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ON EXTREMAL PROBLEMS RELATED TO INVERSE BALAYAGE*

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Dedicated to the memory of Professor Aurel Cornea

Abstract. Suppose G is a body in $\mathbb{R}^d$, $D \subset G$ is compact, and ρ a unit measure on ∂G. Inverse balayage refers to the question of whether there exists a measure ν supported inside D such that ρ and ν produce the same electrostatic field outside G. Establishing a duality principle between two extremal problems, it is shown that such an inverse balayage exists if and only if

$$\sup_{\mu} \left\{ \inf_{y \in D} U^{\mu}(y) - \int U^{\rho}\, d\mu \right\} = 0 ,$$

where the supremum is taken over all unit measures μ on ∂G and U^{μ} denotes the electrostatic potential of μ. A consequence is that pairs (ρ, D) admitting such an inverse balayage can be characterized by a ρ-mean-value principle, namely,

$$\sup_{z \in D} h(z) \geq \int h\, d\rho \geq \inf_{z \in D} h(z)$$

for all h harmonic in G and continuous up to the boundary.

In addition, two approaches for the construction of an inverse balayage related to extremal point methods are presented, and the results are applied to problems concerning the determination of restricted Chebychev constants in the theory of polynomial approximation.

Key words. Logarithmic potential, Newtonian potential, balayage, inverse balayage, linear optimization, duality, Chebychev constant, extremal problem.

AMS subject classifications. 31A15, 30C85, 41A17.

1. Introduction. In the classical physics interpretation, balayage refers to the process of sweeping an electrical charge or gravitational mass distribution from the inside of a body in Euclidean space to the surface or boundary of this body, such that the electrostatic or, respectively, gravitational potential outside remains unchanged. Inverse balayage relates to the inverse process posing the question which electrostatic charge or mass distribution inside the body produces a given or measured potential outside.

To put this into mathematical terminology, denote by $K(\cdot,\cdot)$ the logarithmic $(d = 2)$ or, respectively, Newtonian kernel $(d \geq 3)$

$$K(x,y) := \begin{cases} \log \dfrac{1}{|x-y|} , & \text{if } d = 2 , \\[2ex] \dfrac{1}{|x-y|^{d-2}} , & \text{if } d \geq 3 , \end{cases}$$

and, for each (finite Borel-) measure μ, define the logarithmic or Newtonian potential by

$$U^{\mu}(x) := \int K(x,y)\, d\mu(y) \qquad (x \in \mathbb{R}^d) .$$

Moreover, suppose $\emptyset \neq G \subset \mathbb{R}^d$, $d \geq 2$, is a bounded open set, such that ∂G is also the boundary of the unbounded component of $\mathbb{R}^d \setminus \overline{G}$. For simplicity of presentation and formulation, assume that G is regular with respect to the Dirchlet problem. For the notion of

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regularity and fundamental results from classical potential theory, we refer the reader to the standard literature [5], [12], [14], [16].

Let $\mathcal{M}_1(A)$ stand for the set of (say) unit Borel measures on $A \subset \mathbb{R}^d$, and suppose $\nu \in \mathcal{M}_1(G)$. The balayage of ν to ∂G is a unit measure $\rho = \text{Bal}(\nu, \partial G)$ on ∂G such that

$$U^\rho(x) = U^\nu(x) \qquad (x \in \mathbb{R}^d \setminus \overline{G}) \ .$$

Such a balayage always exists and can, for instance, be characterized in the following two ways (for this and more, see [9, Sect.2], [14, IV§1], [16, II.4]):

(i) Denote by $\mathcal{H}(G)$ the collection of all functions continuous on $\overline{G}$ and harmonic in the interior G. Then $\rho = \text{Bal}(\nu, \partial G)$ is uniquely characterized by the fact that integration leaves the class $\mathcal{H}(G)$ invariant:

$$\int h \, d\rho = \int h \, d\nu \qquad (h \in \mathcal{H}(G)) \ .$$

In particular, it is necessary that

$$\inf_{z \in \text{ supp}(\nu)} h(z) \leq \int h \, d\rho \leq \sup_{z \in \text{ supp}(\nu)} h(z) \qquad (h \in \mathcal{H}(G)) \ . \tag{1.1}$$

(ii) For (possibly signed) measures ν_1, ν_2 define the (mixed) energy

$$\langle \nu_1, \nu_2 \rangle := \int U^{\nu_1} \, d\nu_2 \tag{1.2}$$

(provided this quantity is well-defined). Then ρ is the unique unit measure on ∂G minimizing the energy

$$\langle \mu - \nu, \mu - \nu \rangle$$

among all unit measures μ on ∂G.

Now, suppose to the contrary, that $\rho \in \mathcal{M}_1(\partial G)$ is given, and let $\emptyset \neq D \subset G$ (or $D \subset \overline{G}$). An inverse balayage to ρ on D is a unit measure $\nu \in \mathcal{M}_1(D)$ such that $\rho = \text{Bal}(\nu, \partial G)$. In other words, the potential of ρ can as well be produced by the more densely concentrated charge or mass ν. Clearly, the collection $\text{Bal}^{-1}(\rho, D)$ of such inverse balayages is either empty or a convex set. However, the problem to find $\nu \in \text{Bal}^{-1}(\rho, D)$ is in general ill-conditioned. Important questions are:

(i) For which pairs (ρ, D) is $\text{Bal}^{-1}(\rho, D) \neq \emptyset$?

(ii) Do there exist designated or canonical measures $\nu \in \text{Bal}^{-1}(\rho, D)$ which are in some sense minimal (e.g., with respect to their support, i.e., mother bodies or materic bodies; see [9], [10], [11], [13])?

The paper is organized as follows: In Section 2 we introduce two extremal problems related to inverse balayage and state their basic properties. Section 3 generalizes these extremal problems and proves that their discrete versions are dual to one another. The key point is an interpretation in terms of duality in classical linear optimization. Conclusions providing characterizations for the existence of an inverse balayage on a given set to a given measure are drawn in Section 4. In addition, two approaches for the construction of an inverse balayage are presented. Finally, Section 5 is devoted to relating (restricted) Chebychev constants to minimax problems from the theory of polynomial approximation.

2. Extremal problems for potentials. Let $\rho \in \mathcal{M}_1(\partial G)$ be fixed. In the following, we will investigate the extremal problem

$$I(U^\rho, \partial G, D) := \sup_{\mu \in \mathcal{M}_1(\partial G)} \left\{ \inf_{y \in D} U^\mu(y) - \int U^\rho \, d\mu \right\} \tag{2.1}$$

and relate this to inverse balayage.

REMARK 2.1. Consider $\mu = \mu_{\partial G}$, the Robin equilibrium distribution of ∂G [16] [12]. By Frostman's theorem [6], which describes the Faraday cage effect in mathematical terms, the potential $U^{\mu_{\partial G}}$ is constant on $\overline{G}$. The corresponding value $V_{\partial G}$ is called Robin constant. Then

$$\inf_{y \in D} U^{\mu_{\partial G}}(y) = V_{\partial G} = \int V_{\partial G} \, d\rho = \int U^\rho \, d\mu_{\partial G} \, .$$

Consequently, $I(U^\rho, \partial G, D) \geq 0$. On the other hand, if an inverse balayage to ρ on D exists, then

$$I(U^\rho, \partial G, D) \leq 0 \, . \tag{2.2}$$

In fact, if $\nu \in \mathrm{Bal}^{-1}(\rho, D)$, then for each unit measure $\mu \in \mathcal{M}_1(\partial G)$,

$$\int U^\rho \, d\mu = \int U^\nu \, d\mu = \int U^\mu \, d\nu \geq \int \inf_{y \in D} U^\mu(y) \, d\nu = \inf_{y \in D} U^\mu(y) \, .$$

One result of this paper is that, under decent assumptions on D and G, (2.2) actually characterizes those measures $\rho \in \mathcal{M}_1(\partial G)$, which admit an inverse balayage on D (Theorem 4.1).

NOTATION 2.2. A sequence (μ_n) of measures on $\mathbb{R}^d$ is said to converge to a measure μ in the weak-star sense (symbol: $\mu_n \xrightarrow{*} \mu$), if

$$\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu$$

for all continuous functions f with compact support.

REMARK 2.3. If D is compact, then $\mathrm{Bal}^{-1}(\rho, D)$ is weak-star sequentially compact.

PROPOSITION 2.4. *Suppose U^ρ is continuous and let $D \subset G$. There exists $\mu^* \in \mathcal{M}_1(\partial G)$ such that*

$$I(U^\rho, \partial G, D) = \inf_{y \in D} U^{\mu^*}(y) - \int U^\rho \, d\mu^* \, .$$

Indeed, every weak-star point of accumulation of any sequence $(\mu_n) \subset \mathcal{M}_1(\partial G)$ with

$$I(U^\rho, \partial G, D) = \lim_{n \to \infty} \left\{ \inf_{y \in D} U^{\mu_n}(y) - \int U^\rho \, d\mu_n \right\} \tag{2.3}$$

is such an extremal measure.

Proof. Let $(\mu_n) \subset \mathcal{M}_1(\partial G)$ with (2.3) and suppose $\mu_n \overset{n \in \mathcal{N}}{\longrightarrow} \mu^*$ in the weak-star sense along some subsequence $\mathcal{N} \subset \mathbb{N}$. Note that by Helly's selection theorem, such weak-star points of accumulation do always exist. Then $\mu^* \in \mathcal{M}_1(\partial G)$ and, since U^ρ is continuous,

$$\lim_{\mathcal{N} \ni n \to \infty} \int U^\rho \, d\mu_n = \int U^\rho \, d\mu^* \, . \tag{2.4}$$

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Moreover,

$$\liminf_{\mathcal{N}\ni n\to\infty} \inf_{y\in D} U^{\mu_n}(y) \le \inf_{y\in D} U^{\mu^*}(y) \,. \tag{2.5}$$

In fact, if (2.5) was not true, then for some $\varepsilon_1 > 0$, some $y_0 \in D$ and $n \in \mathcal{N}$ sufficiently large,

$$U^{\mu^*}(y_0) + \varepsilon_1/2 < \inf_{y\in D} U^{\mu^*}(y) + \varepsilon_1 < \inf_{y\in D} U^{\mu_n}(y) \le U^{\mu_n}(y_0)\,,$$

contradicting the fact that by weak-star convergence

$$U^{\mu_n}(y) = \int K(\cdot, y)\, d\mu_n \overset{n\in\mathcal{N}}{\longrightarrow} \int K(\cdot, y)\, d\mu^* = U^{\mu^*}(y)$$

pointwise for y outside ∂G. It follows from (2.3), (2.4), and (2.5) as well as from the definition of $I(U^\rho, \partial G, D)$ that

$$I(U^\rho, \partial G, D) \le \inf_{y\in D} U^{\mu^*}(y) - \int U^\rho\, d\mu^* \le I(U^\rho, \partial G, D)\,. \qquad \square$$

REMARK 2.5. μ^* is, in general, not unique. For example, consider $D = \{z_0\} \subset G$ and $\rho = \omega_{z_0}$, the so-called harmonic measure of z_0. Note that $\omega_{z_0} = \mathrm{Bal}(\delta_{z_0}, \partial G)$, where here and in what follows, δ_{z_0} denotes the unit point mass in z_0. Then, for all $\mu \in \mathcal{M}_1(\partial G)$,

$$\inf_{z\in D} U^\mu(z) = U^\mu(z_0) = \int U^\mu\, d\delta_{z_0} = \int U^\mu\, d\rho\,.$$

The approach to relate the inverse balayage problem with (2.1) is to consider a second extremal problem

$$\delta(U^\rho, \partial G, D) := \inf_{\nu\in\mathcal{M}_1(D)} \sup_{x\in\partial G} \{U^\nu(x) - U^\rho(x)\}\,. \tag{2.6}$$

We will establish that the quantities (2.1) and (2.6) are dual perspectives of the same thing.

REMARK 2.6. By the domination principle [16] [12], it is clear that $\delta(U^\rho, \partial G, D) \ge 0$. Moreover, $\delta(U^\rho, \partial G, D) = 0$ if there exists an inverse balayage to ρ on D. Conversely, if ∂G is also the boundary of the unbounded component of $\mathbb{R}^d \setminus \overline{G}$ and if there is a $\nu \in \mathcal{M}_1(D)$ such that

$$\sup_{x\in\partial G} \{U^\nu(x) - U^\rho(x)\} = 0\,,$$

then by the maximium principle, ν is an inverse balayage to ρ. On the other hand, there are simple examples when $\delta(U^\rho, \partial G, D) = 0$ although an inverse balayage to ρ on D does not exist. For instance, this is the case when ρ is the balayage to ∂G of some Dirac measure δ_z in $z \in \overline{G} \setminus D$, where z is a point of accumulation of D and D is not "vast" enough to carry an inverse balayage to ρ. This is the reason why, henceforth, additional assumptions need to be imposed on the set D.

PROPOSITION 2.7. *Suppose $D \subset \overline{G}$ is compact. There exists $\nu^* \in \mathcal{M}_1(D)$ such that*

$$\delta(U^\rho, \partial G, D) = \sup_{x\in\partial G} \left\{U^{\nu^*}(x) - U^\rho(x)\right\}\,.$$

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Indeed, every weak-star point of accumulation of any sequences $(\nu_n) \subset \mathcal{M}_1(D)$ *with*

$$\delta(U^\rho, \partial G, D) = \lim_{n\to\infty} \sup_{x\in\partial G} \{U^{\nu_n}(x) - U^\rho(x)\} \tag{2.7}$$

is such a measure.

Proof. Suppose $(\nu_n) \subset \mathcal{M}_1(D)$ satisfies (2.7), and let ν^* be any weak-star point of accumulation, say $\nu_n \overset{*}{\to} \nu^*$ along $\mathcal{N} \subset \mathbb{N}$. Then $\nu^* \in \mathcal{M}_1(D)$, since D is compact. By the classical Principle of Descent [16] [12],

$$\liminf_{\mathcal{N}\ni n\to\infty} U^{\nu_n}(x) \geq U^{\nu^*}(x) \quad (x \in \mathbb{R}^d)\,.$$

Therefore,

$$\liminf_{\mathcal{N}\ni n\to\infty} \sup_{x\in\partial G} \{U^{\nu_n}(x) - U^\rho(x)\} \geq \sup_{x\in\partial G} \left\{U^{\nu^*}(x) - U^\rho(x)\right\}\,.$$

Taking into account (2.7) and the definition of $\delta(U^\rho, \partial G, D)$ it follows that

$$\delta(U^\rho, \partial G, D) \geq \sup_{x\in\partial G} \left\{U^{\nu^*}(x) - U^\rho(x)\right\} \geq \delta(U^\rho, \partial G, D)\,. \qquad \square$$

REMARK 2.8. Again, ν^* is — in general — not unique. Note that any inverse balayage to ρ on D (provided existence) is such an extremal measure.

3. Duality of the extremal problems. In order to derive a duality principle related to inverse balayage, we start by formulating the previously introduced extremal problems (2.1) and (2.6) in a slightly more general way.

Let $\emptyset \neq X, Y \subset \mathbb{R}^d$ be (Borel-) sets, and let $k(\cdot,\cdot) : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a kernel, which is continuous on $X \times Y$ as an extended real-valued function and, say, bounded from below. For (finite Borel-) measures μ on X and ν on Y, set

$$u^\mu(y) := \int k(x,y)\, d\mu(x) \quad (y \in Y)\,,$$

$$u_\nu(x) := \int k(x,y)\, d\nu(y) \quad (x \in X)\,.$$

If f is a measurable bounded real-valued function on X, define

$$\delta(f, X, Y) := \inf_{\nu\in\mathcal{M}_1(Y)} \left\{\sup_{x\in X} \{u_\nu(x) - f(x)\}\right\} \tag{P}$$

and

$$I(f, X, Y) := \sup_{\mu\in\mathcal{M}_1(X)} \left\{\inf_{y\in Y} u^\mu(y) - \int f\, d\mu\right\}\,. \tag{P^*}$$

NOTATION 3.1. We call (P^*) the dual problem associated with (P).

REMARK 3.2. Note that in the subsequent statements one could assume $f = 0$ by considering $\tilde{k}(x,y) := k(x,y) - f(x)$ in (P) and (P^*), respectively. We have chosen not to do so, because the initial formulation is closer to the questions concerning the symmetric logarithmic or Newtonian kernel, and since (P) has an interpretation as a problem of one-sided approximation to a function f by potentials with respect to a general kernel. Moreover, note that for $X' \subset X$ and $Y' \subset Y$,

$$\delta(f, X', Y) \leq \delta(f, X, Y'), \quad I(f, X', Y) \leq I(f, X, Y').$$

THEOREM 3.3. *Suppose X and Y are both finite sets. Then*

$$I(f, X, Y) = \delta(f, X, Y).$$

Proof. W.l.o.g. we may assume that at least one of the quantities in (P) or (P^*) is finite. Write $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. The extremal problem (P) can be equivalently stated as the linear program

$$\begin{array}{c} \min \; c^t \cdot \lambda! \\ A \cdot \lambda \geq b \\ \lambda \geq 0 \end{array} \tag{P_1}$$

with

$$A = \begin{pmatrix} -k(x_1, y_1) & \cdots & -k(x_1, y_m) & 1 & -1 \\ \vdots & & \vdots & \vdots & \vdots \\ -k(x_n, y_1) & \cdots & -k(x_n, y_m) & 1 & -1 \\ 1 & \cdots & 1 & 0 & 0 \\ -1 & \cdots & -1 & 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} -f(x_1) \\ \vdots \\ -f(x_n) \\ 1 \\ -1 \end{pmatrix}, \; c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ +1 \\ -1 \end{pmatrix}.$$

Here, the first entries $\lambda_1, \ldots, \lambda_m$ of the vector $\lambda \in \mathbb{R}^{m+2}$ are the masses that a unit measure ν on Y associates with the points $y_1, \ldots, y_m$, and the difference $\lambda_{m+1} - \lambda_{m+2}$ of two intermediate variables is the maximum of $u_\nu - f$ on X. In linear optimization it is common to associate with the linear program (P_1) the dual program (P_1^*) given by

$$\begin{array}{c} \max \; b^t \cdot u! \\ A^t \cdot u \leq c \\ u \geq 0. \end{array} \tag{P_1^*}$$

This is a reformulation of (P^*) with $u_1, \ldots, u_n$ being the masses that a measure μ on X associates with the points $x_1, \ldots, x_n$ and $u_{n+1} - u_{n+2}$ being the minimum of u^μ on Y.

It is well-known [4, §5.2] that admissible vectors λ for (P_1) and admissible u for (P_1^*) are related via

$$c^t \cdot \lambda \geq b^t \cdot u.$$

In addition, if a solution λ^* to (P_1) (or u^* to (P_1^*)) exists, then there exists also a solution u^* to (P_1^*) (or λ^* to (P_1)) and the respective objective functions are the same, i.e.,

$$c^t \cdot \lambda^* = b^t \cdot u^*.$$

Since (P_1) and (P_1^*) have admissible vectors, each problem has a solution, and Theorem 3.3 is proved. ∎

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Our next goal is to extend Theorem 3.3 also to non-finite sets.

LEMMA 3.4 (Principle of Descent). *Suppose X and Y are compact sets. Let $(x_n) \subset X$ be a sequence converging to x and $(\nu_n) \subset \mathcal{M}_1(Y)$ converging to ν in the weak-star sense. Then*

$$u_\nu(x) \le \liminf_{n\to\infty} u_{\nu_n}(x_n)\,.$$

Proof. The proof is classical (see [16, p.71]). From the Montone Convergence Theorem and by weak-star convergence,

$$\begin{aligned} u_\nu(x) &= \lim_{M\to+\infty} \int \min(M, k(x,y))\, d\nu(y) \\ &= \lim_{M\to+\infty} \lim_{n\to\infty} \int \min(M, k(x_n,y))\, d\nu_n(y) \le \liminf_{n\to\infty} u_{\nu_n}(x_n)\,. \end{aligned}$$ ∎

LEMMA 3.5. *Suppose X and Y are compact sets and that f is continuous. Then*

$$\delta(f,X,Y) = \sup_{\substack{X'\subset X \\ X' \text{finite}}} \delta(f,X',Y)\,, \quad I(f,X,Y) = \sup_{\substack{X'\subset X \\ X' \text{finite}}} I(f,X',Y)\,. \tag{3.1}$$

Proof. First, assume to the contrary that the left-hand side equality in (3.1) does not hold. Then there exists $\varepsilon_2 > 0$ such that for all finite sets $X' \subset X$,

$$\inf_{\nu\in\mathcal{M}_1(Y)} \sup_{x\in X} \{u_\nu(x) - f(x)\} > \inf_{\nu'\in\mathcal{M}_1(Y)} \sup_{x'\in X'} \{u_{\nu'}(x') - f(x')\} + \varepsilon_2\,.$$

Now, choose a sequence (X'_n) of finite subsets of X such that, say, $\operatorname{dist}(x, X'_n) \le 1/n$ for all $x \in X$. There are $\nu_n \in \mathcal{M}_1(Y)$ with

$$\inf_{\nu\in\mathcal{M}_1(Y)} \sup_{x\in X} \{u_\nu(x) - f(x)\} > \sup_{x'\in X'_n} \{u_{\nu_n}(x') - f(x')\} + \varepsilon_2\,.$$

By Helly's Selection Theorem, there exists a weak-star limit $\nu^* = \lim_{n\in\Lambda} \nu_n \in \mathcal{M}_1(Y)$ along some subsequence $\Lambda \subset \mathbb{N}$. Let $x^* \in X$ be such that

$$u_{\nu^*}(x^*) - f(x^*) > \sup_{x'\in X'_n} \{u_{\nu_n}(x') - f(x')\} + \varepsilon_2/2 \tag{3.2}$$

and choose $x'_n \in X'_n$ converging to x^*. By the Principle of Descent (Lemma 3.4) and since f is continuous on X,

$$\liminf_{\Lambda\ni n\to\infty} \{u_{\nu_n}(x'_n) - f(x'_n)\} \ge u_{\nu^*}(x^*) - f(x^*)\,.$$

This contradicts (3.2).

Now, assume that the second equality in (3.1) does not hold, say

$$I(f,X,Y) > \sup_{X'} I(f,X',Y) + \varepsilon_3$$

with $\varepsilon_3 > 0$. Let $\mu \in \mathcal{M}_1(X)$ be such that

$$\inf_{y\in Y} u^\mu(y) - \int f\, d\mu > \sup_{X'} \sup_{\mu'\in\mathcal{M}_1(X')} \left\{ \inf_{y\in Y} u^{\mu'}(y) - \int f\, d\mu' \right\} + \varepsilon_3\,.$$

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There exist finite sets $X_n' \subset X$ and measures $\mu_n' \in \mathcal{M}_1(X_n')$ converging to μ in the weak-star sense. Take $y_n \in Y$ such that

$$\inf_{y\in Y} u^{\mu}(y) - \int f\,d\mu > u^{\mu_n'}(y_n) - \int f\,d\mu_n' + \varepsilon_3\,. \tag{3.3}$$

Since Y is compact, we may w.l.o.g. assume that $y = \lim y_n' \in Y$ exists. Then by the Principle of Descent, weak-star convergence and (3.3),

$$u^{\mu}(y) - \int f\,d\mu \le \liminf_{n\to\infty} u^{\mu_n'}(y_n') - \int f\,d\mu_n' \le \inf_{y\in Y} u^{\mu}(y) - \int f\,d\mu - \varepsilon_3\,,$$

giving again a contradiction. ☐

LEMMA 3.6. *Suppose that X and Y are compact sets and that f is continuous. Assume that $k(\cdot,\cdot)$ is finite on $X \times Y$. Then*

$$\delta(f,X,Y) = \inf_{\substack{Y'\subset Y\\ Y' \textit{finite}}} \delta(f,X,Y')\,, \qquad I(f,X,Y) = \inf_{\substack{Y'\subset Y\\ Y' \textit{finite}}} I(f,X,Y')\,. \tag{3.4}$$

Proof. Assume contrary to the first equality in (3.4) that for some $\varepsilon_4 > 0$,

$$\delta(f,X,Y) < \inf_{\substack{Y'\subset Y\\ Y' \text{finite}}} \delta(f,X,Y') - \varepsilon_4\,.$$

Then there exists $\nu \in \mathcal{M}_1(Y)$ such that for each finite set $Y' \subset Y$ and every $\nu' \in \mathcal{M}_1(Y')$,

$$\sup_{x\in X}\{u_\nu(x) - f(x)\} < \sup_{x\in X}\{u_{\nu'}(x) - f(x)\} - \varepsilon_4\,. \tag{3.5}$$

Now, choose a sequence of finite sets $Y_n' \subset Y$ and measures $\nu_n' \in \mathcal{M}_1(Y_n')$ such that $\nu_n' \to \nu$ in the weak-star topology. By (3.5) there exist $x_n' \in X$ with the property that

$$\sup_{x\in X}\{u_\nu(x) - f(x)\} < u_{\nu_n'}(x_n') - f(x_n') - \varepsilon_4\,. \tag{3.6}$$

Since X is compact, we may w.l.o.g. assume that $x^* = \lim x_n' \in X$ exists. Taking into account that $u_{\nu_n'} \to u_\nu$ uniformly on X (recall that $k(\cdot,\cdot)$ is assumed finite and continuous on $X \times Y$) as well as the continuity of f, we deduce

$$\lim_{n\to\infty}\{u_{\nu_n'}(x_n') - f(x_n')\} = u_\nu(x^*) - f(x^*)\,,$$

which contradicts (3.6). Thus, the first equality in (3.4) holds.

Now, assume that the second equality in (3.4) does not hold, say, there exists $\varepsilon_5 > 0$ such that

$$\sup_{\mu\in\mathcal{M}_1(X)}\left\{\inf_{y\in Y} u^{\mu}(y) - \int f\,d\mu\right\} < \sup_{\mu'\in\mathcal{M}_1(X)}\left\{\inf_{y'\in Y'} u^{\mu'}(y') - \int f\,d\mu'\right\} - \varepsilon_5$$

for all finite sets $Y' \subset Y$. Choose a sequence $Y_n' \subset Y$ of finite sets such that $\operatorname{dist}(y, Y_n') \le 1/n$ for all $y \in Y$. Then there are measures $\mu_n \in \mathcal{M}_1(X)$ with

$$\sup_{\mu\in\mathcal{M}_1(X)}\left\{\inf_{y\in Y} u^{\mu}(y) - \int f\,d\mu\right\} < \inf_{y'\in Y_n'} u^{\mu_n}(y') - \int f\,d\mu_n - \varepsilon_5\,. \tag{3.7}$$

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By Helly's Theorem, there exists a weak-star limit $\mu^* = \lim_{\Lambda \ni n \to \infty} \mu_n$ along some subsequence $\Lambda \subset \mathbb{N}$. Since $u^{\mu_n} \to u^{\mu^*}$ along Λ uniformly on Y and because u^{μ^*} is continuous on Y, we deduce that

$$\lim_{\Lambda \ni n \to \infty} \left\{ \inf_{y' \in Y'_n} u^{\mu_n}(y') - \int f \, d\mu_n \right\} \leq \inf_{y \in Y} u^{\mu^*}(y) - \int f \, d\mu^* ,$$

contradicting (3.7). This completes the proof of Lemma 3.6. ☐

THEOREM 3.7. *Suppose that X and Y are compact sets and that f is continuous. Assume that $k(\cdot,\cdot)$ is finite on $X \times Y$. Then*

$$I(f, X, Y) = \delta(f, X, Y) .$$

Proof. By Lemma 3.5, Lemma 3.6, and Theorem 3.3,

$$\begin{aligned} I(f,X,Y) &= \inf_{\substack{Y' \subset Y \\ Y' \text{finite}}} I(f,X,Y') = \inf_{\substack{Y' \subset Y \\ Y' \text{finite}}} \sup_{\substack{X' \subset X \\ X' \text{finite}}} I(f,X',Y') \\ &= \inf_{\substack{Y' \subset Y \\ Y' \text{finite}}} \sup_{\substack{X' \subset X \\ X' \text{finite}}} \delta(f,X',Y') = \inf_{\substack{Y' \subset Y \\ Y' \text{finite}}} \delta(f,X,Y') = \delta(f,X,Y) . \end{aligned}$$ ☐

COROLLARY 3.8. *Suppose $D = \overline{D} \subset G$ and that U^ρ is continuous on ∂G. Then*

$$I(U^\rho, \partial G, D) = \delta(U^\rho, \partial G, D) .$$

Proof. Since $\partial G \cap D = \emptyset$, the kernel $k(\cdot,\cdot) = K(\cdot,\cdot)$ is finite on $X \times Y = \partial G \times D$ so that Theorem 3.7 can be applied to $f = U^\rho$. ☐

It is natural to consider also a corresponding problem where the infimum and supremum in the definition of $\delta(f, X, Y)$ are exchanged. With the same approach one can deduce

THEOREM 3.9. *Suppose X and Y are both finite sets. Then*

$$\inf_{\mu \in \mathcal{M}_1(X)} \left\{ \sup_{y \in Y} u^\mu(y) - \int f \, d\mu \right\} = \sup_{\nu \in \mathcal{M}_1(Y)} \left\{ \inf_{x \in X} \{ u_\nu(x) - f(x) \} \right\} . \tag{3.8}$$

Outline of a proof of Theorem 3.9. Multiply both sides of (3.8) by -1. Writing $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, the linear program associated with the negative right-hand side of (3.8) is

$$\begin{array}{c} \min \;\; c^t \cdot \lambda! \\ A \cdot \lambda \geq b \\ \lambda \geq 0 \end{array} \tag{P_1'}$$

with

$$A = \begin{pmatrix} k(x_1,y_1) & \cdots & k(x_1,y_m) & -1 & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ k(x_n,y_1) & \cdots & k(x_n,y_m) & -1 & 1 \\ 1 & \cdots & 1 & 0 & 0 \\ -1 & \cdots & -1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \\ 1 \\ -1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ +1 \\ -1 \end{pmatrix} .$$

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The corresponding dual program is a reformulation of the negative left-hand side of (3.8). Everything else works as in the proof of Theorem 3.3. ☐

THEOREM 3.10. *Suppose that X and Y are compact sets and that f is continuous. Assume that $k(\cdot,\cdot)$ is finite on $X \times Y$. Then*

$$\inf_{\mu\in\mathcal{M}_1(X)} \left\{ \sup_{y\in Y} u^{\mu}(y) - \int f\, d\mu \right\} = \sup_{\nu\in\mathcal{M}_1(Y)} \left\{ \inf_{x\in X} \{u_\nu(x) - f(x)\} \right\} .$$

Theorem 3.10 follows from Theorem 3.9 in much the same way as Theorem 3.7 follows from Theorem 3.3. We do not dwell on the proof.

Reformulating the extremal problem by taking the absolute value in the definition of $\delta(f,X,Y)$ one can apply the same method to obtain

THEOREM 3.11. *Suppose X and Y are both finite sets. Then*

$$\begin{aligned} &\sup_{\substack{\mu_1+\mu_2\in\mathcal{M}_1(X)\\ \mu_1,\mu_2\ge 0}} \left\{ \inf_{y\in Y} u^{\mu_1}(y) - u^{\mu_2}(y) - \int f\, d(\mu_1-\mu_2) \right\} \\ &\quad = \inf_{\nu\in\mathcal{M}_1(Y)} \left\{ \sup_{x\in X} |u_\nu(x) - f(x)| \right\} . \end{aligned} \tag{3.9}$$

Outline of a proof of Theorem 3.11. Writing $X = \{x_1,\dots,x_n\}$ and $Y = \{y_1,\dots,y_m\}$, (3.9) follows from a consideration of the linear program

$$\begin{array}{c} \min\ c^t\cdot\lambda! \\ A\cdot\lambda \ge b \\ \lambda \ge 0 \end{array} \tag{P_1''}$$

with

$$A = \begin{pmatrix} -k(x_1,y_1) & \cdots & -k(x_1,y_m) & 1 & -1 \\ \vdots & & \vdots & \vdots & \vdots \\ -k(x_n,y_1) & \cdots & -k(x_n,y_m) & 1 & -1 \\ +k(x_1,y_1) & \cdots & +k(x_1,y_m) & 1 & -1 \\ \vdots & & \vdots & \vdots & \vdots \\ +k(x_n,y_1) & \cdots & +k(x_n,y_m) & 1 & -1 \\ 1 & \cdots & 1 & 0 & 0 \\ -1 & \cdots & -1 & 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} -f(x_1) \\ \vdots \\ -f(x_n) \\ +f(x_1) \\ \vdots \\ +f(x_n) \\ 1 \\ -1 \end{pmatrix},\ c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ +1 \\ -1 \end{pmatrix}.$$

Considering the corresponding dual program, the proof of Theorem 3.11 follows the same lines as the proof of Theorem 3.3. ☐

THEOREM 3.12. *Suppose that X and Y are compact sets and that f is continuous. Assume that $k(\cdot,\cdot)$ is finite on $X \times Y$. Then*

$$\sup_{\substack{\mu_1+\mu_2\in\mathcal{M}_1(X)\\ \mu_1,\mu_2\ge 0}} \left\{ \inf_{y\in Y} u^{\mu_1}(y) - u^{\mu_2}(y) - \int f\, d(\mu_1-\mu_2) \right\} = \inf_{\nu\in\mathcal{M}_1(Y)} \left\{ \sup_{x\in X} |u_\nu(x) - f(x)| \right\} .$$

Since Theorem 3.12 follows from Theorem 3.11 just as Theorem 3.7 follows from Theorem 3.3, we omit the proof and leave the details to the reader.

4. Characterization for the existence of an inverse balayage. Recall that, for simplification, ∂G is assumed to be also the boundary of the unbounded component of $\mathbb{R}^d \setminus \overline{G}$. For the same reason, we will impose the natural assumption that U^ρ is continuous on ∂G, and thus in all $\mathbb{R}^d$.

THEOREM 4.1. *Suppose $D = \overline{D} \subset G$ and that U^ρ is continuous on ∂G. Then the following statements are equivalent:*

(i) *$Bal^{-1}(\rho, D) \neq \emptyset$.*
(ii) *$\delta(U^\rho, \partial G, D) = 0$.*
(iii) *$I(U^\rho, \partial G, D) = 0$.*

Proof. It remains to prove (ii) $\Rightarrow$ (i). By Proposition 2.7, there exists $\nu^* \in \mathcal{M}_1(D)$ such that

$$\sup_{x \in \partial G} \left\{ U^\rho(x) - U^{\nu^*}(x) \right\} = 0 \,.$$

In particular, $U^\rho \leq U^{\nu^*}$ on ∂G. Now, $h := U^{\nu^*} - U^\rho$ is harmonic in $\mathbb{R}^d \setminus \overline{G}$, continuous on $\mathbb{R}^d \setminus G$ with non-negative boundary values on ∂G. Moreover,

$$\begin{cases} h(\infty) = 0\,, & \text{if } d = 2\,, \\ \lim\limits_{|x| \to \infty} |x|^{d-2} h(x) = 0\,, & \text{if } d \geq 3\,. \end{cases}$$

By the Minimum Principle, $h = 0$ in $\mathbb{R}^d \setminus G$. Consequently, ν^* is an inverse balayage to ρ. ∎

A reformulation of (i) $\Leftrightarrow$ (iii) is

COROLLARY 4.2. *The measure ρ with continuous potential does admit an inverse balayage on $D = \overline{D} \subset G$ if and only if the mean-value inequality*

$$\inf_{y \in D} U^\mu(y) \leq \int U^\mu \, d\rho$$

holds for every measure μ on ∂G.

Similarly, one can deduce the following consequence of Theorem 3.10.

COROLLARY 4.3. *The measure ρ with continuous potential does admit an inverse balayage on $D = \overline{D} \subset G$ if and only if the mean-value inequality*

$$\sup_{y \in D} U^\mu(y) \geq \int U^\mu \, d\rho$$

holds for every measure μ on ∂G.

The following corollary shows that the existence of an inverse balayage is equivalent to a mean-value inequality in the space of harmonic functions.

COROLLARY 4.4 (Mean-Value Inequality Property). *The measure ρ with continuous potential does admit an inverse balayage on $D = \overline{D} \subset G$ if and only if the mean-value inequality*

$$\inf_{y \in D} h(y) \leq \int h \, d\rho \leq \sup_{y \in D} h(y) \tag{4.1}$$

holds for every function h harmonic in G and continuous on $\overline{G}$.

Proof. First, suppose $\nu^* \in \mathrm{Bal}^{-1}(\rho, D)$. Then, for every $h \in \mathcal{H}(G)$ (see (1.1)),

$$\int h \, d\rho = \int h \, d\nu^* \in \left[\inf_{y \in D} h(y), \sup_{y \in D} h(y) \right] \,.$$

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Conversely, suppose that (4.1) holds, and let $\mu \in \mathcal{M}_1(\partial G)$ be arbitrary. Since U^μ is lower semi-continuous, there exists a sequence $(h_n) \subset \mathcal{H}(G)$ such that $h_n \uparrow U^\mu$ on ∂G. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int h_n \, d\rho = \int U^\mu \, d\rho \tag{4.2}$$

and in the same way, for all $z \in G$,

$$h_n(z) = \int h_n \, d\omega_z \to \int U^\mu \, d\omega_z = U^\mu(z) \, ,$$

where ω_z denotes the harmonic measure. Since D is compact, the latter implies that

$$\liminf_{n\to\infty} \inf_{y\in D} h_n(y) \geq \inf_{y\in D} U^\mu(y) \, . \tag{4.3}$$

The assertion follows from (4.1), (4.2), (4.3) and Corollary 4.2. $\square$

REMARK 4.5. Independently, Sjödin [15] has applied operator theory to the inverse balayage problem, from which an alternative proof for Corollary 4.4 can be obtained: Suppose first that the compact set D is so large that every function $h \in \mathcal{H}(G)$ is uniquely determined by its values on D, e.g., if the interior of the intersection of D with every component of G is non-empty. Despite slight modifications, the following reasoning is essentially due to [15, p.252]. If (4.1) holds, then the linear functional

$$L(h) = \int h \, d\rho \qquad (h \in \mathcal{H}(G))$$

satisfies

$$|L(h)| \leq \sup_{x\in D} |h(x)| \qquad (h \in \mathcal{H}(G)) \, .$$

Considering $\mathcal{H}(G)$ a subspace of $C(D)$, by the Hahn-Banach Theorem one may extend L to a linear functional on $C(D)$ such that

$$|L(f)| \leq \sup_{x\in D} |f(x)| \qquad (f \in C(D)) \, .$$

By the Riesz Theorem, there exists μ on D with

$$L(f) = \int f \, d\mu \qquad (f \in C(D)) \, .$$

In particular, μ is the desired inverse balayage of ρ to D. Finally, in the situation of general compact D one may apply the previous reasoning to $D_n = \{x \in G : \mathrm{dist}(x, D) \leq 1/n\}$, let n tend to ∞ and apply an argument involving weak-star compactness (Remark 2.3), treating separately the case when some component of G does not intersect with D.

EXAMPLE 4.6. Let $z \in G$ and take $D = \{z\}$. By Corollary 4.4, a measure ρ on ∂G admits an inverse balayage on $\{z\}$ if and only if

$$h(z) = \int h \, d\rho$$

for all $h \in \mathcal{H}(G)$. This is the well-known characterization of the harmonic measure $\rho = \omega_z$ by the mean-value principle.

REMARK 4.7. Corollary 4.2 states that an inverse balayage to $\rho \in \mathcal{M}_1(\partial G)$ on $D = \overline{D} \subset G$ exists if and only if for all unit measures μ on ∂G there exists $y \in D$ such that (see (1.2))

$$\langle \mu, \rho - \omega_y \rangle \geq 0 \, .$$

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REMARK 4.8. (Construction of an inverse balayage). Suppose $\emptyset \neq D \subset \overline{G}$. The considerations in Section 3 suggest to use the following approach in order to construct an inverse balayage to given $\rho \in \mathcal{M}_1(\partial G)$:

Step 0. Choose discretizations $X = X_n \subset \partial G$ $Y = Y_m \subset D$ consisting of n, respectively m points, such that they are asymptotically dense in the respective sets.

Step 1. Solve the corresponding problem (P_1) (or (P_1') or (P_1'')) of linear optimization for the coefficients $\lambda_1, \ldots, \lambda_m$. Note that $\lambda_j = \lambda_j(X_n, Y_m)$.

Step 2. Consider the measures

$$\nu_m = \sum_{j=1}^{m} \lambda_j \, \delta_{y_j} \in \mathcal{M}_1(D)$$

and let $n, m \to \infty$. If D is compact and if $\mathrm{Bal}^{-1}(\rho, D) \neq \emptyset$, then each weak-star point of accumulation will be an inverse balayage to ρ on D.

If the problem (P_1) is chosen for the optimization process, then the discrete masses $\lambda_j \, \delta_{y_j}$ will rather tend to stay away from ∂G as the sets X_n get more dense, while the continuous formulation of the extremal problem cannot distinguish between any two inverse balayages. One may therefore be led to conjecture that the corresponding weak-star points of accumulation of the ν_n are in some sense minimal inverse balayages. This should be compared to results of Gustafsson [11] who has shown that for convex polyhedra ∂G in $\mathbb{R}^d$ and ρ being the surface measure on ∂G, there does exist an inverse balayage in the interior G with minimal support (a so-called mother body), and it is supported by hyperplanes consisting of those points inside having at least two closest neighbours on ∂G. The numerical behavior of the proposed algorithm is not considered in this paper, it will be the topic of future research. As pointed out by one of the referees, it may also be useful to incorporate a priori information of the inverse balayage measure in order to obtain good approximations in practical applications.

REMARK 4.9. A similar approach is not to associate optimal masses with given points, but given (equal) masses with optimal points:

Step 1'. Let $m \geq 1$ and choose $y_1, \ldots, y_m \in D$ such that

$$\sup_{x \in \partial G} \left\{ \frac{1}{m} \sum_{j=1}^{m} K(y_j, x) - U^{\rho}(x) \right\} \tag{4.4}$$

is minimal with respect to all (not necessarily distinct) possible points on D.

Step 2'. Consider the measures

$$\nu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{y_j} \in \mathcal{M}_1(D) \, .$$

If D is compact and if $\mathrm{Bal}^{-1}(\rho, D) \neq \emptyset$, then each weak-star point of accumulation will be an inverse balayage to ρ on D.

Again, the extremal problem defining the points y_j is such that they tend to stay away from ∂G. Examples of Borodin [3] give further evidence for the above conjecture: Let $D = \{|z| \leq 1\} \subset \mathbb{R}^2$ and suppose ρ is the equilibrium distribution of D. For $\partial G = \partial D$, the points y_j coincide with the origin, a phenomenon that, by the maximum principle, can be generalized to arbitrary lemniscatic regions. For $\partial G = \partial D \cup \{0\}$ the points y_j are dilated (and possibly rotated) roots of unity. Moreover, if $\partial G \subset \mathbb{R}^2$ is a smooth Jordan curve, then the quality

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of approximation of an inverse balayage to ρ by ν_n can be quantified in relating (4.4) to the discrepancy

$$D[\rho,\hat{\nu}_n] := \sup\{|\rho(J)-\hat{\nu}_n(J)| \ : \ J \text{ any subarc of } \partial G\}\ ,$$

where $\hat{\nu}_n$ stands for the balayage of ν_n to ∂G (cf. [1],[2],[8]).

5. Application to polynomial approximation. Denote by $\mathcal{P}_n(A)$ the set of monic (complex) polynomials of exact degree $n \geq 1$ having all zeros in $A \subset \mathbb{C}$. For $p_n \in \mathcal{P}_n(A)$, let $\nu_{p_n} \in \mathcal{M}_1(A)$ stand for the corresponding normalized zero counting measure associating equal mass $1/n$ with each zero of p_n (taking into account multiple zeros). Then polynomials are related to potentials by the simple identity

$$U^{\nu_{p_n}}(z) = \log\frac{1}{|p_n(z)|^{1/n}} \quad (z\in\mathbb{C})\ .$$

Let $\emptyset \neq X, Y \subset \mathbb{C}$ and suppose w is a function positive and (say) continuous on X.

DEFINITION 5.1. *The number*

$$t_w(X,Y) := \limsup_{n\to\infty} \sqrt[n]{\inf_{q_n\in\mathcal{P}_n(Y)} \sup_{x\in X} |q_n(x)\, w^n(x)|} \tag{5.1}$$

is called restricted (weighted) Chebychev constant for the pair (X,Y).

$$m_w(X,Y) := \limsup_{n\to\infty} \sqrt[n]{\sup_{p_n\in\mathcal{P}_n(Y)} \inf_{x\in X} |p_n(x)\, w^n(x)|} \tag{5.2}$$

is referred to as restricted (weighted) Maximin constant for the pair (X,Y).

REMARK 5.2. Note that (5.2) can be interpreted as a Chebychev problem for reciprocals of monic polynomials. In fact, denoting by $\mathcal{R}_{0,n}(Y)$ the set of reciprocals of monic polynomials of degree n with all poles in Y,

$$\frac{1}{m_w(X,Y)} = \liminf_{n\to\infty} \sqrt[n]{\inf_{r_n\in\mathcal{R}_{0,n}(Y)} \sup_{x\in X} |r_n(x)\, w^{-n}(x)|}\ .$$

The restricted Chebychev and Maximin constant, respectively, can be related to corresponding dual extremal problems

$$t_w^*(X,Y) := \limsup_{n\to\infty} \sqrt[n]{\inf_{q_n\in\mathcal{P}_n(X)} \left(\sup_{y\in Y} |q_n(y)| \exp\left(n\int \log w\, d\nu_{q_n}\right)\right)} \tag{5.3}$$

and

$$m_w^*(X,Y) := \limsup_{n\to\infty} \sqrt[n]{\sup_{p_n\in\mathcal{P}_n(X)} \left(\inf_{y\in Y} |p_n(y)| \exp\left(n\int \log w\, d\nu_{p_n}\right)\right)} \tag{5.4}$$

by means of

PROPOSITION 5.3. *Suppose X and Y are compact and disjoint. Then the limits in (5.1)-(5.4) exist and*

$$t_w(X,Y) = m_w^*(X,Y)\ , \qquad m_w(X,Y) = t_w^*(X,Y)\ .$$

Proof. Let $f := \log w$. By simple calculation,

$$\inf_{p_n \in \mathcal{P}_n(Y)} \sup_{x \in X} \{U^{\nu_{p_n}}(x) - f(x)\} = -\log\left(\sup_{p_n \in \mathcal{P}_n(Y)} \inf_{x \in X} |p_n(x)\, w^n(x)|\right)^{1/n} \tag{5.5}$$

and

$$\begin{aligned} &\sup_{q_n \in \mathcal{P}_n(X)} \left\{\inf_{y \in Y} U^{\nu_{q_n}}(y) - \int f\, d\nu_{q_n}\right\} \\ = \quad &-\log\left(\inf_{q_n \in \mathcal{P}_n(X)} \left(\sup_{y \in Y} |q_n(y)| \exp\left(n \int \log w\, d\nu_{q_n}\right)\right)\right)^{1/n} . \end{aligned} \tag{5.6}$$

Now, it is not hard to see that the left-hand side in (5.5) converges to $\delta(f, X, Y)$, and the left-hand side in (5.6) converges to $I(f, X, Y)$. By Theorem 3.7, the limits coincide, and the second assertion of Proposition 5.3 is proved. The same way, the first assertion of Proposition 5.3 follows from Theorem 3.10. ☐

COROLLARY 5.4. *Suppose X and Y are compact and disjoint. In the unweighted case $w \equiv 1$ the restricted Chebychev- and Maximin-problems are asymptotically dual in the sense that*

$$t_1(X, Y) = m_1(Y, X)\,.$$

REMARK 5.5. Suppose $X \subset Y$. Then $m_1(Y, X) = 0$, while $t_1(X, Y) \neq 0$ provided X is sufficiently large, i.e., of positive logarithmic capacity. Therefore, in Proposition 5.3 the condition that X and Y are disjoint cannot be dropped without further assumptions. On the other hand, it can be deduced from [3] and [16, Theorem III.3.1] that if X is the closure of a simply connected bounded domain and $Y = \partial X$, then $t_1(X, Y) = m_1(Y, X)$. The corresponding relation between inverse balayage and the asymptotics of (weighted) Maximin polynomials from polynomial approximation is described in [7].

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