) \le c(1+\sqrt{n^2/N}),$$

$$\int_0^{2\pi} \left| \sum_{k=0}^{n-1} h_{k+1,n} \sin\theta u_k(\cos\theta) \sin\psi u_k(\cos\psi) \right| d\nu(\theta) \le c(1+\sqrt{n^2/N}). \tag{4.13}$$

The estimates (4.13) imply (2.4). It is clear from (2.2) and the fact that $h(k/(n+1)) = 1$ if $k \le (n+1)/2$ that $T_{n,N}(P) = P$ if $P \in \mathbb{H}_{(n+1)/2}$. Hence, (2.4) leads to (2.5). ☐

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