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APPROXIMATION OF THE HILBERT TRANSFORM VIA USE OF SINC CONVOLUTION*

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Abstract. This paper derives a novel method of approximating the Hilbert transform by the use of sinc convolution. The proposed method may be used to approximate the Hilbert transform over any subinterval Γ of the real line $\mathbb{R} \equiv (-\infty, \infty)$, which means the interval Γ may be a finite or semi-infinite interval, or the entire real line $\mathbb{R}$. Given a column vector $\boldsymbol{f}$ consisting of m values of a function f defined on m sinc points of Γ, we obtain a column vector $\boldsymbol{g} = \boldsymbol{Af}$ whose entries approximate the Hilbert transform on the same set of m sinc points. The present paper describes an explicit method for the construction of such a matrix $\boldsymbol{A}$.

Key words. Sinc methods, Hilbert transform, Cauchy principal value integral

AMS subject classifications. 65R10

1. Introduction and summary. In this paper, we derive a novel collocation formula for approximating the Hilbert transform

$$(\mathcal{H}g)(x) \equiv \frac{\mathrm{P.V.}}{\pi} \int_\Gamma \frac{g(t)}{t-x}\, dt,$$

via use of sinc convolution, which is a method of approximating convolutions [3, §4.6]. The symbol $\mathrm{P.V.}$ denotes the Cauchy principal value as usual.

We restrict our examples to the case in which Γ is either the real line $\mathbb{R} \equiv (-\infty, \infty)$ or a subinterval of $\mathbb{R}$, although we can easily extend the proposed formula to more general analytic arcs. Methods for such approximations have previously been obtained via use of a set of basis functions derived for this purpose in [3, §5.2]. However, the method derived in this paper is more accurate for the purpose of collocating the Hilbert transform.

In Section 2, which follows, we describe the sinc tools which we shall require to derive our formula for approximating $\mathcal{H}g$ on Γ. In Section 3, we derive our novel formula for the Hilbert transform, and prescribe explicit conditions under which we can expect our derived formula to be accurate. In Section 4, we give some illustrative examples using our formula.

2. The requisite sinc tools. We review here some tools for sinc approximation which we shall require to derive our collocation formula for approximating $\mathcal{H}g$ on Γ. We first introduce some spaces for sinc approximation, and then describe sinc interpolation, sinc indefinite integration and sinc convolution. The results in this section are derived in [3], and therefore the proofs are omitted. The reader can also find a more detailed summary of sinc methods in [2, 4].

2.1. Sinc spaces. We introduce here two important spaces of functions $\mathbf{L}_{\alpha,d}(\varphi)$ and $\mathbf{M}_{\alpha,d}(\varphi)$.

Given a positive number d, let φ be a conformal map of a domain $\mathcal{D} \subset \mathbb{C}$ onto the strip region $\mathcal{D}_d$ which is defined by

$$\mathcal{D}_d \equiv \{w \in \mathbb{C} : |\Im w| < d\}$$

($\mathbb{C}$ denotes the complex plane). Now, Γ is a general analytic arc which is defined by

$$\Gamma \equiv \{z \in \mathbb{C} : z = \varphi^{-1}(w),\ w \in \mathbb{R}\}.$$

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Let Γ have end points $a \equiv \varphi^{-1}(-\infty)$ and $b \equiv \varphi^{-1}(\infty)$, and define ρ by $\rho(z) = e^{\varphi(z)}$. We then let the class $\mathbf{L}_{\alpha,d}(\varphi)$ denote the family of all functions f that are analytic and uniformly bounded in $\mathcal{D}$ with

$$f(z) = \begin{cases} \mathcal{O}\left(\rho(z)^{\alpha}\right), & z \to a, \\ \mathcal{O}\left(\rho(z)^{-\alpha}\right), & z \to b. \end{cases}$$

Instead of using the strip region $\mathcal{D}_d$, the parameter d is also determined by the condition

$$\tilde{F}(\zeta) = \mathcal{O}(e^{-d|\zeta|}), \qquad \zeta \to \pm\infty,$$

where $\tilde{F}$ is the Fourier transform of $f \circ \varphi^{-1}$, i.e.,

$$\tilde{F}(\zeta) \equiv \int_{\mathbb{R}} f \circ \varphi^{-1}(w)\, e^{i\zeta w}\, dw.$$

The second class $\mathbf{M}_{\alpha,d}(\varphi)$ will denote the family of all functions g that are analytic and bounded in $\mathcal{D}$, and satisfy $f \equiv g - \mathcal{L}g \in \mathbf{L}_{\alpha,d}(\varphi)$, where

$$(\mathcal{L}g)(z) \equiv \frac{g(a) + \rho(z)\, g(b)}{1 + \rho(z)}.$$

Note that $(\mathcal{L}g)(a) = g(a)$ and $(\mathcal{L}g)(b) = g(b)$ because $\rho(a) = 0$ and $\rho(b) = \infty$. Then, it follows that $f(a) = f(b) = 0$.

THEOREM 2.1. *Let $d' \in (0, d)$. The spaces $\mathbf{L}_{\alpha,d}(\varphi)$ and $\mathbf{M}_{\alpha,d}(\varphi)$ have the following properties:*

i. If $f \in \mathbf{M}_{\alpha,d}(\varphi)$, then $f'/\varphi' \in \mathbf{L}_{\alpha,d'}(\varphi)$;
ii. If $f'/\varphi' \in \mathbf{L}_{\alpha,d}(\varphi)$, then $f \in \mathbf{M}_{\alpha,d}(\varphi)$;
iii. If $f \in \mathbf{L}_{\alpha,d}(\varphi)$, then $\int_{\Gamma} |\varphi'(x)\, f(x)|\, |dx| < \infty$;
iv. If $f \in \mathbf{L}_{\alpha,d}(\varphi)$, then $\mathcal{H}f \in \mathbf{M}_{\alpha,d'}(\varphi)$.

2.2. Sinc interpolation on Γ. The sinc function $\mathrm{sinc}(x)$ and the sinc function $S(k,h)(x)$ are defined by

$$\mathrm{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x},$$
$$S(k,h)(x) \equiv \mathrm{sinc}\left(\frac{x}{h} - k\right),$$

where $h > 0$ and $k \in \mathbb{Z}$ ($\mathbb{Z}$ denotes the set of all integers). We then define the sinc basis $\{\omega_j\}_{j=-N}^{N}$ including its end bases and an error term ε_N as follows:

$$h \equiv \left(\frac{\pi d}{\alpha N}\right)^{1/2},$$
$$\gamma_j \equiv S(j,h) \circ \varphi, \qquad (j = -N, -N+1, \ldots, N),$$
$$\omega_j \equiv \gamma_j, \qquad (j = -N+1, -N+2, \ldots, N-1),$$
$$\omega_{-N} \equiv \frac{1}{1+\rho} - \sum_{j=-N+1}^{N} \frac{1}{1+e^{jh}}\, \gamma_j,$$
$$\omega_N \equiv \frac{\rho}{1+\rho} - \sum_{j=-N}^{N-1} \frac{e^{jh}}{1+e^{jh}}\, \gamma_j,$$
$$\varepsilon_N \equiv N^{1/2} e^{-(\pi \alpha d N)^{1/2}},$$

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where α and d are positive numbers used in Section 2.1.

We let $\boldsymbol{w}$ denote the row vector of basis functions

$$\boldsymbol{w}(z) \equiv (\omega_{-N}(z), \omega_{-N+1}(z), \ldots, \omega_N(z)) .$$

We then also define the sinc points on Γ by $z_j \equiv \varphi^{-1}(jh)$ $(j = -N, -N+1, \ldots, N)$, where h is as above. Given a function u defined on Γ, we define a diagonal matrix $\boldsymbol{D}$ and an operator V by

$$\boldsymbol{D}(u) \equiv \mathrm{diag}[u(z_{-N}), u(z_{-N+1}), \ldots, u(z_N)],$$
$$\mathrm{V}u \equiv (u(z_{-N}), u(z_{-N+1}), \ldots, u(z_N))^T.$$

We also define a norm by

$$\|f\| \equiv \sup_{z\in\Gamma} |f(z)|.$$

For sinc interpolation, we have the following theorem.

THEOREM 2.2. *If $f \in \mathbf{M}_{\alpha,d}(\varphi)$, then there exists a constant C, independent of N, such that*

$$\|f - \boldsymbol{w}\,\mathrm{V}f\| < C\,\varepsilon_N.$$

2.3. Sinc collocation on Γ. While sinc interpolation, as implied above, requires the exact values of the interpolated function at the sinc points, sinc methods also have methods of approximating calculus operations on a function, which yield the results by approximate values at the sinc points. We then consider here the interpolation for the case in which the given values are approximate ones.

THEOREM 2.3. *If $f \in \mathbf{M}_{\alpha,d}(\varphi)$, and $\boldsymbol{c} \equiv (c_{-N}, c_{-N+1}, \ldots, c_N)^T$ is a complex vector of order $m \equiv 2N+1$ such that*

$$\left(\sum_{j=-N}^{N} |f(z_j) - c_j|^2\right)^{1/2} < \delta, \qquad \textit{for } \delta > 0,$$

then

$$\|f - \boldsymbol{w}\,\boldsymbol{c}\| < C\varepsilon_N + \delta.$$

When we compute an approximation $\boldsymbol{c}$ to $\mathrm{V}f$ via sinc methods, then the error of this approximation is at most $N^\beta\varepsilon_N$ with β a small positive number. The above error bound on $f - \boldsymbol{w}\,\boldsymbol{c}$ thus shows that the uniform error on Γ in our approximation of f by $\boldsymbol{w}\,\boldsymbol{c}$ is at most twice as large as the error of approximation at the sinc points, i.e., of $\mathrm{V}f$ by $\boldsymbol{c}$.

2.4. Sinc indefinite integration on Γ. Let numbers σ_k and e_k be defined by

$$\sigma_k = \int_0^k \mathrm{sinc}(x)\,dx, \qquad k \in \mathbb{Z},$$
$$e_k = 1/2 + \sigma_k,$$

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and then define a *Toeplitz* matrix $\boldsymbol{I}^{(-1)} = [e_{i-j}]$ with e_{i-j} denoting the $(i,j)^{\text{th}}$ element of $\boldsymbol{I}^{(-1)}$. We furthermore define operators $\mathcal{J}^+$, $\mathcal{J}^-$, $\mathcal{J}_m^+$ and $\mathcal{J}_m^-$, and matrices $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$ as follows:

$$
\begin{aligned}
(\mathcal{J}^+ f)(x) &\equiv \int_a^x f(t)\,dt, \\
(\mathcal{J}^- f)(x) &\equiv \int_x^b f(t)\,dt, \\
(\mathcal{J}_m^+ f)(x) &\equiv \boldsymbol{w}(x)\,\boldsymbol{A}^+\,\mathrm{V} f, \qquad \boldsymbol{A}^+ \equiv h\,\boldsymbol{I}^{(-1)}\,\boldsymbol{D}(1/\varphi'), \\
(\mathcal{J}_m^- f)(x) &\equiv \boldsymbol{w}(x)\,\boldsymbol{A}^-\,\mathrm{V} f, \qquad \boldsymbol{A}^- \equiv h\,(\boldsymbol{I}^{(-1)})^T\boldsymbol{D}(1/\varphi'),
\end{aligned}
$$

where the dependence of $\mathcal{J}_m^\pm$ on m is through the dependence of $\boldsymbol{A}^\pm$, V and $\boldsymbol{w}$ on $m \equiv 2N+1$. We then have the following theorem.

THEOREM 2.4. *If $f/\varphi' \in \mathbf{L}_{\alpha,d}(\varphi)$, then, for all $N > 1$,*

$$
\begin{aligned}
\|\mathcal{J}^+ f - \mathcal{J}_m^+ f\| &= \mathcal{O}\left(\varepsilon_N\right), \\
\|\mathcal{J}^- f - \mathcal{J}_m^- f\| &= \mathcal{O}\left(\varepsilon_N\right).
\end{aligned}
$$

2.5. Sinc convolution on $(a,b) \subseteq \mathbb{R}$. The methods of this section are derived in [3, §4.6]. They enable us to approximate two model integrals which are given by

$$
\begin{aligned}
p(x) &= \int_a^x f(x-t)\,g(t)\,dt, \\
q(x) &= \int_x^b f(t-x)\,g(t)\,dt,
\end{aligned}
$$

where $x \in (a,b)$. The sum of these two integrals constructs the definite convolution

$$
p(x) + q(x) = \int_a^b f(|x-t|)\,g(t)\,dt.
$$

Also, if f is odd, the difference of these two integrals constructs another definite convolution

$$
q(x) - p(x) = \int_a^b f(t-x)\,g(t)\,dt.
$$

We first define the "variant" Laplace transform of f by

$$
\mathcal{F}(s) = \int_E f(t)\,e^{-t/s}\,dt,
$$

where E is any subset of $\mathbb{R}$ such that $E \supseteq (0, b-a)$ and $\Re s > 0$. It may then be shown that

$$
p = \mathcal{F}(\mathcal{J}^+)g, \qquad \text{and also} \qquad q = \mathcal{F}(\mathcal{J}^-)g.
$$

THEOREM 2.5. *Under suitable conditions, e.g., that the integrals p and q should belong to $\mathbf{M}_{\alpha,d}(\varphi)$, we have*

$$
\begin{aligned}
\|p - \mathcal{F}(\mathcal{J}_m^+)g\| &= \mathcal{O}\left(\varepsilon_N\right), \\
\|q - \mathcal{F}(\mathcal{J}_m^-)g\| &= \mathcal{O}\left(\varepsilon_N\right).
\end{aligned}
$$

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We may note here that these approximations can now be evaluated as follows. If $\boldsymbol{A}^+ = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues of $\boldsymbol{A}^+$ and $\boldsymbol{X}$ is the corresponding matrix of eigenvectors, then

$$p \approx \mathcal{F}(\mathcal{J}_m^+)g = \boldsymbol{w}\,\mathcal{F}(\boldsymbol{A}^+)\,\mathrm{V}g = \boldsymbol{w}\boldsymbol{X}\,\mathcal{F}(\boldsymbol{\Lambda})\,\boldsymbol{X}^{-1}\,\mathrm{V}g,$$

and similarly for q. The matrix $\mathcal{F}(\boldsymbol{\Lambda})$ is a diagonal matrix whose diagonal components are the eigenvalues corresponding to $\boldsymbol{\Lambda}$.

3. The Hilbert transform via sinc convolution. In this section, we shall derive our novel collocation formula for approximating the Hilbert transform.

THEOREM 3.1. *If* $\mathcal{H}g \in \mathbf{M}_{\alpha,d}$, *then*

$$\left\| \mathcal{H}g - \frac{1}{\pi}(\log(\mathcal{J}_m^-) - \log(\mathcal{J}_m^+))g \right\| = \mathcal{O}(\varepsilon_N). \tag{3.1}$$

Proof. Let $\Gamma = (a,b) \subseteq \mathbb{R}$. Consider the approximation of the integral

$$H_\varepsilon(x) = \int_a^b f_\varepsilon(t-x)\,g(t)\,dt,$$

where $\varepsilon > 0$, and f_ε is defined by

$$f_\varepsilon(t) = \begin{cases} 0, & t \in (-\varepsilon,\varepsilon), \\ 1/t, & t \in (a,b)\backslash(-\varepsilon,\varepsilon). \end{cases}$$

In this case, we obtain the "variant" Laplace transform of f_ε by

$$\mathcal{F}_\varepsilon(s) \equiv \int_0^\infty f_\varepsilon(t)\,e^{-t/s}\,dt = \int_\varepsilon^\infty \frac{e^{-t/s}}{t}\,dt = E_1(\varepsilon/s),$$

where

$$E_1(z) = -\gamma - \log(z) - G(z),$$
$$G(z) = \sum_{n=1}^\infty \frac{(-1)^n z^n}{n \cdot n!}.$$

The last equality above follows from the equations (5.1.1) and (5.1.11) in [1], and γ denotes Euler's constant.

By proceeding as outlined in Section 2.5, we get

$$H_\varepsilon(x) = \{(\mathcal{F}_\varepsilon(\mathcal{J}^-) - \mathcal{F}_\varepsilon(\mathcal{J}^+))g\}(x),$$

which is approximated by

$$\begin{aligned} H_\varepsilon(x) &\approx \boldsymbol{w}(x)\,\{\mathcal{F}_\varepsilon(\boldsymbol{A}^-) - \mathcal{F}_\varepsilon(\boldsymbol{A}^+)\}\,\mathrm{V}g \\ &= \boldsymbol{w}(x)\,\{-\log(\varepsilon(\boldsymbol{A}^-)^{-1}) + \log(\varepsilon(\boldsymbol{A}^+)^{-1}) - G(\varepsilon(\boldsymbol{A}^-)^{-1}) + G(\varepsilon(\boldsymbol{A}^+)^{-1})\}\,\mathrm{V}g. \end{aligned}$$

By assumption, all eigenvalues of $\boldsymbol{A}^\pm$ lie in the open right half plane of the complex plain[1], and therefore it follows that

$$\lim_{\varepsilon\to 0+} \{-\log(\varepsilon(\boldsymbol{A}^-)^{-1}) + \log(\varepsilon(\boldsymbol{A}^+)^{-1})\} = \log(\boldsymbol{A}^-) - \log(\boldsymbol{A}^+),$$

[1]A conjecture of F. Stenger. To date, it has been shown true up to order $m = 1024$ by direct computation.

and also

$$\lim_{\varepsilon\to 0+}\{-G(\varepsilon(\boldsymbol{A}^-)^{-1})+G(\varepsilon(\boldsymbol{A}^+)^{-1})\}=0.$$

We thus get our approximation of the Hilbert transform

$$(\mathcal{H}g)(x)=\frac{1}{\pi}\lim_{\varepsilon\to 0+}H_\varepsilon(x)\approx\frac{1}{\pi}\{(\log(\mathcal{J}_m^-)-\log(\mathcal{J}_m^+))g\}(x).$$

From Theorem 2.5, we obtain (3.1). ☐

4. Some examples. In this section, we show some illustrative examples of approximation of the Hilbert transform by the use of our novel formula (3.1) which are implemented in MATLAB version 7.0.

Fig. 4.1–4.8 show the shapes of the results of the following examples. We set $N = 20$, and calculated the approximate values on the set of points that uniformly divides the interval of each figure into $2N+1$. The interval Γ was set to be $(0,1)$, $(-1,1)$, $(0,\infty)$, or $(-\infty,\infty)$. For $(0,\infty)$ and $(-\infty,\infty)$, we chose $(0,\pi)$ and $(-\pi,\pi)$ for the interval of figures, respectively.

EXAMPLE 4.1. $\Gamma=(0,1)$, $h=4/\sqrt{N}$, $\varphi(x)=\log(x/(1-x))$,

$$g(t)=1,\quad \textit{and}\quad (\mathcal{H}g)(x)=\frac{1}{\pi}\log\left(\frac{1-x}{x}\right).$$

EXAMPLE 4.2. $\Gamma=(0,1)$, $h=4/\sqrt{N}$, $\varphi(x)=\log(x/(1-x))$,

$$g(t)=t(1-t),\quad \textit{and}\quad (\mathcal{H}g)(x)=\frac{x-x^2}{\pi}\log\left(\frac{1-x}{x}\right)-\frac{x}{\pi}+\frac{1}{2\pi}.$$

EXAMPLE 4.3. $\Gamma=(-1,1)$, $h=4/\sqrt{N}$, $\varphi(x)=\log((x+1)/(1-x))$,

$$g(t)=\sqrt{1-t^2},\quad \textit{and}\quad (\mathcal{H}g)(x)=-x.$$

EXAMPLE 4.4. $\Gamma=(0,\infty)$, $h=4/\sqrt{N}$, $\varphi(x)=\log(x)$,

$$g(t)=\frac{1}{1+t},\quad \textit{and}\quad (\mathcal{H}g)(x)=\frac{1}{\pi(1+x)}\log\frac{1}{x}.$$

EXAMPLE 4.5. $\Gamma=(-\infty,\infty)$, $h=3/\sqrt{N}$, $\varphi(x)=x$,

$$g(t)=\cos t,\quad \textit{and}\quad (\mathcal{H}g)(x)=-\sin x.$$

EXAMPLE 4.6. $\Gamma=(-\infty,\infty)$, $h=2/\sqrt{N}$, $\varphi(x)=x$,

$$g(t)=\frac{1}{1+t^2},\quad \textit{and}\quad (\mathcal{H}g)(x)=\frac{-x}{1+x^2}.$$

EXAMPLE 4.7. $\Gamma=(-\infty,\infty)$,

$$g(t)=\frac{1-t}{1+t^2},\quad \textit{and}\quad (\mathcal{H}g)(x)=\frac{1-x}{1+x^2}.$$

In the last example, $g(t)$ and $(\mathcal{H}g)(x)$ slowly converge to zero at $t=\pm\infty$ and $x=\pm\infty$, respectively. Hence, as seen in Fig. 4.7, in which $h=4/\sqrt{N}$ and $\varphi(x)=x$, we could not obtain an accurate approximation with the identity map. In Fig. 4.8, in which $h=2/\sqrt{N}$ and $\varphi(x)=\log(x+\sqrt{1+x^2})$, instead of the identity map, we adopted a double exponential transformation, and then obtained a good result.

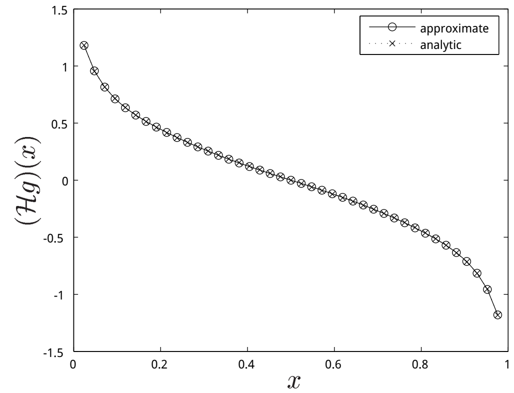

FIG. 4.1. *Example 4.1.*

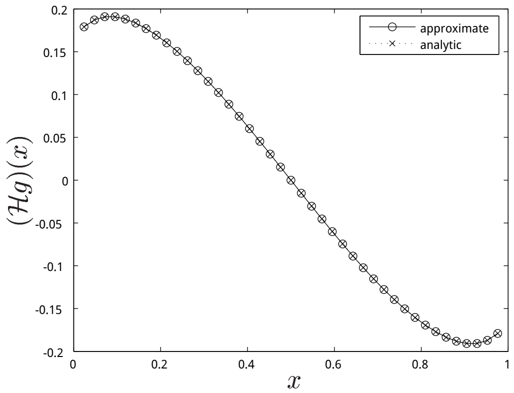

FIG. 4.2. *Example 4.2.*

5. Conclusion. In this paper, we proposed a novel method of approximating the Hilbert transform. This method is applicable to the Hilbert transform over any subinterval of $\mathbb{R}$. In this method, the Hilbert transform is evaluated at the discrete points that other sinc numerical methods exploit, without dividing the interval by a singular point. Therefore, this approximation of the Hilbert transform is easily combined with other sinc numerical methods. Also, using double exponential transformations, we can obtain an accurate approximation even if the Hilbert transform has an algebraic convergence rate at end points.

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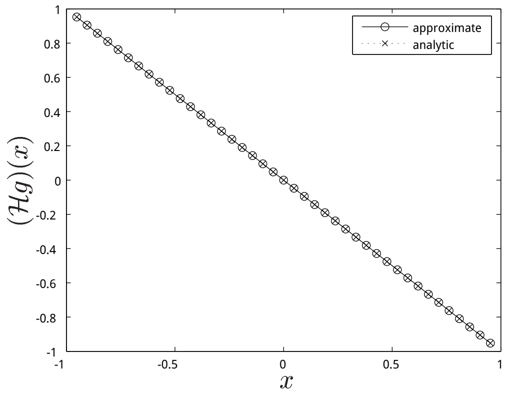

FIG. 4.3. *Example 4.3.*

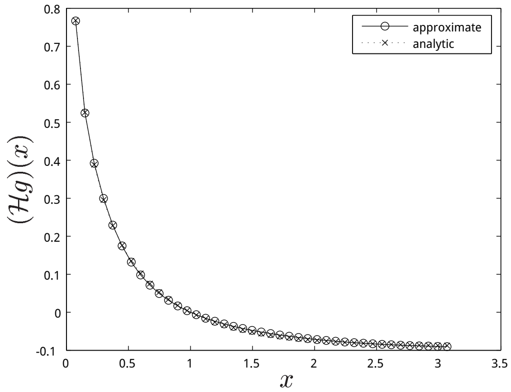

FIG. 4.4. *Example 4.4.*

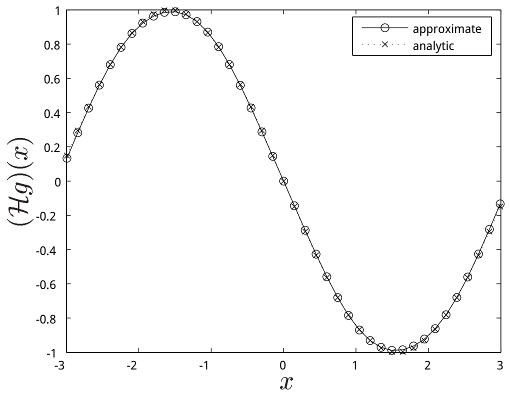

FIG. 4.5. *Example 4.5.*

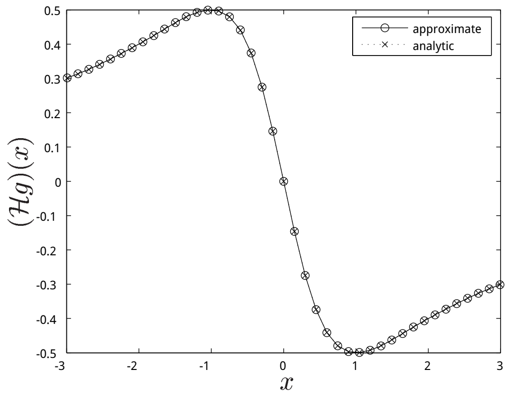

FIG. 4.6. *Example* 4.6.

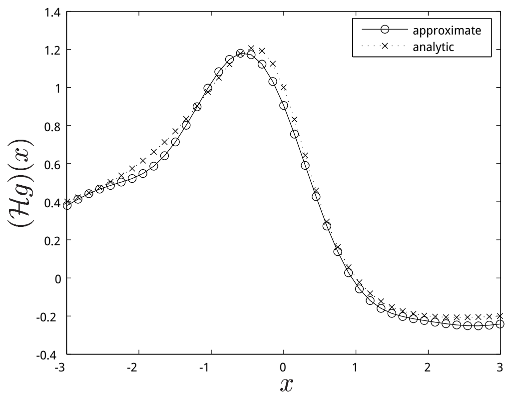

FIG. 4.7. *Example* 4.7 (1).

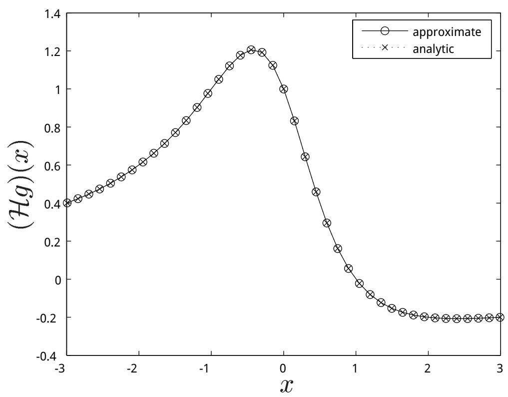

FIG. 4.8. *Example* 4.7 (2).