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Contents

1 Bivariate interpolation at Xu points: results, extensions and applications. *Len Bos, Marco Caliari, Stefano De Marchi, and Marco Vianello*.

Abstract.
In a recent paper, Y. Xu proposed a set of Chebyshev-like points for polynomial interpolation on the square $[-1,1]^2$. We have recently proved that the Lebesgue constant of these points grows like $\log^2$ of the degree (as with the best known points for the square), and we have implemented an accurate version of their Lagrange interpolation formula at linear cost. Here we construct non-polynomial Xu-like interpolation formulas on bivariate compact domains with various geometries, by means of composition with suitable smooth transformations. Moreover, we show applications of Xu-like interpolation to the compression of surfaces given as large scattered data sets.

Key Words.
bivariate polynomial interpolation, Xu points, Lebesgue constant, domains transformations, generalized rectangles, generalized sectors, large scattered data sets, surface compression

AMS(MOS) Subject Classifications.
65D05

17 Alternative orthogonal polynomials and quadratures. *Vladimir S. Chelyshkov*.

Abstract.
A bidirectional orthogonalization algorithm is applied to construct sequences of polynomials, which are orthogonal over the interval [0, 1] with the weighting function 1. Functional and recurrent relations are derived for the sequences that are the result of inverse orthogonalization procedure. Quadratures, generating by the sequences, are described. An example on approximation of the Cauchy problem is given.

Key Words.
orthogonal polynomial, recurrence relation, quadrature, initial value problem

AMS(MOS) Subject Classifications.
33C45

27 On the support of the equilibrium measure for arcs of the unit circle and for real intervals. *D. Benko, S. B. Damelin, and P. D. Dragnev*.

Abstract.
We study the support of the equilibrium measure for weights defined on arcs of the unit circle and on intervals of the compactified real line. We provide several conditions to ensure that the support of the equilibrium measure is one interval or one arc.

Key Words.
logarthmic potential theory, external fields, equilibrium measure, equilibrium support

AMS(MOS) Subject Classifications.
31A15, 30C15, 78A30

41 Inverse source problem in a 3D ball from best meromorphic approximation on 2D slices. *L. Baratchart, J. Leblond, and J.-P. Marmorat*.

Abstract.
We show that the inverse monopolar or dipolar source problem in a 3D ball from overdetermined Dirichlet-Neumann data on the boundary sphere reduces to a family of 2D inverse branchpoint problems in cross sections of the sphere, at least when there are finitely many sources. We adapt from [L. Baratchart et al., *Recovery of pointwise sources or small inclusions in 2D domains and rational approximation*, Inverse Problems, 21 (2005), pp. 51–74] an approach to these 2D inverse problem which is based on meromorphic approximation, and we present numerical results.

Key Words.
inverse source problems, potential theory, meromorphic approximation

AMS(MOS) Subject Classifications.
31A25, 30E10, 30E25, 35J05

54 A remark on uniqueness of best rational approximants of degree 1 in L^2 of the circle. *L. Baratchart*.

Abstract.
We derive a criterion for uniqueness of a critical point in H^2 rational approximation of degree 1. Although narrowly restricted in scope because it deals with degree 1 only, this criterion is interesting because it addresses a large class of functions. The method elaborates on the topological approach in [L. Baratchart and F. Wielonsky, *Rational approximation in the real Hardy space H^2 and Stieltjes integrals: a uniqueness theorem*, Constr. Approx., 9 (1993), pp. 1–21] and [L. Baratchart et al., *A criterion for uniqueness of a critical points in H^2 rational approximation,* Canad. J. Math., 47 (1995), pp. 1121–1147].

Key Words.
rational approximation, uniqueness, Hardy spaces, critical points

AMS(MOS) Subject Classifications.
31A25, 30E10, 30E25, 35J05

67 Error controlled regularization by projection. *Wolfgang Dahmen and Markus Jürgens*.

Abstract.
The paper is concerned with regularization concepts for the inversion of diffusion processes. The application of the involved evolution operators is based on Dunford integral representations combined with the adaptive application of resolvents using recent wavelet method s. In particular, this allows us to develop and realize numerically, to our knowledge for the first time in this context, an SVD projection method which is compared to several versions of Tikhonov-type schemes. The theoretical

findings are complemented by numerical tests shedding some light on the quantitative performance of the schemes.

Key Words.
inverse problems, Dunford integrals, quadrature, Tikhonov method, projection methods, truncated SVD expansion, adaptive wavelet methods

AMS(MOS) Subject Classifications.
47A52, 65J20, 65J22

101 Quadrature-free quasi-interpolation on the sphere. *M. Ganesh and H. N. Mhaskar.*

Abstract.
We construct certain quasi-interpolatory operators for approximation of functions on the sphere, using tensor product scattered data satisfying certain symmetry conditions. Our operators are constructed without using any quadrature formulas. We use instead a special class of orthonormal bivariate trigonometric polynomials. These polynomials are functions on the sphere, and are constructed in a numerically stable way, based on the data locations. The quasi-interpolatory operators give near best approximation to every continuous function.
We demonstrate our constructions numerically with several benchmark functions using randomly generated data locations.

Key Words.
function approximation on the sphere, scattered data, quasi-interpolation, Jacobi matrices

AMS(MOS) Subject Classifications.
42A15, 65D32, 33C55

115 An integral representation of some hypergeometric functions. *K. A. Driver and S. J. Johnston.*

Abstract.
The Euler integral representation of the ${}_2F_1$ Gauss hypergeometric function is well known and plays a prominent role in the derivation of transformation identities and in the evaluation of ${}_2F_1(a, b; c; 1)$, among other applications. The general ${}_{p+k}F_{q+k}$ hypergeometric function has an integral representation where the integrand involves ${}_pF_q$. We give a simple and direct proof of an Euler integral representation for a special class of ${}_{q+1}F_q$ functions for $q \geq 2$. The values of certain ${}_3F_2$ and ${}_4F_3$ functions at $x = 1$, some of which can be derived using other methods, are deduced from our integral formula.

Key Words.
3F2 hypergeometric functions, general hypergeometric functions, integral representation

AMS(MOS) Subject Classifications.
15A15

121 Weighted approximation of derivatives on the half-line. *Katherine Balázs and Theodore Kilgore.*

Abstract.
Weighted polynomial approximation of derivatives on the half line $[0, \infty)$ is considered. The weight function will be of the form $e^{-R(t)}$, a “folded” Freud weight.

That is, that $R(x^2) = Q(x)$, where $e^{-Q(x)}$ is a Freud weight on $(-\infty, \infty)$. Linear processes which can be used for approximation of derivatives include interpolation, in particular using node-sets recently developed by J. Szabados.

Key Words.
Freud weights, derivatives, weighted approximation

AMS(MOS) Subject Classifications.
41A10, 41A05, 65D05

129 The circle theorem and related theorems for Gauss-type quadrature rules. *Walter Gautschi*.

Abstract.
In 1961, P.J. Davis and P. Rabinowitz established a beautiful "circle theorem" for Gauss and Gauss–Lobatto quadrature rules. They showed that, in the case of Jacobi weight functions, the Gaussian weights, suitably normalized and plotted against the Gaussian nodes, lie asymptotically for large orders on the upper half of the unit circle centered at the origin. Here analogous results are proved for rather more general weight functions—essentially those in the Szegö class—, not only for Gauss and Gauss–Lobatto, but also for Gauss–Radau formulae. For much more restricted classes of weight functions, the circle theorem even holds for Gauss–Kronrod rules. In terms of potential theory, the semicircle of the circle theorem can be interpreted as the reciprocal density of the equilibrium measure of the interval $[-1, 1]$. Analogous theorems hold for weight functions supported on any compact subset Δ of $(-1, 1)$, in which case the (normalized) Gauss points approach the reciprocal density of the equilibrium measure of Δ. Many of the results are illustrated graphically.

Key Words.
Gauss quadrature formulae, circle theorem, Gauss–Radau, Gauss–Lobatto and Gauss–Kronrod formulae, Christoffel function, potential theory, equilibrium measure

AMS(MOS) Subject Classifications.
65D32, 42C05

138 New constructions of piecewise-constant wavelets. *Youngmi Hur and Amos Ron*.

Abstract.
The classical Haar wavelet system of $L_2(\mathbb{R}^n)$ is commonly considered to be very local in space. We introduce and study in this paper piecewise-constant framelets (PCF) that include the Haar system as a special case. We show that any bi-framelet pair consisting of PCFs provides the same Besov space characterizations as the Haar system. In particular, it has *Jackson-type performance* $s_J = 1$ and *Bernstein-type performance* $s_B = 0.5$. We then construct two PCF systems that are either, in high spatial dimensions, far more local than Haar, or are as local as Haar while delivering better performance: $s_J = s_B = 1$. Both representations are computed and inverted by fast algorithms.

Key Words.
frames, framelets, wavelets, Haar wavelets, piecewise-constant wavelets, PCF, Besov spaces, Unitary Extension Principle

AMS(MOS) Subject Classifications.
42C15, 42C40

158 Orthogonal polynomials and Ramanujan's q-continued fractions. *Mourad E. H. Ismail and Xin Li*.

Abstract.
We give new and simple proofs to some famous q-continued fraction identities of Ramanujan by using the theory of orthogonal polynomials.

Key Words.
orthogonal polynomials, continued fraction

AMS(MOS) Subject Classifications.
33C47, 11A55

166 Bernstein's weighted approximation on $\mathbb{R}$ still has problems. *D. S. Lubinsky*.

Abstract.
Let $W : \mathbb{R} \to (0, 1]$ be continuous. Bernstein's approximation problem, posed in 1924, dealt with approximation by polynomials in the norm

$$\|f\|_W := \|fW\|_{L_\infty(\mathbb{R})}.$$

The qualitative form of this problem was solved by Achieser, Mergelyan, and Pollard, in the 1950's. Quantitative forms of the problem were actively investigated starting from the 1960's. We survey old and recent aspects of this topic. One recent finding is that there are weights for which the polynomials are dense, but which do not admit a Jackson-Favard inequality. In fact the weight $W(x) = \exp(-|x|)$ exhibits this peculiarity. Moreover, not all L_p spaces are the same when degree of approximation is considered. We also pose some open problems.

Key Words.
weighted approximation, polynomial approximation, Jackson-Bernstein theorems

AMS(MOS) Subject Classifications.
41A17

178 On Euler's differential methods for continued fractions. *Sergey Khrushchev*.

Abstract.
A differential method discovered by Euler is justified and applied to give simple proofs to formulas relating important continued fractions with Laplace transforms. They include Stieltjes formulas and some Ramanujan formulas. A representation for the remainder of Leibniz's series as a continued fraction is given. We also recover the original Euler's proof for the continued fraction of hyperbolic cotangent.

Key Words.
continued fractions, Ramanujan formulas, Laplace transform

AMS(MOS) Subject Classifications.
30B70

201 On norms of factors of multivariate polynomials on convex bodies. *András Kroó*.

Abstract.
Estimation of norms of factors of polynomials is a widely investigated extremal problem with numerous applications in functional analysis, number theory, approximation theory. In this note we study the following problem: let K be a convex body

in $\mathbb{R}^d$ and consider a product of polynomials qr, where q is arbitrary and r is a monic multivariate polynomial. The goal is to find an upper bound for the uniform norm of q on K provided that such bound for qr is known.

Key Words.
multivariate polynomials, norms of factors, convex bodies

AMS(MOS) Subject Classifications.
41A17, 41A63

206 On weighted $(0, 2)$-type interpolation. *Margit Lénárd.*

Abstract.
The weighted (0,2)-interpolation with additional Hermite-type conditions is studied in a unified way with respect to the existence, uniqueness and representation (explicit formulae). Sufficient conditions are given on the nodes and the weight function, for the problem to be regular. Examples are presented on the zeros of the classical orthogonal polynomials.

Key Words.
Birkhoff interpolation, Pál-type interpolation, Hermite interpolation, weighted (0,2)-interpolation, regularity, explicit formulae

AMS(MOS) Subject Classifications.
41A05

224 Stable multiresolution analysis on triangles for surface compression. *Jan Maes and Adhemar Bultheel.*

Abstract.
Recently we developed multiscale spaces of C^1 piecewise quadratic polynomials on the Powell–Sabin 6-split of a triangulation relative to arbitrary polygonal domains $\Omega \subset \mathbb{R}^2$. These multiscale bases are weakly stable with respect to the L_2 norm. In this paper we prove that these multiscale spaces form a multiresolution analysis for the Banach space $C^1(\overline{\Omega})$ and we show that the multiscale basis forms a strongly stable Riesz basis for the Sobolev spaces $H^s(\Omega)$ with $s \in (2, \frac{5}{2})$. In other words, the norm of a function $f \in H^s(\Omega)$ can be determined from the size of the coefficients in the multiscale representation of f. This property makes the multiscale basis suitable for surface compression. A simple algorithm for compression is proposed and we give an optimal a priori error bound that depends on the smoothness of the input surface and on the number of terms in the compressed approximant.

Key Words.
hierarchical bases, Powell–Sabin splines, wavelets, stable approximation by splines, surface compression

AMS(MOS) Subject Classifications.
41A15, 65D07, 65T60, 41A63

259 Distribution of primes and a weighted energy problem. *Igor E. Pritsker.*

Abstract.
We discuss a recent development connecting the asymptotic distribution of prime numbers with weighted potential theory. These ideas originated with the Gelfond-Schnirelman method (circa 1936), which used polynomials with integer coefficients

and small sup norms on $[0, 1]$ to give a Chebyshev-type lower bound in prime number theory. A generalization of this method for polynomials in many variables was later studied by Nair and Chudnovsky, who produced tight bounds for the distribution of primes. Our main result is a lower bound for the integral of Chebyshev's ψ-function, expressed in terms of the weighted capacity for polynomial-type weights. We also solve the corresponding potential theoretic problem, by finding the extremal measure and its support. This new connection leads to some interesting open problems on weighted capacity.

Key Words.
distribution of prime numbers, polynomials, integer coefficients, weighted transfinite diameter, weighted capacity, potentials

AMS(MOS) Subject Classifications.
11N05, 31A15, 11C08

278 A note on the sharpness of the Remez-type inequality for homogeneous polynomials on the sphere. *M. Yattselev*.

Abstract.
Remez-type inequalities provide upper bounds for the uniform norms of polynomials p on given compact sets K, provided that $|p(x)| \leq 1$ for every $x \in K \setminus E$, where E is a subset of K of small measure. In this note we obtain an asymptotically sharp Remez-type inequality for homogeneous polynomials on the unit sphere in $\mathbb{R}^d$.

Key Words.
Remez-type inequalities, homogeneous polynomials

AMS(MOS) Subject Classifications.
41A17

284 Analysis on the unit ball and on the simplex. *Yuan Xu*.

Abstract.
Many results on the unit ball and those on the simplex can be deduced from each other or from the corresponding results on the unit sphere. The areas in which such a connection appears include orthogonal polynomials, approximation, cubature formulas and polynomial interpolation. We explain this phenomenon in some detail.

Key Words.
analysis, ball, simplex, orthogonal polynomials, approximation, cubature

AMS(MOS) Subject Classifications.
41A10, 42C10, 41A63

302 More examples on general order multivariate Padé approximants for pseudo-multivariate functions. *Ping Zhou*.

Abstract.
Although general order multivariate Padé approximants have been introduced some decades ago, very few explicit formulas have been given so far. We show in this paper that, for any given pseudo-multivariate function, we can compute its (M, N) general order multivariate Padé approximant for some given index sets M, N with

the usage of Maple or other software. Examples include a multivariate form of the sine function

$$S\left(x,y\right)=\left(x+y\right)\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\frac{x^{2i}y^{2j}}{\left(2\left(i+j\right)+1\right)!},$$

a multivariate form of the logarithm function

$$L(x,y)=\sum_{i+j\geq 1}\frac{x^{i}y^{j}}{i+j},$$

a multivariate form of the inverse tangent function

$$T\left(x,y\right)=\left(x+y\right)\sum_{i,j=0}^{\infty}\left(-1\right)^{i+j}\frac{x^{2i}y^{2j}}{2\left(i+j\right)+1},$$

and many others.

Key Words.
multivariate Padé approximant; pseudo-multivariate function

AMS(MOS) Subject Classifications.
41A21

309 A partition of the unit sphere into regions of equal area and small diameter. *Paul Leopardi*.

Abstract.
The recursive zonal equal area sphere partitioning algorithm is a practical algorithm for partitioning higher dimensional spheres into regions of equal area and small diameter. This paper describes the partition algorithm and its implementation in Matlab, provides numerical results and gives a sketch of the proof of the bounds on the diameter of regions. A companion paper gives details of the proof.

Key Words.
sphere, partition, area, diameter, zone

AMS(MOS) Subject Classifications.
11K38, 31-04, 51M15, 52C99, 74G65

328 Fine structure of the zeros of orthogonal polynomials, I. A tale of two pictures. *Barry Simon*.

Abstract.
Mhaskar-Saff found a kind of universal behavior for the bulk structure of the zeros of orthogonal polynomials for large n. Motivated by two plots, we look at the finer structure for the case of random Verblunsky coefficients and for what we call the BLS condition: $\alpha_n = Cb^n + O((b\Delta)^n)$. In the former case, we describe results of Stoiciu. In the latter case, we prove asymptotically equal spacing for the bulk of zeros.

Key Words.
OPUC, clock behavior, Poisson zeros, orthogonal polynomials

AMS(MOS) Subject Classifications.
42C05, 30C15, 60G55

369 Szegő polynomials: a view from the Riemann-Hilbert window. *A. Martínez-Finkelshtein*.

Abstract.
This is an expanded version of the talk given at the conference "Constructive Functions Tech-04". We survey some recent results on canonical representation and asymptotic behavior of polynomials orthogonal on the unit circle with respect to an analytic weight. These results are obtained using the steepest descent method based on the Riemann-Hilbert characterization of these polynomials.

Key Words.
zeros, asymptotics, Riemann-Hilbert problem, Szegő polynomials, Verblunsky coefficients

AMS(MOS) Subject Classifications.
33C45

393 Remarks on restriction eigenfunctions in $\mathbf{C}^n$. *Gabriela Putinar and Mihai Putinar*.

Abstract.
An elementary inquiry, based on examples and counterexamples, of some qualitative properties of doubly orthogonal systems of analytic functions on domains in $\mathbf{C}^n$ leads to a better understanding of the deviation from the classical Hardy space of the disk setting. The main results relay on Hilbert space with reproducing kernel techniques.

Key Words.
Hilbert space with reproducing kernel, restriction operator, doubly orthogonal system, min-max principle

AMS(MOS) Subject Classifications.
47A75, 32A25, 65F15

409 Fourier–Bessel functions of singular continuous measures and their many asymptotics. *Giorgio Mantica*.

Abstract.
We study the Fourier transform of polynomials in an orthogonal family, taken with respect to the orthogonality measure. Mastering the asymptotic properties of these transforms, that we call Fourier–Bessel functions, in the argument, the order, and in certain combinations of the two is required to solve a number of problems arising in quantum mechanics. We discuss known results, new approaches and open conjectures, hoping to justify our belief that these investigations may involve interesting discoveries, well beyond the quantum mechanical applications.

Key Words.
singular measures, Fourier transform, orthogonal polynomials, almost periodic Jacobi matrices, Fourier-Bessel functions, quantum intermittency, Julia sets, iterated function systems, generalized dimensions, potential theory

AMS(MOS) Subject Classifications.
42C05, 33E20, 28A80, 30E15, 30E20

431 On the eigenstructure of the Bernstein kernel. *Uri Itai*.

Abstract.
In approximation theory a common technique is to assume duality behavior between the Bernstein operator and the transformation matrix between the standard basis and the Bernstein basis. In this paper we shall produce an example that this assumption is not always correct. In particular, the eigenstructures of the operator and the transformation Matrix are distinguished.

Key Words.
eigenstructure, Bernstein operator, Bernstein polynomial basis

AMS(MOS) Subject Classifications.
42C15, 42C30

439 On one question of Ed Saff. *Boris Shekhtman*.

Abstract.
In relation to Fourier-Padé approximation, Ed Saff observed that Taylor and Lagrange interpolation projections satisfy the following property:

$$P(f) \cdot P(g) \in \Pi_n \Longrightarrow P(f \cdot g) = P(f) \cdot P(g).$$

We classify all projections that satisfy this property, thus answering a question of Saff. Some error formulas for approximation with the above-mentioned projections are also produced.

Key Words.
ideal projection, Hermite interpolation, error formula

AMS(MOS) Subject Classifications.
41A05, 41A10, 41A45, 41A80

446 On the exact estimates of the best spline approximations of functions. *Akhtam Khatamov*.

Abstract.
In the paper the exact (in the sense of the order of smallness) estimates of the best spline approximations of functions of one variable from different functional classes on a finite segment in uniform and integral metrics are obtained.

Key Words.
spline, polynomial spline, best spline approximation, uniform and integral metrics, class of convex function, class of function with convex derivatives, class of function with generalized finite variation, module of continuity, module of variation, spline of the minimal defect with free knots

AMS(MOS) Subject Classifications.
41A15

454 The properties, inequalities and numerical approximation of modified Bessel functions. *Juri M. Rappoport*.

Abstract.
Some new properties of kernels of modified Kontorovitch–Lebedev integral transforms — modified Bessel functions of the second kind with complex order

$K_{\frac{1}{2}+i\beta}(x)$ are presented. Inequalities giving estimations for these functions with argument x and parameter β are obtained. The polynomial approximations of these functions as a solutions of linear differential equations with polynomial coefficients and their systems are proposed.

Key Words.
Chebyshev polynomials, modified Bessel functions, Lanczos Tau method, Kontorovich-Lebedev integral transforms

AMS(MOS) Subject Classifications.
33C10, 33F05, 65D20

467 On convergence of orthonormal expansions for exponential weights. *H. P. Mashele*.

Abstract.
Let $I = (-d, d)$ be a real interval, finite or infinite, and let $W : I \to (0, \infty)$. Assume that W^2 is a weight, so that we may define orthonormal polynomials corresponding to W^2. For $f : I \to \mathbb{R}$, let $s_m[f]$ denote the mth partial sum of the orthonormal expansion of f with respect to these polynomials. We show that if $f'W \in L_\infty(I) \cap L_2(I)$, then $\|(s_m[f] - f)W\|_{L_\infty(I)} \to 0$ as $m \to \infty$. The class of weights considered includes even exponential weights.

Key Words.
orthonormal polynomials, de la Vallée Poussin means

AMS(MOS) Subject Classifications.
65N12, 65F35, 65J20, 65N55

480 From Taylor to quadratic Hermite–Padé polynomials. *Herbert Stahl*.

Abstract.
Taylor polynomials, Padé approximants, and algebraic Hermite-Padé approximants form a hierarchy of approximation concepts of growing complexity. In the present contribution we climb this ladder of concepts by reviewing results about the asymptotic behaviour of polynomials that are connected with the three concepts. In each case the concepts are used for the approximation of the exponential function. The review starts with a classical result by G. Szegö about the asymptotic behaviour of zeros of the Taylor polynomials, it is then continued with asymptotic results by E.B. Saff and R.S. Varga about the asymptotic behaviour of zeros and poles of Padé approximants, and in the last part, analogous results are considered with respect to quadatic Hermite-Padé polynomials. Here, known results are reviewed and some new ones are added. The new results are concerned with the non-diagonal case of quadatic Hermite-Padé polynomials.

Key Words.
Taylor series, Padé approximants, Hermite–Padé polynomials

AMS(MOS) Subject Classifications.
41A21, 41A58, 41A63, 30B10

511 Open problems in constructive function theory. *L. Baratchart, A. Martínez-Finkelshtein, D. Jimenez, D. S. Lubinsky, H. N. Mhaskar, I. Pritsker, M. Putinar, N. Stylianopoulos, V. Totik, P. Varju, and Y. Xu*.

Abstract.
A number of open problems on constructive function theory are presented. These were submitted by participants of Constructive Function Theory Tech-04.

Key Words.
constructive function theory, potential theory, orthogonal polynomials, quadrature formulae, integer polynomials

AMS(MOS) Subject Classifications.
41A17, 41A21, 41A55, 41A99, 42C15