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ISOTROPIC AND ANISOTROPIC A POSTERIORI ERROR ESTIMATION OF THE MIXED FINITE ELEMENT METHOD FOR SECOND ORDER OPERATORS IN DIVERGENCE FORM*

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Abstract. This paper presents an a posteriori residual error estimator for the mixed FEM of second order operators using isotropic or anisotropic meshes in $\mathbb{R}^d$, $d = 2$ or 3. The reliability and efficiency of our estimator is established without any regularity assumptions on the solution of our problem.

Key words. error estimator, stretched elements, mixed FEM, anisotropic solution.

AMS subject classifications. 65N15, 65N30.

1. Introduction. Let us fix a bounded domain Ω of $\mathbb{R}^d$, $d = 2$ or 3 with a polygonal boundary ($d = 2$) or a polyhedral one ($d = 3$). In this paper we consider the following second order problem: For $f \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be the unique solution of

$$\operatorname{div}(A\nabla u) = -f \text{ in } \Omega, \tag{1.1}$$

where the matrix $A \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ is supposed to be symmetric and uniformly positive definite.

The mixed formulation of that problem is well-known [27, 31, 28, 7, 8], and consists in finding $(\underline{p}\,, u)$ in $X \times M$ solution of :

$$\begin{cases} \int_\Omega (A^{-1}\underline{p}) \cdot \underline{q}\, dx + \int_\Omega u \operatorname{div} \underline{q}\, dx = 0, \forall \underline{q} \in X, \\ \int_\Omega v \operatorname{div} \underline{p}\, dx = -\int_\Omega fv\, dx, \quad \forall v \in M, \end{cases} \tag{1.2}$$

where

$$X = H(\operatorname{div}, \Omega) := \{\underline{q} \in [L^2(\Omega)]^d : \operatorname{div} \underline{q} \in L^2(\Omega)\},$$

endowed with the natural norm

$$\|\underline{q}\|^2_{H(\operatorname{div},\Omega)} := \|\underline{q}\|^2_{L^2(\Omega)} + \|\operatorname{div} \underline{q}\|^2_{L^2(\Omega)},$$

and $M = L^2(\Omega)$. Since this problem has at most one solution [31, p.16], the unique solution $(\underline{p}\,, u)$ is given by $\underline{p} = A\nabla u$, when u is the unique solution of (1.1).

Problem (1.2) is approximated in a conforming finite element subspace $X_h \times M_h$ of $X \times M$ based on a triangulation $\mathcal{T}$ of the domain made of isotropic or anisotropic elements. Under the property $\operatorname{div} X_h = M_h$, the discrete problem has a unique discrete solution $(\underline{p}_h, u_h) \in X_h \times M_h$. We then consider an efficient and reliable residual anisotropic a posteriori error estimator for the error $\underline{\epsilon} = \underline{p} - \underline{p}_h$ in the $H(\operatorname{div}, \Omega)$-norm and $e = u - u_h$ in the $L^2(\Omega)$-norm.

Anisotropic a posteriori error estimations are highly recommended for problem (1.2) since the solution presents edge and corner singularities [14, 17, 13, 22, 25] or boundary

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layers [23, 24], for which the use of such elements is more appropriate than isotropic ones (see [3, 18] for the treatment of standard elliptic problems). For corner singularities in 2D or edge singularities in 3D a priori error estimations are available in special geometries [15, 30] but require the explicit knowledge of the singularities which may require some numerical efforts.

Isotropic a posteriori error estimators for standard elliptic boundary value problems are currently well understood (see for instance [32] and the references cited there). The extension of these methods to anisotropic meshes starts with the recent works [29, 18, 16, 12]. The analysis of isotropic a posteriori error estimators for the mixed finite element method were initiated in [6, 2, 8] but the estimator is efficient and reliable in a non-natural norm [6, 2] or it is efficient and reliable but under the H^2-regularity of the solution of (1.1) [8] (which is often not the case, see also [9] for the elasticity system). Therefore the goal of this paper is to extend the method from [8] to the case of isotropic or anisotropic meshes in 2D and 3D, using some techniques from [18], and moreover without any regularity assumptions on the solution of (1.1).

The organization of the paper is the following: Section 2 recalls the discretization of our problem, introduces some anisotropic quantities, some mild assumptions on the meshes and some natural conditions on the finite element spaces. In Section 3 we give some anisotropic interpolation error estimates for Clément type interpolation and prove the uniform discrete inf-sup condition. Some examples of elements satisfying our theoretical assumptions are presented in Section 4. There we further give sufficient conditions on the meshes ensuring the stability of the scheme. The efficiency and reliability of the error are established in Section 5. Finally Section 6 is devoted to numerical tests which confirm our theoretical analysis.

Let us finish this introduction with some notation used in the whole paper: The $L^2(D)$-norm will be denoted by $\|\cdot\|_D$. In the case $D = \Omega$, we will drop the index Ω. The usual norm and seminorm of $H^1(D)$ are denoted by $\|\cdot\|_{1,D}$ and $|\cdot|_{1,D}$. The notation $\underline{u}$ means that the quantity u is a vector and $\nabla\underline{u}$ means the matrix $(\partial_j u_i)_{1\le i,j\le d}$ (i being the index of row and j the index of column). For a vector function $\underline{u}$ we denote by $\underline{\text{curl}}\ \underline{u} = \partial_1 u_2 - \partial_2 u_1$ in 2D and $\underline{\text{curl}}\ \underline{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T$ in 3D. On the other hand in 2D for a scalar function ϕ we write $\underline{\text{curl}}\ \phi = (\partial_2\phi, -\partial_1\phi)^T$ (note that the curl of a two-dimensional vector field is a scalar but in order to avoid a multiplicity of notation we denote it as a vector since no confusion is possible). Finally, the notation $a \lesssim b$ and $a \sim b$ means the existence of positive constants C_1 and C_2 (which are independent of $\mathcal{T}$ and of the function under consideration) such that $a \le C_2 b$ and $C_1 b \le a \le C_2 b$, respectively.

2. Discretization of the problem. The domain Ω is discretized by a conforming mesh $\mathcal{T}$, cf. [10]. In 2D, all elements are either triangles or rectangles. In 3D the mesh consists either of tetrahedra, of rectangular hexahedra, or of rectangular pentahedra (i.e. prisms where the triangular faces are perpendicular to the rectangular faces), cf. also the figures of Section 2.2. The restriction to rectangles, rectangular hexahedra or rectangular pentahedra is only made for the sake of simplicity; the extension to parallelogram, hexahedra or pentahedra is straightforward using affine transformations.

Elements will be denoted by T, T_i or T', its edges (in 2D) or faces (in 3D) are denoted by E. The set of all (interior and boundary) edges (2D) or faces (3D) of the triangulation will be denoted by $\mathcal{E}$. Let $\underline{x}$ denote a nodal point, and let $\mathcal{N}_{\bar{\Omega}}$ be the set of nodes of the mesh. The measure of an element or edge/face is denoted by $|T| := \text{meas}_d(T)$ and $|E| := \text{meas}_{d-1}(E)$, respectively.

For an edge E of a 2D element T introduce the *outer normal vector* by $\underline{n} = (n_x, n_y)^\top$. Similarly, for a face E of a 3D element T set $\underline{n} = (n_x, n_y, n_z)^\top$. From now, the word "face" will denote either an edge in the 2D case or a face in the 3D case. Furthermore, for each

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face E we fix one of the two normal vectors and denote it by $\underline{n}_E$. In the 2D case introduce additionally the *tangent vector* $\underline{t} = \underline{n}^\perp := (-n_y, n_x)^\top$ such that it is oriented positively (with respect to T). Similarly set $\underline{t}_E := \underline{n}_E^\perp$.

The *jump* of some (scalar or vector valued) function v across a face E at a point $\underline{y} \in E$ is then defined as

$$[\![v(\underline{y})]\!]_E := \begin{cases} \lim\limits_{\alpha\to+0} v(\underline{y} + \alpha\underline{n}_E) - v(\underline{y} - \alpha\underline{n}_E) & \text{for an interior face } E, \\ v(y) & \text{for a boundary face } E. \end{cases}$$

Note that the sign of $[\![v]\!]_E$ depends on the orientation of $\underline{n}_E$. However, terms such as a gradient jump $[\![\nabla v\,\underline{n}_E]\!]_E$ are independent of this orientation.

Furthermore one requires local subdomains (also known as patches). As usual, let ω_T be the union of all elements having a common face with T. Similarly let ω_E be the union of the elements having E as face. By $\omega_{\underline{x}}$ we denote the union of all elements having $\underline{x}$ as node.

Later on we specify additional, mild mesh assumptions that are partially due to the anisotropic discretization.

2.1. Discrete formulation. The discrete problem associated with (1.2) is to find $(\underline{p}_h, u_h) \in X_h \times M_h$ such that

$$\begin{cases} \int_\Omega (A^{-1}\underline{p}_h)\cdot\underline{q}_h\,dx + \int_\Omega u_h \operatorname{div}\underline{q}_h\,dx = 0, \forall \underline{q}_h \in X_h, \\ \int_\Omega v_h \operatorname{div}\underline{p}_h\,dx = -\int_\Omega f v_h\,dx, \quad \forall v_h \in M_h, \end{cases} \tag{2.1}$$

where X_h (resp. M_h) is a finite dimensional subspace of X (resp. M).

Recall that the errors are defined by

$$\underline{\epsilon} := \underline{p} - \underline{p}_h,\ e := u - u_h.$$

Therefore subtracting (1.2) with $\underline{q} = \underline{q}_h$ and $v = v_h$ from (2.1) we obtain the 'Galerkin orthogonality' relations

$$\int_\Omega (A^{-1}\underline{\epsilon})\cdot\underline{q}_h\,dx + \int_\Omega e \operatorname{div}\underline{q}_h\,dx = 0, \forall \underline{q}_h \in X_h, \tag{2.2}$$

$$\int_\Omega v_h \operatorname{div}\underline{\epsilon}\,dx = 0, \quad \forall v_h \in M_h. \tag{2.3}$$

2.2. Some anisotropic quantities. In our exposition T can be a triangle or rectangle (2D case), or a tetrahedron, a (rectangular) hexahedron, or a prismatic pentahedron (3D case).

Parts of the analysis require *reference elements* $\bar{T}$ that can be obtained from the actual element T via some affine linear transformation F_T. The table below lists the reference elements for each case. Furthermore for an element T we define 2 or 3 *anisotropy vectors* $\underline{p}_{i,T}, i = 1 \ldots d$, that reflect the main anisotropy directions of that element. These anisotropy vectors are defined and visualized in the table below as well.

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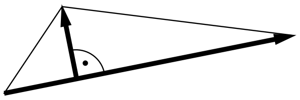
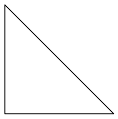
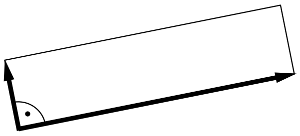
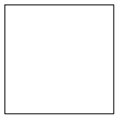
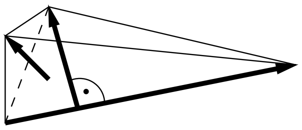
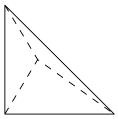
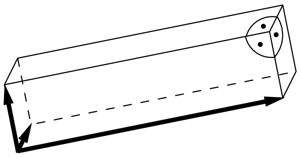
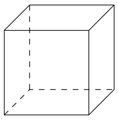
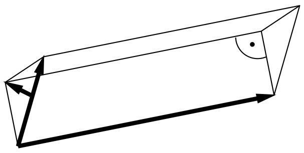
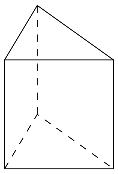
Element T	Reference element $\bar{T}$	Anisotropy vectors $\underline{p}_{i,T}$
Triangle 	 $0 \le \bar{x}, \bar{y}$ $\bar{x} + \bar{y} \le 1$	$\underline{p}_{1,T}$ longest edge $\underline{p}_{2,T}$ height vector
Rectangle 	 $0 \le \bar{x}, \bar{y} \le 1$	$\underline{p}_{1,T}$ longest edge $\underline{p}_{2,T}$ height vector
Tetrahedron 	 $0 \le \bar{x}, \bar{y}, \bar{z}$ $\bar{x} + \bar{y} + \bar{z} \le 1$	$\underline{p}_{1,T}$ longest edge $\underline{p}_{2,T}$ height in largest face that contains $\underline{p}_{1,T}$ $\underline{p}_{3,T}$ remaining height
Hexahedron 	 $0 \le \bar{x}, \bar{y}, \bar{z} \le 1$	$\underline{p}_{1,T}$ longest edge $\underline{p}_{2,T}$ height in largest face that contains $\underline{p}_{1,T}$ $\underline{p}_{3,T}$ remaining height
Pentahedron (Prism) 	 $0 \le \bar{x}, \bar{y}, \bar{z} \le 1$ $\bar{x} + \bar{y} \le 1$	longest edge in triangle; height in triangle; height over triangle (see figure, vectors ordered by length)

The anisotropy vectors $\underline{p}_{i,T}$ are enumerated such that their lengths are decreasing, i.e. $|\underline{p}_{1,T}| \ge |\underline{p}_{2,T}| \ge |\underline{p}_{3,T}|$ in the 3D case, and analogously in 2D. The *anisotropic lengths* of an element T are now defined by

$$h_{i,T} := |\underline{p}_{i,T}|$$

which implies $h_{1,T} \ge h_{2,T} \ge h_{3,T}$ in 3D. The smallest of these lengths is particularly important; thus we introduce

$$h_{min,T} := h_{d,T} \equiv \min_{i=1\dots d} h_{i,T}.$$

Finally the anisotropy vectors $\underline{p}_{i,T}$ are arranged columnwise to define a matrix

$$\left.\begin{array}{rcll} C_T & := & [\underline{p}_{1,T}, \underline{p}_{2,T}] \in \mathbb{R}^{2\times 2} & \text{in 2D} \\ C_T & := & [\underline{p}_{1,T}, \underline{p}_{2,T}, \underline{p}_{3,T}] \in \mathbb{R}^{3\times 3} & \text{in 3D.} \end{array}\right\} \tag{2.4}$$

Note that C_T is orthogonal since the anisotropy vectors $\underline{p}_{i,T}$ are orthogonal too, and

$$C_T^\top C_T = \operatorname{diag}\{h_{1,T}^2, \dots, h_{d,T}^2\}.$$

Furthermore we introduce the *height* $h_{E,T} = \frac{|T|}{|E|}$ over an edge/face E of an element T.

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2.3. Mesh assumptions. The mesh has to satisfy some mild assumptions.

- The mesh is conforming in the standard sense of [10].
- A node $\underline{x}_j$ of the mesh is contained only in a bounded number of elements (uniformly in h).
- The size of neighbouring elements does not change rapidly, i.e.

$$h_{i,T_1} \sim h_{i,T_2} \qquad \forall i = 1 \ldots d, \forall T_1 \cap T_2 \neq \emptyset.$$

Sometimes it is more convenient to have *face related* data instead of *element related* data. Hence for an interior face $E = T_1 \cap T_2$ we introduce

$$h_{min,E} := \frac{h_{min,T_1} + h_{min,T_2}}{2} \qquad \text{and} \qquad h_E := \frac{h_{E,T_1} + h_{E,T_2}}{2}.$$

For boundary faces $E \subset \partial T$ simply set $h_{min,E} := h_{min,T}$, $h_E := h_{E,T}$. The last assumption from above readily implies

$$h_E \sim h_{E,T_1} \sim h_{E,T_2} \qquad \text{and} \qquad h_{min,E} \sim h_{min,T_1} \sim h_{min,T_2}.$$

2.4. Finite element spaces assumptions. We assume that the element spaces X_h, M_h satisfy

$$\{\underline{q} \in H(\operatorname{div}, \Omega) : \underline{q}_{|T} \in [\mathbb{P}_0(T)]^d, \forall T \in \mathcal{T}\} \subset X_h, \tag{2.5}$$

$$X_h \subset \{\underline{q} \in H(\operatorname{div}, \Omega) : \underline{q}_{|T} \in [H^1(T)]^d, \forall T \in \mathcal{T}\},$$

$$\operatorname{div} X_h = M_h. \tag{2.6}$$

We suppose that the commuting diagram property holds [7, 8]: There exists an interpolation operator $\Pi_h : W \to X_h$, where $W = H(\operatorname{div}, \Omega) \cap L^s(\Omega)$, with $s > 2$, such that the next diagram commutes

$$\begin{array}{ccc} W & \stackrel{\operatorname{div}}{\rightarrow} & M \\ \Pi_h \downarrow & & \downarrow \rho_h \\ X_h & \stackrel{\operatorname{div}}{\rightarrow} & M_h, \end{array} \tag{2.7}$$

where ρ_h is the $L^2(\Omega)$-orthogonal projection on M_h. This property implies in particular

$$\operatorname{div} (Id - \Pi_h) W \perp M_h, \tag{2.8}$$

the orthogonality being in the $L^2(\Omega)$-sense and Id meaning the identity operator.

We further assume that the interpolant satifies the global stability estimate

$$\|\Pi_h \underline{q}\| \lesssim \|\underline{q}\|_{1,\Omega}, \forall \underline{q} \in [H^1(\Omega)]^d. \tag{2.9}$$

We will see that this assumption added to (2.6) and (2.7) leads to the uniform discrete inf-sup condition. Even if our further method does not require this condition, it is recommended to have a robust discrete analysis.

Finally we assume that Π_h satisfies the approximation property

$$\int_E v_h (\underline{q} - \Pi_h \underline{q}) \cdot \underline{n}_E = 0, \forall \underline{q} \in W, v_h \in M_h, E \in \mathcal{E}. \tag{2.10}$$

Such properties will be checked in some particular cases in Section 4.

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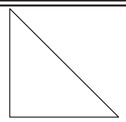
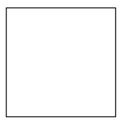
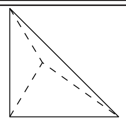
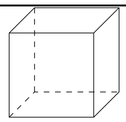
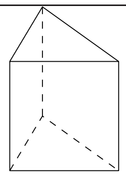
3. Analytical tools. Since we treat *anisotropic elements*, some analytical tools which are known from the standard theory have to be reinvestigated. This is mainly due to the fact that the aspect ratio of the elements is no longer bounded, as it is the case with isotropic elements.This leads to the introduction of a so-called *alignment measure* and a *approximation measure*, cf. below. It is important to notice that these measures are not a (theoretical or practical) obstacle to efficient and reliable error estimation; furthermore for isotropic meshes they are equivalent to 1.

3.1. Bubble functions, extension operator, inverse inequalities. For the analysis we require bubble functions and extension operators that satisfy certain properties. We start with the reference element $\bar{T}$ and define an *element bubble function* $b_{\bar{T}} \in C(\bar{T})$. We also require an *edge bubble function* $b_{\bar{E},\bar{T}} \in C(\bar{T})$ for an edge $\bar{E} \subset \partial\bar{T}$ (2D case), and a *face bubble function* $b_{\bar{E},\bar{T}} \in C(\bar{T})$ for a face $\bar{E} \subset \partial\bar{T}$ (3D case). Without loss of generality assume that $\bar{E}$ is on the $\bar{x}$ axis (2D case) or in the $\bar{x}\bar{y}$ plane (tetrahedral and hexahedral case). For the pentahedral case, the triangular face $\bar{E}_\triangle$ is also in the $\bar{x}\bar{y}$ plane but the rectangular face $\bar{E}_\Box$ is in the $\bar{x}\bar{z}$ plane.

Furthermore an *extension operator* $\mathrm{F}_{\mathrm{ext}} : C(\bar{E}) \to C(\bar{T})$ will be necessary that acts on some function $v_{\bar{E}} \in C(\bar{E})$. The table below gives the definitions in each case. For vector valued functions apply the extension operator componentwise.

Ref. element $\bar{T}$	Bubble functions	Extension operator
	$b_{\bar{T}} := 3^3\bar{x}\bar{y}(1-\bar{x}-\bar{y})$ $b_{\bar{E},\bar{T}} := 2^2\bar{x}(1-\bar{x}-\bar{y})$	$\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}})(\bar{x},\bar{y}) := v_{\bar{E}}(\bar{x})$
	$b_{\bar{T}} := 2^4\bar{x}(1-\bar{x})\bar{y}(1-\bar{y})$ $b_{\bar{E},\bar{T}} := 2^2\bar{x}(1-\bar{x})(1-\bar{y})$	$\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}})(\bar{x},\bar{y}) := v_{\bar{E}}(\bar{x})$
	$b_{\bar{T}} := 4^4\bar{x}\bar{y}\bar{z}(1-\bar{x}-\bar{y}-\bar{z})$ $b_{\bar{E},\bar{T}} := 3^3\bar{x}\bar{y}(1-\bar{x}-\bar{y}-\bar{z})$	$\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}})(\bar{x},\bar{y},\bar{z}) := v_{\bar{E}}(\bar{x},\bar{y})$
	$b_{\bar{T}} := 2^6\bar{x}(1-\bar{x})\bar{y}(1-\bar{y})\bar{z}(1-\bar{z})$ $b_{\bar{E},\bar{T}} := 2^4\bar{x}(1-\bar{x})\bar{y}(1-\bar{y})(1-\bar{z})$	$\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}})(\bar{x},\bar{y},\bar{z}) := v_{\bar{E}}(\bar{x},\bar{y})$
	$b_{\bar{T}} := 3^3 2^2\bar{x}\bar{y}(1-\bar{x}-\bar{y})\bar{z}(1-\bar{z})$ $b_{\bar{E},\bar{T},\triangle} := 3^3\bar{x}\bar{y}(1-\bar{x}-\bar{y})(1-\bar{z})$ $b_{\bar{E},\bar{T},\Box} := 2^4\bar{x}(1-\bar{x}-\bar{y})\bar{z}(1-\bar{z})$	$\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}_\triangle})(\bar{x},\bar{y},\bar{z}) := v_{\bar{E}_\triangle}(\bar{x},\bar{y})$ $\mathrm{F}_{\mathrm{ext}}(v_{\bar{E}_\Box})(\bar{x},\bar{y},\bar{z}) := v_{\bar{E}_\Box}(\bar{x},\bar{z})$

The element bubble function b_T for the actual element T is obtained simply by the corresponding affine linear transformation. Similarly the edge/face bubble function $b_{E,T}$ is defined. Later on an edge/face bubble function b_E is needed on the domain $\omega_E = T_1 \cup T_2$. This is achieved by an elementwise definition, i.e.

$$b_E|_{T_i} := b_{E,T_i}, \qquad i = 1, 2.$$

Analogously the extension operator is defined for functions $v_E \in C(E)$. By the same ele-

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mentwise definition obtain then $\mathrm{F}_{\mathrm{ext}}(v_E) \in C(\omega_E)$. With these definitions one easily checks

$$b_T = 0 \text{ on } \partial T, \qquad b_E = 0 \text{ on } \partial\omega_E, \qquad \|b_T\|_{\infty,T} = \|b_E\|_{\infty,\omega_E} = 1.$$

Next, one needs so-called *inverse inequalities* proved for instance in Lemma 4.1 of [12].

LEMMA 3.1 (Inverse inequalities). *Let $E \subset \partial T$ be an edge/face of an element T. Consider $v_T \in \mathbb{P}^{k_0}(T)$ and $v_E \in \mathbb{P}^{k_1}(E)$. Then the following equivalences/inequalities hold. The inequality constants depend on the polynomial degree k_0 or k_1 but not on T, E or v_T, v_E.*

$$\|v_T b_T^{1/2}\|_T \sim \|v_T\|_T \tag{3.1}$$

$$\|\nabla(v_T b_T)\|_T \lesssim h_{min,T}^{-1}\|v_T\|_T \tag{3.2}$$

$$\|v_E b_E^{1/2}\|_E \sim \|v_E\|_E \tag{3.3}$$

$$\|\mathrm{F}_{\mathrm{ext}}(v_E) b_E\|_T \lesssim h_{E,T}^{1/2}\|v_E\|_E \tag{3.4}$$

$$\|\nabla(\mathrm{F}_{\mathrm{ext}}(v_E) b_E)\|_T \lesssim h_{E,T}^{1/2} h_{min,T}^{-1}\|v_E\|_E. \tag{3.5}$$

3.2. Clément interpolation. For our analysis we need some interpolation operator that maps a function from $H^1(\Omega)$ to the usual space $S(\Omega,\mathcal{T})$ made of continuous and piecewise polynomial functions on the triangulation. Hence Lagrange interpolation is unsuitable, but Clément like interpolant is more appropriate. Recall that the nodal basis function $\varphi_{\underline{x}} \in S(\Omega,\mathcal{T})$ associated with a node $\underline{x}$ is uniquely determined by the condition

$$\varphi_{\underline{x}}(\underline{y}) = \delta_{\underline{x},\underline{y}} \qquad \forall \underline{y} \in \mathcal{N}_{\bar{\Omega}},$$

and by the polynomial space of $\varphi_{\underline{x}}|T$:

Finite element domain T	Local space $\mathcal{P}_T$ of $\varphi_{\underline{x}}\vert T \circ F_T$
Triangle, Tetrahedron	$\mathbb{P}^1(\bar{T})$
Rectangle, Hexahedron	$\mathbb{Q}^1(\bar{T})$
Pentahedron	span$\{1, \bar{x}, \bar{y}, \bar{z}, \bar{x}\bar{z}, \bar{y}\bar{z}\}$

Then $S(\Omega,\mathcal{T})$ is defined as the space spanned by the functions $\varphi_{\underline{x}}$, for all nodes $\underline{x} \in \mathcal{N}_{\bar{\Omega}}$. Equivalently, it can be expressed as

$$S(\Omega,\mathcal{T}) := \{v_h \in C(\bar{\Omega}) : v_h\big|_T \circ F_T \in \mathcal{P}_T\} \subset H^1(\Omega), \tag{3.6}$$

with $\mathcal{P}_T$ as described in the above table.

Next, the Clément interpolation operator will be defined via the basis functions $\varphi_{\underline{x}} \in S(\Omega,\mathcal{T})$.

DEFINITION 3.2 (Clément interpolation operator). *We define the* Clément interpolation operator $\mathrm{I}_{\mathrm{Cl}} : H^1(\Omega) \to S(\Omega,\mathcal{T})$ *by*

$$\mathrm{I}_{\mathrm{Cl}} v := \sum_{\underline{x} \in \mathcal{N}_{\bar{\Omega}}} \frac{1}{|\omega_{\underline{x}}|} \left(\int_{\omega_{\underline{x}}} v \right) \varphi_{\underline{x}}.$$

The interpolation error estimates on anisotropic triangulations are different to the isotropic case. The anisotropic elements have to be aligned with the anisotropy of the function in order to obtain sharp estimates. To this end we introduce a quantity which measures the alignment of mesh and function.

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DEFINITION 3.3 (alignment measure). *For $v \in H^1(\Omega)$, set*

$$m_1(v, \mathcal{T}) := \frac{\left(\sum_{T\in\mathcal{T}} h_{min,T}^{-2} \|C_T^\top \nabla v\|_T^2\right)^{1/2}}{\|\nabla v\|} \tag{3.7}$$

From that definition we see that

$$1 \le m_1(v, \mathcal{T}) \le \max_{T\in\mathcal{T}} \frac{h_{1,T}}{h_{min,T}}.$$

These estimates imply that for isotropic meshes $m_1(v, \mathcal{T}) \sim 1$ and consequently for such meshes the alignment measure disappears in other constants.

For anisotropic meshes the term $C_T^\top \nabla v$ contains directional derivatives of v along the main anisotropic directions $\underline{p}_{i,T}$ of T. Therefore T will be aligned with v if long (resp. small) anisotropic direction $\underline{p}_{1,T}$ (resp. $\underline{p}_{3,T}$) is associated with small (resp. large) directional derivative $\underline{p}_{1,T}^\top \cdot \nabla v$ (resp. $\underline{p}_{3,T}^\top \cdot \nabla v$). If all elements are aligned with v then the numerator and denominator of $m_1(v, \mathcal{T})$ will be of the same size and consequently $m_1(v, \mathcal{T}) \sim 1$. We refer to [18, 19] for more details.

Finally we may state the interpolation estimates.

LEMMA 3.4 (Clément interpolation estimates). *For any $v \in H^1(\Omega)$ it holds*

$$\sum_{T\in\mathcal{T}} h_{min,T}^{-2} \|v - \mathrm{I}_{\mathrm{Cl}} v\|_T^2 \le m_1^2(v, \mathcal{T})\, \|\nabla v\|^2 \tag{3.8}$$

$$\sum_{F\in\mathcal{E}} \frac{h_F}{h_{min,F}^2} \|v - \mathrm{I}_{\mathrm{Cl}} v\|_F^2 \le m_1^2(v, \mathcal{T})\, \|\nabla v\|^2. \tag{3.9}$$

Proof. The proof of the estimates (3.8) and (3.9) is given in [18] and simply use some scaling arguments. □

At the end for $\underline{q} \in [H^1(\Omega)]^d$ we introduce its *approximation measure*

$$a(\underline{q}, \mathcal{T}) := \frac{\left(\sum_{T\in\mathcal{T}} h_{min,T}^{-2} \|\underline{q} - \Pi_h \underline{q}\|_T^2\right)^{1/2}}{\|\underline{q}\|_{1,\Omega}}. \tag{3.10}$$

Roughly speaking this quantity measures the alignement of the mesh $\mathcal{T}$ with $\underline{q}$. For isotropic meshes it is then bounded from above by 1 (see Section 4).

3.3. Surjectivity of the divergence operator. Here we focus on the surjectivity of the divergence operator from $[H^1(\Omega)]^d$ to $L^2(\Omega)$. This result will be used in the next subsection as well as in Subsection 5.3.

LEMMA 3.5. *Let g be an arbitrary function in $L^2(\Omega)$, then there exists $\underline{v} \in [H^1(\Omega)]^d$ such that*

$$\operatorname{div} \underline{v} = g \text{ in } \Omega, \tag{3.11}$$

$$\|\underline{v}\|_{1,\Omega} \lesssim \|g\|. \tag{3.12}$$

Proof. Consider a domain D with a smooth boundary such that $\bar{\Omega} \subset D$. We extend g by zero outside Ω to get $\tilde{g}$ in $L^2(D)$. Let $\psi \in H_0^1(D)$ be the unique weak solution of

$$\Delta\psi = \tilde{g} \text{ in } D.$$

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As $\tilde{g} \in L^2(D)$ and D has a smooth boundary, ψ belongs to $H^2(D)$ with the estimate

$$\|\psi\|_{2,D} \lesssim \|\tilde{g}\|_D = \|g\|. \tag{3.13}$$

Therefore $\underline{v}$ defined in Ω by

$$\underline{v} = \nabla\psi \text{ in } \Omega$$

belongs to $[H^1(\Omega)]^d$ and satisfies (3.11) as well as (3.12) as a consequence of (3.13). □

This lemma differs from the classical result on the divergence operator [17] by the fact that $\underline{v}$ is no more zero on the boundary and then allows to leave the zero mean condition on g.

3.4. Uniform discrete inf-sup condition. We end this section by showing that the commuting diagram property and the continuity of Π_h from $H^1(\Omega)$ into $L^2(\Omega)$ guarantee the uniform discrete inf-sup condition.

LEMMA 3.6. *If (2.7) and (2.9) hold then there exists a constant $\beta^* > 0$ independent of h such that for every $v_h \in M_h$*

$$\sup_{\underline{q}_h \in X_h} \frac{\int_\Omega v_h \operatorname{div} \underline{q}_h \, dx}{\|\underline{q}_h\|_{H(\operatorname{div},\Omega)}} \geq \beta^* \|v_h\|. \tag{3.14}$$

Proof. Let us fix $v_h \in M_h$. It suffices to show that there exists $\underline{q}_h \in X_h$ such that

$$\operatorname{div} \underline{q}_h = v_h \text{ in } \Omega, \tag{3.15}$$

$$\|\underline{q}_h\| \lesssim \|v_h\|. \tag{3.16}$$

Let $\underline{v} \in [H^1(\Omega)]^d$ be the solution of (3.11) with $g = v_h$ obtained in Lemma 3.5. Take

$$\underline{q}_h = \Pi_h \underline{v}.$$

By (2.7) it satisfies (3.15). Indeed by (2.8), we have

$$\int_\Omega \operatorname{div} (\underline{v} - \underline{q}_h) w_h = 0, \forall w_h \in M_h,$$

or equivalently

$$\int_\Omega (v_h - \operatorname{div} \underline{q}_h) w_h = 0, \forall w_h \in M_h,$$

which leads to (3.15) since $\operatorname{div} \underline{q}_h$ belongs to M_h by the assumption (2.6).

The estimate (3.16) directly follows from (2.9) and (3.12). □

4. Examples. In this section we present a list of finite element pairs fulfilling the theoretical assumptions of the previous sections. For an easier readibility, since our a posteriori error analysis from section 5 is independent of the choice of the elements, the reader not interested in all the details from this section may skip the remainder of this section.

For any element $T \in \mathcal{T}$, we describe in the next table the finite dimensional spaces $D_k(T)$ and $M_k(T)$, where $k \in \mathbb{N}$, for the Raviart-Thomas elements (in short RT), the Brezzi-Douglas-Marini elements (BDM), and the Brezzi-Douglas-Fortin-Marini elements (BDFM).

Name	Element	$M_k(T)$	$D_k(T)$
RT	Triangle/Tetra	$RT_k := [\mathbb{P}_k]^d + \underline{x}\,\widetilde{\mathbb{P}}_k$	$\mathbb{P}_k$
RT	Rectangle	$\mathbb{P}_{k+1,k} \times \mathbb{P}_{k,k+1}$	$\mathbb{Q}_k$
RT	Hexahedra	$\mathbb{P}_{k+1,k,k} \times \mathbb{P}_{k,k+1,k} \times \mathbb{P}_{k,k,k+1}$	$\mathbb{Q}_k$
RT	Pentahedra	$RT_0(x_1,x_2) \times \mathbb{P}_1(x_3)$	$\mathbb{P}_0$
BDM	Triangle/Tetra	$[\mathbb{P}_{k+1}]^d$	$\mathbb{P}_k$
BDFM	Triangle/Tetra	$\{\underline{q} \in [\mathbb{P}_{k+1}]^d : \underline{q}\cdot\underline{n} \in \mathcal{R}_k(\partial T)\}$	$\mathbb{P}_k$

Here $\mathbb{P}_{k+1,k,k}$ means the space of polynomials of degree $k+1$ in x_1 and of degree k in x_2 and x_3, $\widetilde{\mathbb{P}}_k$ means the space of homogeneous polynomials of degree k, while $\mathcal{R}_k(\partial T)$ denotes the space of functions defined in ∂T which are a polynomial of degree at most k on each edge/face of T. With these sets we may define

$$M_h := \{v_h \in M : v_{h|T} \in D_k(T), \forall T \in \mathcal{T}\}, \tag{4.1}$$

$$X_h := \{\underline{p}_h \in X : \underline{p}_{h|T} \in M_k(T), \forall T \in \mathcal{T}\}. \tag{4.2}$$

For these element pairs (X_h, M_h), except the pentahedral case, the assumptions (2.6), (2.7) and (2.10) are checked in Section III.3 of [7]. The case of pentahedra is proved similarly by using the standard degrees of freedom

$$\int_E \underline{q}\cdot\underline{n}, \forall E \in \mathcal{E}, E \subset \partial T.$$

We now show that the stability estimate (2.9) holds in some particular situations.

We start with a general result.

LEMMA 4.1. *If the elements $T \in \mathcal{T}$ satisfy*

$$h_{1,T}^2 \lesssim h_{min,T}, \tag{4.3}$$

then (2.9) holds.

Proof. Using the affine transformation $\underline{x} = A_T\underline{\bar{x}} + P_0$ which maps the reference element $\bar{T}$ to T and Piola's transformation

$$\underline{\bar{q}}(\underline{\bar{x}}) = A_T^{-1}\underline{q}(\underline{x}),$$

which preserves the degree of freedom, we have

$$\begin{aligned}
\|\underline{q} - \Pi_h\underline{q}\|_T^2 &= |T|\int_{\bar{T}} |A_T(\underline{\bar{q}} - \bar{\Pi}\underline{\bar{q}})|^2 \\
&\le |T|\|A_T\|^2\int_{\bar{T}} |\underline{\bar{q}} - \bar{\Pi}\underline{\bar{q}}|^2 \\
&\lesssim |T|\|A_T\|^2\int_{\bar{T}} |\nabla\underline{\bar{q}}|^2 \\
&\lesssim \|A_T\|^2\int_T |\nabla(A_T^{-1}\underline{q})A_T|^2 \\
&\lesssim \|A_T\|^4\|A_T^{-1}\|^2\int_T |\nabla\underline{q}|^2.
\end{aligned}$$

Since by Lemma 2.2 of [18] we have $\|A_T\| \sim h_{1,T}$ and $\|A_T^{-1}\| \sim h_{min,T}^{-1}$, the above estimate and the assumption (4.3) yields

$$\|\underline{q} - \Pi_h\underline{q}\|_T^2 \lesssim \int_T |\nabla\underline{q}|^2.$$

The sum of this estimate on $T \in \mathcal{T}$ leads to the conclusion. □

For boundary layer meshes $h_{min,T} = \tau h$ and $h_{1,T} = h$, where $\tau \sim \sqrt{\epsilon}|\ln\sqrt{\epsilon}|$, the thickness of the layer being $\sqrt{\epsilon}$ (see [3, 18, 20]), therefore the assumption (4.3) becomes then $h \leq \tau$ and could be too restrictive. Similarly for refined meshes along edge singularities, then $h_{min,T} = h^{\frac{1}{\lambda}}$ and $h_{1,T} = h$, where $\lambda > 0$ is the smallest edge singular exponent [4, 3, 18, 5], in that case (4.3) reduces to $\lambda \geq 1/2$. Again this condition is too restrictive for strong edge singularities (λ is always $\geq 1/2$ for the Laplace equation, but for general transmission problems (A piecewise constant), λ could be as small as we want [14, 22, 25, 26, 11]). These considerations motivate the use of finer arguments to get (2.9), namely adapting the arguments of Sections 4 and 5 of [1], we can prove the following results.

LEMMA 4.2. *Assume given a 2D triangulation $\mathcal{T}$ made of triangles T which satisfy*

$$h_{1,T} \lesssim \sin\theta_{max,T}, \tag{4.4}$$

where $\theta_{max,T}$ is the maximal angle of T. Assume that (X_h, M_h) corresponds to the Raviart-Thomas element of order 0 (i.e. defined by (4.1)-(4.2) with $k = 0$). Then (2.9) holds.

Proof. By Lemmas 4.1 and 4.2 of [1] for any $T \in \mathcal{T}$, we have

$$\|\underline{q} - \Pi_h \underline{q}\|_T \lesssim \frac{h_{1,T}}{\sin\theta_{max,T}} \|\nabla \underline{q}\|_T.$$

The assumption (4.4) directly yields the desired estimate. □

Remark that the assumption (4.4) is much weaker than (4.3). Indeed it is satisfied for any tensor product meshes, for any meshes satisfying the maximal angle condition (i.e. there exists $\gamma^* < \pi$ such that $\theta_{max,T} \leq \gamma^*$), while such meshes may not satisfy (4.3). The condition (4.4) is weaker than the maximal angle condition since it is equivalent to

$$\frac{\pi}{3} \leq \theta_{max,T} \leq \pi - c h_{1,T},$$

for some $c > 0$ and then allows $\theta_{max,T}$ to tend to π.

In a similar manner we prove the

LEMMA 4.3. *Assume given a 3D triangulation $\mathcal{T}$ made of tetrahedra T satisfying*

$$h_{1,T} \lesssim (\det M)^2, \tag{4.5}$$

where M is a matrix made of three vectors $\underline{v}_i$, $i = 1, 2, 3$, where $\underline{v}_i$ are the direction of the edges sharing a common vertex and such that $|\underline{v}_i| = 1$. Assume that (X_h, M_h) corresponds to the Raviart-Thomas element of order 0 (i.e. defined by (4.1)-(4.2) with $k = 0$). Then (2.9) holds.

Proof. By Lemmas 5.1 and 5.2 of [1] for any $T \in \mathcal{T}$, we have

$$\|\underline{q} - \Pi_h \underline{q}\|_T \lesssim \frac{h_{1,T}}{(\det M)^2} \|\nabla \underline{q}\|_T,$$

and we conclude with the assumption (4.5). □

Note that the regular vertex property introduced in [1] implies (4.5), note furthermore that Theorem 5.10 of [1] implies that (2.9) holds under the maximal angle condition introduced by Krizek [21] and quite often used for anisotropic meshes [4, 3].

Let us now pass to rectangular meshes.

LEMMA 4.4. *Assume given a 2D triangulation $\mathcal{T}$ made of rectangles such that the edges of the elements are parallel to the x_1 or x_2 axis. Assume that (X_h, M_h) corresponds to the*

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Raviart-Thomas element of order 0 or 1 (i.e. defined by (4.1)-(4.2) with $k = 0$ or 1). Then (2.9) holds.

Proof. Denote by E_1, E_3 the edges of T parallel to the x_1 axis. Then by definition of the interpolant $\Pi_h \underline{q}$ of $\underline{q}$ we remark that $\underline{q}_1 - (\Pi_h \underline{q})_1$ has a mean zero on E_1 and $\underline{q}_2 - (\Pi_h \underline{q})_2$ has a mean zero on E_2, therefore by a standard scaling argument we have

$$\|\underline{q} - \Pi_h \underline{q}\|_T \lesssim \sum_{j=1,2} h_j \|\partial_j(\underline{q} - \Pi_h \underline{q})\|_T, \tag{4.6}$$

where h_j means here the length of E_j, $j = 1, 2$. It then remains to estimate $\|\partial_j \Pi_h \underline{q}\|_T$. For that purpose we distinguish between the cases $k = 0$ and $k = 1$.

For $k = 0$ we shall prove that

$$\|\nabla \Pi_h \underline{q}\|_T \lesssim \|\nabla \underline{q}\|_T, \tag{4.7}$$

while for $k = 1$, we shall prove that

$$\|\partial_j \Pi_h \underline{q}\|_T \lesssim \|\partial_j \underline{q}\|_T + h_j^{-1} \|\underline{q}\|_T. \tag{4.8}$$

In both cases these estimates yield

$$\|\underline{q} - \Pi_h \underline{q}\|_T \lesssim h_{1,T} \|\nabla \underline{q}\|_T + \|\underline{q}\|_T.$$

and the conclusion follows by summing the square of this estimate on $T \in \mathcal{T}$.

In the case $k = 0$, we remark that

$$\Pi_h \underline{q}(x) = \begin{pmatrix} a_0 + a_1 x_1 \\ b_0 + b_1 x_2 \end{pmatrix},$$

for some real numbers $a_i, b_i, i = 0, 1$. Consequently we get

$$\partial_1 \Pi_h \underline{q} = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \partial_2 \Pi_h \underline{q} = b_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now by Green's formula, the fact that the edges of T are parallel to the axes and the interpolation properties, we may successively write

$$\begin{aligned} \int_T \partial_1 (\Pi_h \underline{q})_1 &= \int_{\partial T} n_1 (\Pi_h \underline{q})_1 = \int_{E_2 \cup E_4} n_1 (\Pi_h \underline{q})_1 \\ &= \int_{E_2 \cup E_4} \Pi_h \underline{q} \cdot n = \int_{E_2 \cup E_4} \underline{q} \cdot n \\ &= \int_{E_2 \cup E_4} n_1 \underline{q}_1 = \int_T \partial_1 \underline{q}_1. \end{aligned}$$

By the fact that $\partial_1 (\Pi_h \underline{q})_1$ is constant and by Cauchy-Schwarz's inequality we obtain

$$|\partial_1 (\Pi_h \underline{q})_1| \le |T|^{-1/2} \|\partial_1 \underline{q}_1\|_T.$$

Integrating the square of this estimate on T we arrive at

$$\|\partial_1 (\Pi_h \underline{q})_1\|_T^2 \le \|\partial_1 \underline{q}_1\|_T^2.$$

Since a similar argument yields

$$\|\partial_2 (\Pi_h \underline{q})_2\|_T^2 \le \|\partial_2 \underline{q}_2\|_T^2,$$

we have proved (4.7) (recalling the form of $\partial_j(\Pi_h \underline{q})$).

For $k = 1$, $\Pi_h \underline{q}$ has the form

$$\Pi_h \underline{q}(x) = \begin{pmatrix} a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_2 + a_4 x_1 x_2 + a_5 x_1^2 x_2 \\ b_0 + b_1 x_2 + b_2 x_2^2 + b_3 x_1 + b_4 x_1 x_2 + b_5 x_1 x_2^2 \end{pmatrix},$$

for some real numbers $a_i, b_i, i = 0, \cdots, 5$. Consequently we get

$$\partial_1 \Pi_h \underline{q} = \begin{pmatrix} a_1 + 2a_2 x_1 + a_4 x_2 + 2a_5 x_1 x_2 \\ b_3 + b_4 x_2 + b_5 x_2^2 \end{pmatrix}.$$

For the estimation of $\partial_1(\Pi_h \underline{q})_1$, applying Green's formula and the interpolation properties we have

$$\begin{aligned} \int_T |\partial_1(\Pi_h \underline{q})_1|^2 &= -\int_T (\Pi_h \underline{q})_1 \partial_1^2 (\Pi_h \underline{q})_1 + \int_{\partial T} n_1 (\Pi_h \underline{q})_1 \partial_1 (\Pi_h \underline{q})_1 \\ &= -\int_T (\Pi_h \underline{q})_1 \partial_1^2 (\Pi_h \underline{q})_1 + \int_{E_2 \cup E_4} n_1 (\Pi_h \underline{q})_1 \partial_1 (\Pi_h \underline{q})_1 \\ &= -\int_T (\Pi_h \underline{q})_1 \partial_1^2 (\Pi_h \underline{q})_1 + \int_{E_2 \cup E_4} (\Pi_h \underline{q}) \cdot n \partial_1 (\Pi_h \underline{q})_1 \\ &= -\int_T (\Pi_h \underline{q})_1 \partial_1^2 \underline{q}_1 + \int_{E_2 \cup E_4} \underline{q} \cdot n \partial_1 (\Pi_h \underline{q})_1 \\ &= \int_T \partial_1 (\Pi_h \underline{q})_1 \partial_1 \underline{q}_1. \end{aligned}$$

By Cauchy-Schwarz's inequality we obtain

$$\|\partial_1(\Pi_h \underline{q})_1\|_T \leq \|\partial_1 \underline{q}_1\|_T.$$

By symmetry we actually have

$$\|\partial_j(\Pi_h \underline{q})_j\|_T \leq \|\partial_j \underline{q}_j\|_T \text{ for } j = 1, 2. \tag{4.9}$$

For the estimation of $\partial_1(\Pi_h \underline{q})_2$, recalling that it is constant we may start with

$$\partial_1(\Pi_h \underline{q})_2 \int_T x_1(h_1 - x_1) = \int_T \partial_1(\Pi_h \underline{q})_2 x_1(h_1 - x_1),$$

where (x_1, x_2) are local Cartesian coordinates such that E_4 is a subset of the x_2 axis and E_2 is a subset of the line $x_1 = h_1$. In the above right-hand side, applying Green's formula we get

$$\partial_1(\Pi_h \underline{q})_2 \int_T x_1(h_1 - x_1) = -\int_T (\Pi_h \underline{q})_2 \partial_1 [x_1(h_1 - x_1)],$$

since the boundary term is zero. Using the interpolation properties we obtain

$$\begin{aligned} \partial_1(\Pi_h \underline{q})_2 \int_T x_1(h_1 - x_1) &= -\int_T (\Pi_h \underline{q})_2 (h_1 - 2x_1) \\ &= -\int_T (\Pi_h \underline{q}) \cdot \begin{pmatrix} 0 \\ h_1 - 2x_1 \end{pmatrix} \\ &= -\int_T \underline{q} \cdot \begin{pmatrix} 0 \\ h_1 - 2x_1 \end{pmatrix} \\ &= -\int_T \underline{q}_2 (h_1 - 2x_1). \end{aligned}$$

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This proves the identity

$$\partial_1(\Pi_h \underline{q})_2 = -\frac{\int_T \underline{q}_2(h_1 - 2x_1)}{\int_T x_1(h_1 - x_1)}.$$

Cauchy-Schwarz's inequality and direct calculations yield

$$|\partial_1(\Pi_h \underline{q})_2| \lesssim h_1^{-1}|T|^{-1/2}\|\underline{q}_2\|_T.$$

Integrating the square of this inequality on T leads to

$$\|\partial_1(\Pi_h \underline{q})_2\|_T \lesssim h_1^{-1}\|\underline{q}_2\|_T.$$

Exchanging the role of 1 and 2, we have proved that

$$\|\partial_j(\Pi_h \underline{q})_k\|_T \lesssim h_j^{-1}\|\underline{q}_k\|_T \text{ for } j \neq k. \tag{4.10}$$

The estimates (4.9) and (4.10) immediately give (4.8). □

Obviously the above result is still valid for a 3D triangulation made of rectangular hexahedra with RT_0 or RT_1.

Let us go on with the case of pentahedra.

LEMMA 4.5. *Assume given a 3D triangulation $\mathcal{T}$ made of rectangular pentahedra $T = T_1 \times I$, where I is a real interval and T_1 is a 2D triangle, which satisfies*

$$h_{1,T} \lesssim \sin\theta_{max,T_1}. \tag{4.11}$$

Assume that (X_h, M_h) corresponds to the Raviart-Thomas element of order 0 (i.e. defined by (4.1)-(4.2) with $k = 0$). Then (2.9) holds.

Proof. Arguments like Lemmas 4.1 and 4.2 of [1] yield

$$\|\underline{q}_i - (\Pi_h \underline{q})_i\|_T \lesssim \frac{h_{1,T}}{\sin\theta_{max,T_1}}\|\nabla(\underline{q} - \Pi_h \underline{q})\|_T \text{ for } i = 1, 2,$$
$$\|\underline{q}_3 - (\Pi_h \underline{q})_3\|_T \lesssim h_{1,T}\|\nabla(\underline{q} - \Pi_h \underline{q})\|_T.$$

The assumption (4.11) then yields

$$\|\underline{q} - \Pi_h \underline{q}\|_T \lesssim \|\nabla(\underline{q} - \Pi_h \underline{q})\|_T. \tag{4.12}$$

It then remains to estimate $\|\nabla \Pi_h \underline{q}\|_T$. Remarking that

$$\Pi_h \underline{q}(x) = \begin{pmatrix} a_0 + a_1 x_1 \\ b_0 + a_1 x_2 \\ c_0 + c_1 x_3 \end{pmatrix},$$

for some real numbers $a_i, b_i, c_i, i = 0, 1, 2$, we see that

$$\partial_1 \Pi_h \underline{q} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \partial_2 \Pi_h \underline{q} = a_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \partial_3 \Pi_h \underline{q}(x) = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which in particular imply div $\Pi_h \underline{q} = 2a_1 + c_1$.

Denote by $E_i, i = 1, 2$ the two faces of T perpendicular to the x_3 axis. As before by Green's formula and the interpolation properties, we may successively write

$$\begin{aligned}\int_T \partial_3(\Pi_h \underline{q}\,)_3 &= \int_{\partial T} n_3(\Pi_h \underline{q}\,)_3 = \int_{E_1\cup E_2} n_3(\Pi_h \underline{q}\,)_3 \\ &= \int_{E_1\cup E_2} \Pi_h \underline{q}\,\cdot n = \int_{E_1\cup E_2} \underline{q}\,\cdot n \\ &= \int_T \partial_3 \underline{q}\,_3.\end{aligned}$$

By the fact that $\partial_3(\Pi_h \underline{q}\,)_3$ is constant and by Cauchy-Schwarz's inequality we obtain

$$|\partial_3(\Pi_h \underline{q}\,)_3| \le |T|^{-1/2} \|\partial_3 \underline{q}\,_3\|_T.$$

Integrating the square of this estimate on T we arrive at

$$\|\partial_3(\Pi_h \underline{q}\,)_3\|_T^2 \le \|\partial_3 \underline{q}\,_3\|_T^2.$$

A similar argument leads to

$$\|\mathrm{div}\,\Pi_h \underline{q}\,\|_T^2 \le \|\mathrm{div}\,\underline{q}\,\|_T^2.$$

By the form of $\partial_i(\Pi_h \underline{q}\,)$, the two above estimates imply that

$$\|\nabla(\Pi_h \underline{q}\,)\|_T^2 \lesssim \|\nabla \underline{q}\,\|_T^2.$$

This estimate in (4.12) gives

$$\|\underline{q}\, - \Pi_h \underline{q}\,\|_T \lesssim \|\nabla \underline{q}\,\|_T,$$

which leads to the conclusion. □

We end this section by showing that the approximation measure a is bounded from above by 1 for isotropic meshes:

LEMMA 4.6. *For any isotropic mesh $\mathcal{T}$ and the above finite element spaces,*

$$a(\underline{q}\,, \mathcal{T}\,) \lesssim 1, \forall \underline{q}\, \in [H^1(\Omega)]^d.$$

Proof. By the proof of Lemma 4.1, we have

$$\|\underline{q}\, - \Pi_h \underline{q}\,\|_T^2 \lesssim \|A_T\|^4 \|A_T^{-1}\|^2 \int_T |\nabla \underline{q}\,|^2.$$

Since for an isotropic mesh we have $\|A_T\| \sim h_{1,T}$ and $\|A_T^{-1}\| \sim h_{min,T}^{-1} \sim h_{1,T}^{-1}$, we get

$$h_{min,T}^{-2} \|\underline{q}\, - \Pi_h \underline{q}\,\|_T^2 \sim h_{1,T}^{-2} \|\underline{q}\, - \Pi_h \underline{q}\,\|_T^2 \lesssim \int_T |\nabla \underline{q}\,|^2.$$

We conclude by summing this estimate on $T \in \mathcal{T}$. □

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5. Error estimators.

5.1. Residual error estimators.
For $\underline{p}_h \in X_h$ we define the jump of $A^{-1}\underline{p}_h$ in the tangential direction across a face E by

$$\underline{J}_{E,t}(\underline{p}_h) := \begin{cases} [\![A^{-1}\underline{p}_h \cdot \underline{t}_E]\!]_E & \text{in 2D,} \\ [\![A^{-1}\underline{p}_h \times \underline{n}_E]\!]_E & \text{in 3D.} \end{cases}$$

In 2D, $\underline{J}_{E,t}(\underline{p}_h)$ is a scalar quantity, but for shortness we write it as a vector, allowing us to treat the 2D and 3D cases in the same time.

DEFINITION 5.1 (Residual error estimator). *For any $T \in \mathcal{T}$, the local residual error estimator is defined by*

$$\begin{aligned}\eta_T^2 &:= \|f + \operatorname{div} \underline{p}_h\|_T^2 + h_{min,T}^2 \|\underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h)\|_T^2 \\ &+ h_{min,T}^2 \min_{v_h \in M_h} \|A^{-1}\underline{p}_h - \nabla v_h\|_T^2 + \sum_{E \subset \partial T} \frac{h_{min,T}^2}{h_E} \|\underline{J}_{E,t}(\underline{p}_h)\|_E^2.\end{aligned}$$

The global residual error estimator is simply

$$\eta^2 := \sum_{T \in \mathcal{T}} \eta_T^2.$$

5.2. Proof of the lower error bound.
We proceed as in [8] with the necessary adaptation due to the anisotropy of the meshes (compare with [18, 12]).

THEOREM 5.2 (Lower error bound). *Assume that there exists $k \in \mathbb{N}$ such that $(A^{-1}\underline{p}_h)_{|T}$ belongs to $\mathbb{P}^k$, for all $T \in \mathcal{T}$. Then for all elements T, the following local lower error bound holds:*

$$\eta_T \lesssim \|\underline{\epsilon}\|_{H(\operatorname{div},\omega_T)} + \|e\|_T. \tag{5.1}$$

Proof. **Curl residual** By the inverse inequality (3.1) and Green's formula, one has

$$\begin{aligned}\|\underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h)\|_T^2 &\sim \int_T b_T |\underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h)|^2 \\ &= -\int_T b_T \underline{\operatorname{curl}}\,(A^{-1}\underline{\epsilon}) \cdot \underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h) \\ &= -\int_T (A^{-1}\underline{\epsilon}) \cdot \underline{\operatorname{curl}}\,(b_T \underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h)) \\ &\le \|A^{-1}\underline{\epsilon}\|_T \|\underline{\operatorname{curl}}\,(b_T \underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h))\|_T.\end{aligned}$$

The inverse inequality (3.2) yields

$$h_{min,T} \|\underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h)\|_T \lesssim \|\underline{\epsilon}\|_T. \tag{5.2}$$

Tangential jump Set

$$\underline{w}_E := \mathrm{F}_{\text{ext}}(\underline{J}_{E,t}(\underline{p}_h)) b_E,$$

which belongs to $H_0^1(\omega_E)$ in 2D and to $[H_0^1(\omega_E)]^3$ in 3D. The inverse inequality (3.3) yields

$$\|\underline{J}_{E,t}(\underline{p}_h)\|_E^2 \lesssim \int_E \underline{J}_{E,t}(\underline{p}_h) \cdot \underline{w}_E = -\int_E \underline{J}_{E,t}(\underline{\epsilon}) \cdot \underline{w}_E.$$

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Elementwise integration yields

$$\begin{aligned}\|\underline{J}_{E,t}(\underline{p}_h)\|_E^2 &\lesssim \sum_{T\subset\omega_E}\int_T\{(A^{-1}\underline{\epsilon})\cdot \underline{\text{curl}}\ \underline{w}_E - \underline{\text{curl}}\ (A^{-1}\underline{\epsilon})\cdot \underline{w}_E\}\\ &= \sum_{T\subset\omega_E}\int_T\{(A^{-1}\underline{\epsilon})\cdot \underline{\text{curl}}\ \underline{w}_E + \underline{\text{curl}}\ (A^{-1}\underline{p}_h)\cdot \underline{w}_E\}\\ &\lesssim \|A^{-1}\underline{\epsilon}\|_{\omega_E}\|\underline{\text{curl}}\ \underline{w}_E\|_{\omega_E} + \sum_{T\subset\omega_E}\|\underline{\text{curl}}\ (A^{-1}\underline{p}_h)\|_T\|\underline{w}_E\|_{\omega_E}.\end{aligned}$$

By the estimate (5.2) we get

$$\|\underline{J}_{E,t}(\underline{p}_h)\|_E^2 \lesssim \|\underline{\epsilon}\|_{\omega_E}(\|\underline{\text{curl}}\ \underline{w}_E\|_{\omega_E} + h_{min,T}^{-1}\|\underline{w}_E\|_{\omega_E}).$$

The inverse inequalities (3.4) and (3.5) lead to

$$\|\underline{J}_{E,t}(\underline{p}_h)\|_E \lesssim \frac{h_E^{1/2}}{h_{min,T}}\|\underline{\epsilon}\|_{\omega_E}. \tag{5.3}$$

Element residual The inverse inequality (3.1) and the fact that $\underline{p} = A\nabla u$ yield

$$\begin{aligned}\|A^{-1}\underline{p}_h - \nabla u_h\|_T^2 &\sim \int_T b_T(A^{-1}\underline{p}_h - \nabla u_h)\cdot(A^{-1}\underline{p}_h - \nabla u_h)\\ &\sim \int_T b_T(A^{-1}\underline{\epsilon} - \nabla e)\cdot(A^{-1}\underline{p}_h - \nabla u_h).\end{aligned}$$

Using Green's formula we get

$$\|A^{-1}\underline{p}_h - \nabla u_h\|_T^2 \lesssim \int_T b_T(A^{-1}\underline{\epsilon})\cdot(A^{-1}\underline{p}_h - \nabla u_h) + \int_T e\ \text{div}\,(b_T(A^{-1}\underline{p}_h - \nabla u_h)).$$

Cauchy-Schwarz's inequality and the inverse inequality (3.2) lead to

$$h_{min,T}\|A^{-1}\underline{p}_h - \nabla u_h\|_T \lesssim \|\underline{\epsilon}\|_T + \|e\|_T. \tag{5.4}$$

Using the estimates (5.2) and (5.3) and (5.4) provides the desired bound (5.1). □

REMARK 5.3. The assumption of theorem 5.2 is not always fulfilled, even if $\underline{p}_h$ is elementwise polynomial, since A^{-1} is not necessarily elementwise polynomial. However, it holds if A is piecewise constant.

5.3. Proof of the upper error bound. The use of Lemma 3.5 allows to prove the following error bound on e.

LEMMA 5.4. *Let $\underline{v} \in [H^1(\Omega)]^d$ be the solution of (3.11) with $g = e$ and satisfying (3.12), obtained in Lemma 3.5. Then the next estimate holds*

$$\|e\| \lesssim \|\underline{\epsilon}\| + a(\underline{v},\mathcal{T})\eta. \tag{5.5}$$

Proof. By (3.11) we may write

$$\|e\|^2 = \int_\Omega (u-u_h)\text{div}\ \underline{v}.$$

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By Green's formula and the fact that $\nabla u = A^{-1}\underline{p}$ (recall that $u = 0$ on $\partial\Omega$) we get

$$\|e\|^2 = -\int_\Omega (A^{-1}\underline{p}) \cdot \underline{v} - \int_\Omega u_h \mathrm{div}\, \underline{v}.$$

Now using the commuting property (2.8) we obtain

$$\|e\|^2 = -\int_\Omega (A^{-1}\underline{p}) \cdot \underline{v} - \int_\Omega u_h \mathrm{div}\, \Pi_h \underline{v}.$$

The discrete mixed formulation (2.1) then leads to

$$\|e\|^2 = -\int_\Omega (A^{-1}(\underline{p} - \underline{p}_h)) \cdot \underline{v} - \int_\Omega (A^{-1}\underline{p}_h) \cdot (\underline{v} - \Pi_h \underline{v}).$$

Since Green's formula on each element and the properties (2.8) and (2.10) imply that

$$\sum_{T\in\mathcal{T}} \int_T \nabla v_h \cdot (\underline{v} - \Pi_h \underline{v}) = 0, \forall v_h \in M_h,$$

we have shown that

$$\begin{aligned}\|e\|^2 = &-\int_\Omega (A^{-1}(\underline{p} - \underline{p}_h)) \cdot \underline{v} \\ &- \sum_{T\in\mathcal{T}} \int_T (A^{-1}\underline{p}_h - \nabla v_h) \cdot (\underline{v} - \Pi_h \underline{v}), \forall v_h \in M_h.\end{aligned}$$

Now Cauchy-Schwarz's inequality leads to

$$\begin{aligned}\|e\|^2 \leq\ &\|A^{-1}(\underline{p} - \underline{p}_h)\|\|\underline{v}\| \\ &+ \sum_{T\in\mathcal{T}} \|A^{-1}\underline{p}_h - \nabla v_h\|_T \|\underline{v} - \Pi_h \underline{v}\|_T, \forall v_h \in M_h.\end{aligned}$$

Using the definition of the approximation measure a we obtain

$$\|e\|^2 \leq \left(\|A^{-1}(\underline{p} - \underline{p}_h)\| + a(\underline{v}, \mathcal{T})(\sum_{T\in\mathcal{T}} h^2_{min,T} \|A^{-1}\underline{p}_h - \nabla v_h\|^2_T)^{1/2} \right) \|\underline{v}\|_{1,\Omega},$$

for any $v_h \in M_h$. The conclusion follows from the estimate (3.12). ◻

Comparing the above lemma with Lemma 5.2 of [8], we remark that the use of Lemma 3.5 allows to avoid the H^2-regularity of the solution of (1.1).

It remains to estimate the error bound on $\underline{\epsilon}$, which is obtained by adapting Lemma 5.1 of [8]. We start with a Helmholtz like decomposition of this error.

LEMMA 5.5. *There exist $z \in H^1_0(\Omega)$ and $\underline{\beta} \in H^1(\Omega)$ in 2D or $\underline{\beta} \in [H^1(\Omega)]^3$ in 3D such that*

$$\underline{\epsilon} = A\nabla z + \underline{\mathrm{curl}}\ \underline{\beta}, \tag{5.6}$$

with the estimates

$$\|z\|_{1,\Omega} \lesssim \|\underline{\epsilon}\| \tag{5.7}$$

$$\|\underline{\beta}\|_{1,\Omega} \lesssim \|\underline{\epsilon}\|. \tag{5.8}$$

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Proof. Firstly we consider $z \in H_0^1(\Omega)$ as the unique solution of $\operatorname{div}(A\nabla z) = \operatorname{div} \underline{\epsilon}$, i.e., solution of

$$\int_\Omega (A\nabla z)\cdot\nabla w = \int_\Omega \underline{\epsilon}\cdot\nabla w, \forall w \in H_0^1(\Omega),$$

which clearly satisfies (5.7). Secondly we remark that $\underline{\epsilon} - A\nabla z$ is divergence free so by Theorem I.3.1 or I.3.4 of [17], there exists $\underline{\beta} \in H^1(\Omega)$ in 2D or $\underline{\beta} \in [H^1(\Omega)]^3$ in 3D such that

$$\underline{\operatorname{curl}}\, \underline{\beta} = \underline{\epsilon} - A\nabla z$$

with the estimate

$$\|\underline{\beta}\|_{1,\Omega} \lesssim \|\underline{\epsilon} - A\nabla z\|,$$

which leads to (5.8) thanks to (5.7). □

For the sake of shortness, in the above lemma, we use exceptionally the notation $\underline{\beta}$ in 2D for the scalar function appearing in the decomposition (5.6).

LEMMA 5.6. *If z and $\underline{\beta}$ are from Lemma 5.5 then the next estimate holds*

$$\|\underline{\epsilon}\| \lesssim (1 + m_1(\underline{\beta}, \mathcal{T}))\eta. \tag{5.9}$$

Proof. Since Green's formula yields

$$\int_\Omega \nabla z \cdot \underline{\operatorname{curl}}\, \underline{\beta} = 0,$$

we may write

$$\int_\Omega (A^{-1}\underline{\epsilon})\cdot\underline{\epsilon} = \int_\Omega (\nabla z)\cdot\underline{\epsilon} + \int_\Omega (A^{-1}\underline{\operatorname{curl}}\, \underline{\beta})\cdot\underline{\operatorname{curl}}\, \underline{\beta}. \tag{5.10}$$

We now estimate separetely the two terms of this right-hand side. For the first one applying Green's formula we get

$$\int_\Omega (\nabla z)\cdot\underline{\epsilon} = -\int_\Omega z \operatorname{div} \underline{\epsilon}$$

By Cauchy-Schwarz's inequality we obtain

$$\left|\int_\Omega (\nabla z)\cdot\underline{\epsilon}\right| \lesssim \|\operatorname{div} \underline{\epsilon}\| \|z\|_{1,\Omega}.$$

Using finally the fact that $\operatorname{div} \underline{p} = -f$ and the estimate (5.7), we conclude

$$\left|\int_\Omega (\nabla z)\cdot\underline{\epsilon}\right| \lesssim \|f + \operatorname{div} \underline{p}_h\| \|\underline{\epsilon}\|. \tag{5.11}$$

For the second term of the right-hand side of (5.10) we take $\underline{\beta}_h = \mathrm{I}_{\mathrm{Cl}}\underline{\beta}$. By (5.6) and Green's formula, we have

$$\int_\Omega (A^{-1}\underline{\operatorname{curl}}\, \underline{\beta})\cdot\underline{\operatorname{curl}}\, \underline{\beta}_h = \int_\Omega (A^{-1}\underline{\epsilon})\cdot\underline{\operatorname{curl}}\, \underline{\beta}_h.$$

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As $\underline{\operatorname{curl}}\ \underline{\beta}_h$ belongs to X_h (due to (2.5)), by the orthogonality relation (2.2) the above identity becomes

$$\int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta}_h = \int_\Omega e \operatorname{div} \underline{\operatorname{curl}}\ \underline{\beta}_h = 0.$$

This identity allows to write

$$\int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta} = \int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ (\underline{\beta} - \underline{\beta}_h).$$

Using the Helmholtz decomposition (5.6) and the fact that $\underline{p} = A\nabla u$ it becomes

$$\int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta} = \int_\Omega (\nabla(u-z) - A^{-1}\underline{p}_h) \cdot \underline{\operatorname{curl}}\ (\underline{\beta} - \underline{\beta}_h).$$

Green's formula in Ω (the boundary term being zero since $u - z = 0$ on the boundary) leads to

$$\int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta} = -\int_\Omega (A^{-1}\underline{p}_h) \cdot \underline{\operatorname{curl}}\ (\underline{\beta} - \underline{\beta}_h).$$

Now applying Green's formula on each element T we get

$$\begin{aligned}\int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta} &= -\sum_{T\in\mathcal{T}} \int_T \underline{\operatorname{curl}}\ (A^{-1}\underline{p}_h) \cdot (\underline{\beta} - \underline{\beta}_h) \\ &\quad + \sum_{E\in\mathcal{E}} \int_E \underline{J}_{E,t}(\underline{p}_h) \cdot (\underline{\beta} - \underline{\beta}_h).\end{aligned}$$

Continuous and discrete Cauchy-Schwarz's inequalities yield

$$\begin{aligned}&\left| \int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta} \right| \le \\ &\quad \left(\sum_{T\in\mathcal{T}} h_{min,T}^2 \|\underline{\operatorname{curl}}\ (A^{-1}\underline{p}_h)\|_T^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}} h_{min,T}^{-2} \|\underline{\beta} - \underline{\beta}_h\|_T^2 \right)^{1/2} \\ &\quad + \left(\sum_{E\in\mathcal{E}} h_{min,E}^2 h_E^{-1} \|\underline{J}_{E,t}(\underline{p}_h)\|_E^2 \right)^{1/2} \left(\sum_{E\in\mathcal{E}} h_{min,E}^{-2} h_E \|\underline{\beta} - \underline{\beta}_h\|_E^2 \right)^{1/2}.\end{aligned}$$

By Lemma 3.4 we obtain

$$\left| \int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta} \right| \lesssim m_1(\underline{\beta}, \mathcal{T})\eta \|\nabla \underline{\beta}\|.$$

According to (5.8) we arrive at the estimate

$$\left| \int_\Omega (A^{-1}\underline{\operatorname{curl}}\ \underline{\beta}) \cdot \underline{\operatorname{curl}}\ \underline{\beta} \right| \lesssim m_1(\underline{\beta}, \mathcal{T})\eta \|\underline{\epsilon}\|. \tag{5.12}$$

The conclusion directly follows from the identity (5.10) and the estimates (5.11) and (5.12). □

Using the two above Lemmas and recalling that $\operatorname{div} \underline{\epsilon} = -(f + \operatorname{div} \underline{p}_h)$ we have obtained the

THEOREM 5.7 (Upper error bound). *Let $\underline{v} \in [H^1(\Omega)]^d$ be the function from Lemma 5.4 and $\underline{\beta}$ the function from Lemma 5.5. Then the error is bounded globally from above by*

$$\|e\| + \|\underline{\epsilon}\|_{H(\operatorname{div},\Omega)} \lesssim (1 + a(\underline{v}, \mathcal{T}) + m_1(\underline{\beta}, \mathcal{T}))\eta. \tag{5.13}$$

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5.4. Applications to isotropic meshes. Our results apply to any element pairs from Section 4 on isotropic meshes. In that case we have $h_{min,T} \sim h_T \sim h_F$ for all faces F of T (recall that h_T is the diameter of T), $m_1(\cdot, \mathcal{T}) \sim 1$ and $a(\cdot, \mathcal{T}) \lesssim 1$. As a consequence the above results may rephrased as follows: the local residual error estimator is given by (see [8])

$$\eta_T^2 := \|f + \operatorname{div} \underline{p}_h\|_T^2 + h_T^2 \|\underline{\operatorname{curl}}\,(A^{-1}\underline{p}_h)\|_T^2 \\ + h_T^2 \min_{v_h \in M_h} \|A^{-1}\underline{p}_h - \nabla v_h\|_T^2 + h_T \sum_{E \subset \partial T} \|\underline{J}_{E,t}(\underline{p}_h)\|_E^2.$$

With this definition the lower error bound (5.1) holds under the same assumption on $\underline{p}_h$ than in Theorem 5.2, while the upper error bound (5.13) reduces to

$$\|e\| + \|\underline{\epsilon}\|_{H(\operatorname{div},\Omega)} \lesssim \eta,$$

without any regularity assumption on the solution of (1.1).

6. Numerical experiments. In this section, we present two $3D$ experiments which confirm the efficiency and reliability of our estimator. The first example treats the case of a smooth solution presenting a boundary layer, while the second example considers the case of a singular solution (not in $H^2(\Omega)$) having an edge singularity. The first example was chosen to show that the alignment and approximation measures are not an obstacle for the efficiency and reliability of the estimator, while the choice of the second example is motivated by the relaxation of the H^2-regularity of the solution.

6.1. Solution with a boundary layer. The present experiments consist in solving the three dimensional mixed problem (2.1) with $A = Id$ on the unit cube $\Omega = (0,1)^3$. Here, we use the Raviart-Thomas element RT_0 described in Section 4, on anisotropic Shishkin type meshes composed of tetrahedra. Each mesh is the tensor product of a 1D Shishkin type mesh and of a uniform 2D mesh, both with n subintervals. With $\tau \in (0,1)$ being a transition point parameter, the coordinates (x_i, y_j, z_k) of the nodes of the hexahedra are defined by

$$dx_1 := 2\tau/n, \quad dx_2 := 2(1-\tau)/n, \quad dy = 1/n, \quad dz = 1/n,$$

$$\left\{\begin{array}{lcll} x_i & := & i\,dx_1 & (0 \le i \le n/2), \\ x_i & := & \tau + (i - n/2)\,dx_2 & (n/2 + 1 \le i \le n), \\ y_j & := & j\,dy & (0 \le j \le n), \\ z_k & := & k\,dz & (0 \le k \le n). \end{array}\right.$$

Each hexahedron is then divided in three tetrahedra, without adding any node (see Figure 6.1).

The discrete problem (2.1) is solved with an Uzawa-type algorithm. The number of degrees of freedom for the determination of $\underline{p}_h$ is equal to the number of faces NF of the mesh. The tests are performed with the following prescribed exact solution u :

$$u(x,y,z) \quad = \quad x(1-x)y(1-y)z(1-z)e^{-\frac{x}{\sqrt{\varepsilon}}}.$$

This allows to have in particular $u_{|\Gamma} = 0$. Note that $\frac{\partial u}{\partial x}$ presents an exponential boundary layer along the line $x = 0$ that does not converge uniformly towards zero when ε goes towards zero. The transition parameter τ involved in the construction of the Shishkin-type mesh is defined by $\tau := \min\{1/2, 2\sqrt{\varepsilon}|\ln\sqrt{\varepsilon}|\}$, which is roughly twice the boundary layer width. The maximal aspect ratio in the mesh is equal to $1/(2\tau)$.

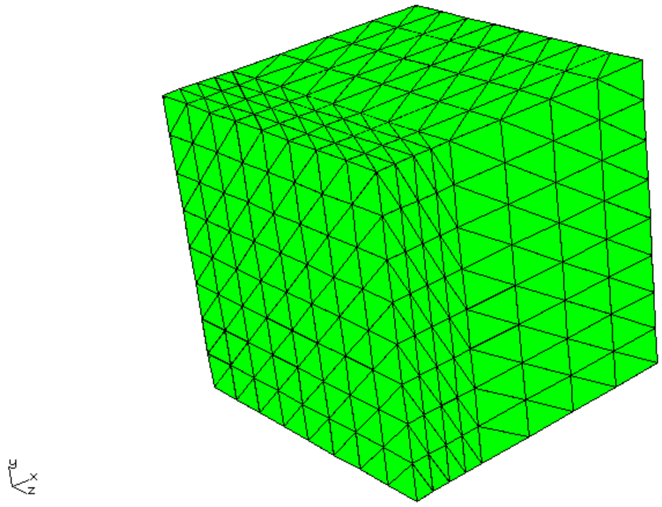

FIG. 6.1. *Shishkin type mesh on the unit cube with $n = 8$ and $\tau = 0.25$.*

Now we investigate the main theoretical results which are the upper and the lower error bounds. In order to present the underlying inequalities (5.13) and (5.1) appropriately, we reformulate them by defining the ratios of left-hand side and right-hand side, respectively:

- $q_{\text{up}} = \dfrac{||e|| + ||\underline{\epsilon}||_{H(\text{div},\Omega)}}{\eta}$ as a function of NF,
- $q_{\text{low}} = \max\limits_{T\in\mathcal{T}} \dfrac{\eta_T}{||\underline{\epsilon}||_{H(\text{div},\omega_T)} + ||e||_T}$ as a function of NF.

The first ratio q_{up} is frequently referred to as *effectivity index*. It measures the *reliability* of the estimator and is related to the global upper error bound. In order to investigate this error bound, recall first that the factor $(1 + a(\underline{v}\,, \mathcal{T}) + m_1(\beta, \mathcal{T}))$ is expected to be of moderate size since we employ well adapted meshes (cf. Theorem 5.7). Hence the corresponding ratio q_{up} should be bounded from above. This is actually confirmed by the experiments (left part of Figure 6.2), where we even notice that the quality of the upper error bound is independent of ε. Thus the estimator is *reliable*.

The second ratio is related to the local lower error bound and measures the *efficiency* of the estimator. According to Theorem 5.2, q_{low} has to be bounded from above. This can be observed indeed in the right part of Figure 6.2, as soon as a sufficiently resolution of the boundary layer is achieved (the smaller ε is, the larger NF must be). Hence the estimator is *efficient*.

6.2. Singular solution. Let us now consider the three dimensional mixed problem (2.1) with $A = Id$ on the truncated cylinder domain Ω defined in the usual cylindrical system of

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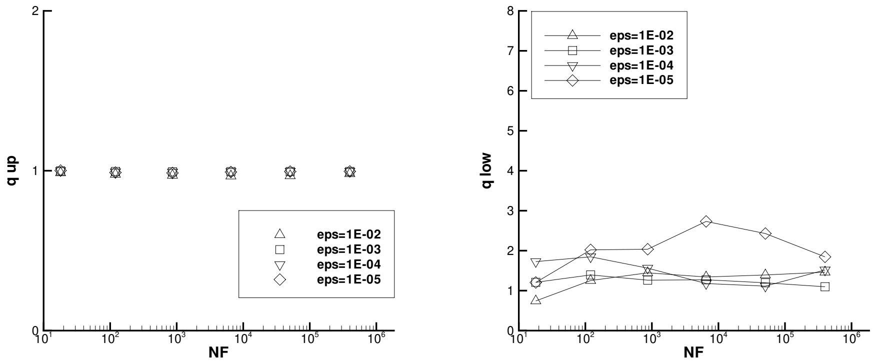

FIG. 6.2. $q_{\rm up}$ *(left) and* $q_{\rm low}$ *(right) in dependence of* NF*, anisotropic solutions.*

coordinates (r, θ, z) by :

$$\begin{cases} 0 \le r \le 0.1, \\ 0 \le \theta \le \frac{3\pi}{2}, \\ 0 \le z \le 0.1. \end{cases}$$

The tests are performed with the following prescribed exact solution u satisfying the homogeneous Dirichlet boundary conditions on $\partial\Omega$ and defined by :

$$u(r, \theta, z) = r^{\frac{2}{3}}(0.1 - r) \sin\left(\frac{2\,\theta}{3}\right) z(0.1 - z).$$

This solution u does not belong to $H^2(\Omega)$, and has the typical edge singular behaviour near the edge $r = 0$. Because of this edge singularity, the mesh is refined in the radial direction near the axis of the cylinder, making it anisotropic (see Figure 6.3). The finite element and the algorithm are the same as in Section 6.1.

Once again, we plot $q_{\rm up}$ and $q_{\rm low}$ defined in Section 6.1 versus NF. This is done in Figure 6.4. Each of these two parameters is bounded from above. That confirms that the estimator is actually reliable and efficient, even for a singular solution as theoretically expected.

The tests presented in this section have been performed with the help of the NETGEN mesh generator (Johannes Kepler University of Linz in Austria) and the SIMULA+ finite element code (MACS, University of Valenciennes and LPMM, University and ENSAM of Metz, both in France).

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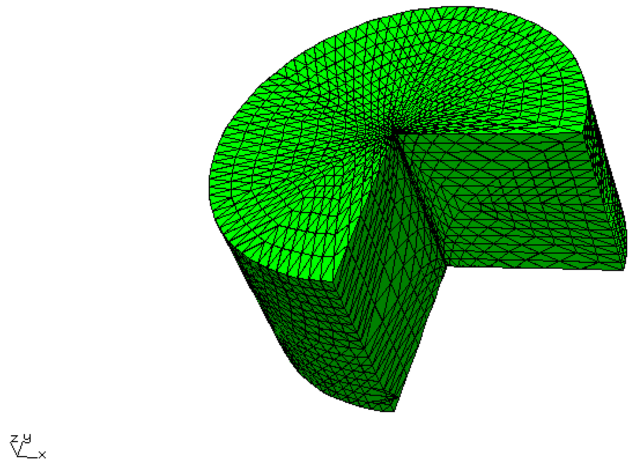

FIG. 6.3. *truncated cylinder mesh refined near the axis.*

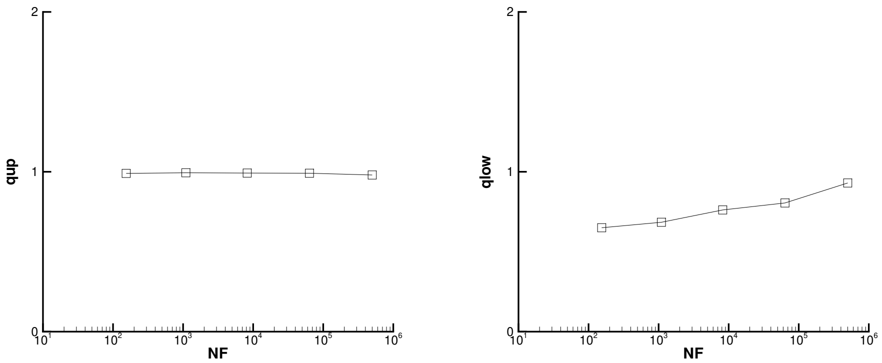

FIG. 6.4. $q_{\rm up}$ *(left) and* $q_{\rm low}$ *(right) in dependence of* NF*, singular solution.*

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