Electronic Transactions on Numerical Analysis.
Volume 23, pp. 288-303, 2006.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

UNIFORMLY CONVERGENT DIFFERENCE SCHEME FOR SINGULARLY PERTURBED PROBLEM OF MIXED TYPE*

ILIYA A. BRAYANOV†

Abstract. A one dimensional singularly perturbed elliptic problem with discontinuous coefficients is considered. The domain under consideration is partitioned into two subdomains. In the first subdomain a convection-diffusion-reaction equation is posed. In the second one we have a pure reaction-diffusion equation. The problem is discretized using an inverse-monotone finite volume method on Shishkin meshes. We establish an almost second-order global pointwise convergence that is uniform with respect to the perturbation parameter. Numerical experiments that support the theoretical results are given.

Key words. convection-diffusion problems, singular perturbation, asymptotic analysis, finite volume methods, modified upwind approximations, uniform convergence, Shishkin mesh

AMS subject classifications. 34A36,34E05, 34E15, 65L10, 65L12, 65L20, 65L50

1. Introduction.

Let us consider the following one dimensional elliptic problem

$$L^-u \equiv -\varepsilon u'' + r(x)u' + q(x)u = f(x),\ x \in \Omega^- = (0,\xi), \tag{1.1}$$

$$L^+u \equiv -\varepsilon u'' + q(x)u = f(x),\ x \in \Omega^+ = (\xi,1), \tag{1.2}$$

$$[u]_{x=\xi} = u(\xi+0) - u(\xi-0) = 0, \quad [u']_{x=\xi} = 0, \tag{1.3}$$

$$u(0) = \psi_0, \quad u(1) = \psi_1, \tag{1.4}$$

where $0 < \varepsilon << 1$ and

$$0 < r_0 \le r(x) \le r_1, \quad 0 < q_0 \le q(x) \le q_1. \tag{1.5}$$

The functions q and f could be discontinuous at the interface point ξ. Note that the sign pattern of the convection coefficients is essential for the behavior of the solution; see [7, 8] for more detailed discussion for convection-diffusion problem with discontinuous convection coefficient. The solution of this problem has a boundary layer at $x = 1$ and interior layers with different widths at $x = \xi$. In Ω^-, where convection-diffusion problems are present, the width of the layer is $O(\varepsilon)$. Since the convection coefficient is positive, the characteristics of the reduced problem point toward the interface point and an interior layer appears on the right part of the boundary of Ω^-. In Ω^+, we have a reaction-diffusion problem. Now the reduced problem is of zero order and boundary layers of width $O(\sqrt{\varepsilon})$ appear on both boundaries of Ω^+. The character of the layers can be readily seen on Fig. 6.1 below.

There is a vast literature dealing with convection-diffusion and reaction-diffusion problems with smooth coefficients and smooth right hand side (see [10, 11, 13, 17] for surveys), but there are only a handful of papers dealing with problems with discontinuous coefficients. Such problems usually present interior layers. The one dimensional convection-diffusion problem with discontinuous input data is discussed recently by several authors; see [4, 7, 8, 9, 10, 15] and references there. The one dimensional reaction-diffusion problem with discontinuous input data and concentrated source is discussed in [6, 12, 14].

Our objective in this paper is to derive global pointwise convergence of almost second order that is uniform with respect to the small parameter ε. We construct partially uniform

*Received October 24, 2005. Accepted for publication March 13, 2006. Recommended by L. Reichel. This work was supported by the National Science Fund, Sofia, Bulgaria, project HS-MI-106/2005.

†Department of Applied Mathematics and Informatics, University of Rousse "Angel Kanchev", Studentska str. 8, 7017 Rousse, Bulgaria (brayanov@ru.acad.bg).

ETNA
Kent State University
etna@mcs.kent.edu

Shishkin meshes in each subdomain Ω^- and Ω^+. The resulting global mesh is condensed closely to the boundary and interior layers. To derive a uniform difference scheme we use an inverse-monotone finite volume discretization on layer adapted meshes. To obtain a second order scheme in Ω^- we use a modification of the monotone Samarski's scheme analogous to [4] . In Ω^+, where a reaction-diffusion problem is present, we modify the scheme at the interface point, similarly to [5]. To prove ε-uniform convergence we use two types of techniques. The first one uses the discrete Green function to obtain a hybrid stability inequality that shows that the maximal nodal error is bounded by a discrete L_1 norm of its truncation error. A result like this is proved by Andreev in [1, 2] and used later in [3] to prove second order ε-uniform convergence in maximum norm for convection-diffusion problem. The second one uses the embedding inequality and estimates in negative norm to prove second order uniform convergence for reaction-diffusion problem on non-uniform meshes; see [16]. This technique is used in [3, 5] to prove almost second order ε-uniform convergence in maximum norm, for the corresponding problems.

An outline of the paper is as follows. In Section 2, we describe some properties of the differential solution, construct Shishkin decomposition of the solution, and derive ε uniform bounds for their derivatives. In Section 3, we construct Shishkin mesh condensed near the boundary and interface layers and obtain a finite volume difference scheme. Some a priori bounds of the discrete problem are given in Section 4. The uniform convergence of the constructed scheme is proved in Section 5. In Section 6 we give some numerical experiments that support the theoretical results.

2. Properties of analytical solution. In this section we establish some properties of solution to problem (1.1)-(1.4) to be used in the analysis of difference scheme. We begin with the existence of a solution of a problem (1.1)-(1.4).

LEMMA 2.1. *Suppose that*

$$r \in C^{k-2}(\Omega^-),\ q, f \in C^{k-2}(\Omega^- \cup \Omega^+),\ k \geq 2 - \textit{integer}. \tag{2.1}$$

Then, problem (1.1)-(1.4) has a solution $u \in C^1(\overline{\Omega}) \cap C^k(\Omega^- \cup \Omega^+)$.

Proof. The construction of a solution is similar to that described in [4] for a convection-diffusion equation. Let y_1 and y_2 be particular solutions to the differential problems

$$L^- y_1 = f,\ x \in \Omega^-, \quad L^+ y_2 =,\ x \in \Omega^+.$$

Consider the function

$$y(x) = \begin{cases} y_1(x) + (\psi_0 - y_1(0))\varphi_1(x) + (A - y_1(\xi))\varphi_2(x), x \in \Omega^-, \\ y_2(x) + (\psi_1 - y_2(1))\varphi_4(x) + (A - y_2(\xi))\varphi_3(x), x \in \Omega^+, \end{cases}$$

where the functions φ_i, $(i = 1, 2, 3, 4)$ satisfy the problems

$$\begin{aligned} &L^- \varphi_i = 0,\ i = 1, 2,\ \varphi_1(0) = 1,\ \varphi_1(\xi) = 0,\ \varphi_2(0) = 0,\ \varphi_2(\xi) = 1, \\ &L^+ \varphi_i = 0,\ i = 3, 4,\ \varphi_3(\xi) = 0,\ \varphi_3(1) = 0,\ \varphi_4(\xi) = 0,\ \varphi_4(1) = 1. \end{aligned}$$

From (2.1) follows that the functions φ_i $(i = 1, 2, 3, 4)$ exist and have derivatives up to order k within their domains. By the maximum principle, $0 \leq \varphi_i \leq 1$ $(i = 1, 2, 3, 4)$ and

$$\varphi_1', \varphi_3' \leq 0, \quad \varphi_2', \varphi_3' \geq 0. \tag{2.2}$$

The second conjugation condition in (1.3) yields the following condition for A:

ETNA
Kent State University
etna@mcs.kent.edu

$$A[\varphi_2'(\xi)-\varphi_3'(\xi)] = y_2'(\xi)-y_1'(\xi)+[\psi_1-y_2(1)]\varphi_4'(\xi) \\ -[\psi_0-y_1(0)]\varphi_1'(\xi)+y_1(\xi)\varphi_2'(\xi)-y_2(\xi)\varphi_3'(\xi).$$

By virtue of (2.2), A is uniquely defined. □

The uniqueness of the solution follows from Lemma 2.2 below.

LEMMA 2.2. *(the maximum principle). Suppose that a function* $w \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ *satisfies*

$$L^-w(x) \le 0,\ x\in\Omega^-, \quad L^+w(x)\le 0,\ x\in\Omega^+, \\ [w']_{x=\xi} \ge 0,\ w(0)\le 0,\ w(1)\le 0,$$

then $w \le 0$ *for all* $x \in \overline{\Omega}$.

Proof. Let x_0 be a point at which w attains its maximal value in $\overline{\Omega}$. If $w(x_0) \le 0$, there is nothing to prove. Suppose therefore that $w(x_0) > 0$, the proof is completed by showing that this leads to contradiction. With the above assumption on the boundary value, either $x_0 \in \Omega^- \cup \Omega^+$ or $x_0 = \xi$. If $x_0 \in \Omega^\pm$ then $w''(x_0) \le 0$, $w'(x_0) = 0$. Therefore $L^\pm w(x_0) > 0$, which is a contradiction.

The only possibility remaining is that $x_0 = \xi$. There are two possible cases.

1. The function $w(x)$ is not differentiable at x_0. Since x_0 is a point at which w attains its maximal value, then $w'(x_0-0) \ge 0$, $w'(x_0+0) \le 0$ and $[w']_{x=x_0} < 0$, which is the required contradiction.

2. The function $w(x)$ is differentiable at x_0. Then $w'(x_0) = 0$ and there exists a subinterval $\sigma_\delta \equiv \{x \in (x_0, x_0+\delta), \delta > 0\}$ such that

$$w(x) > 0, \text{ and } w(x) < w(x_0),\ \forall x \in \sigma_\delta.$$

Let $x_1 \in \sigma_\delta$. There exists $x_2 \in \sigma_\delta$, such that

$$w'(x_2) = \frac{w(x_1)-w(x_0)}{x_1-x_0} < 0,$$

and $x_3 \in \sigma_\delta$, satisfying

$$w''(x_3) = \frac{w'(x_2)-w'(x_0)}{x_2-x_0} < 0.$$

Then $x_3 \in \Omega^+$ and $L^+w(x_3) > 0$, which is the required contradiction. □

Denote by $\|.\|_{L_\infty(\overline{\Omega})}$ the maximum norm in $\overline{\Omega}$. An immediate consequence of the maximum principle is the following stability result.

LEMMA 2.3. *Let* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ *be a solution of problem (1.1)-(1.4), then*

$$\|u\|_{L_\infty(\overline{\Omega})} \le \max\left\{|\psi_0|, |\psi_1|, \frac{\|f\|_{L_\infty(\overline{\Omega})}}{q_0}\right\}, \tag{2.3}$$

and

$$|u^{(k)}(x)| \le C\left(1+\varepsilon^{-k}e^-(x,\varepsilon)\right),\ x\in\overline{\Omega}^-, \tag{2.4}$$

$$|u^{(k)}(x)| \le C\left(1+\varepsilon^{-k/2}e^+(x,\varepsilon)\right),\ x\in\overline{\Omega}^+, \tag{2.5}$$

ETNA
Kent State University
etna@mcs.kent.edu

where

$$e^-(x,\varepsilon) = \exp\left(-\frac{(\xi - x)r_0}{\varepsilon}\right),\ e^+(x,\varepsilon) = e_1^+(x,\varepsilon) + e_2^+(x,\varepsilon),$$
$$e_1^+(x,\varepsilon) = \exp\left(-\frac{(x-\xi)\sqrt{q_0}}{\sqrt{\varepsilon}}\right),\ e_2^+(x,\varepsilon) = \exp\left(-\frac{(1-x)\sqrt{q_0}}{\sqrt{\varepsilon}}\right),$$

and C is independent of ε positive constant.

Proof. Put $\Psi_\pm(x) = -M \pm u$, where $M = \max\left\{\|\psi_0\|, \|\psi_1\|, \frac{\|f\|_{L_\infty(\overline{\Omega})}}{q_0}\right\}$. Clearly $\Psi_\pm \in C^1(\overline{\Omega})$, $\Psi_\pm(0) \le 0$, $\Psi_\pm(1) \le 0$ and for each $x \in \Omega^\pm$,

$$L^\pm \Psi_\pm(x) \le 0.$$

It follows from the maximum principle that $\Psi_\pm(x) \ge 0$ for all $x \in \overline{\Omega}$, which leads to the desired bound (2.3) on u. Now we consider the solution independently in each subdomain $\Omega^\pm$ and using the arguments in [4, 11], to obtain the remaining bounds in (2.4) and (2.5). □

But for the numerical analysis below we shall need a decomposition of the solution into regular and singular part. Using the results in Lemma 2.2 and Lemma 2.3 we obtain the following estimates.

THEOREM 2.4. *Let the functions r, q and f be sufficiently smooth in each subdomain Ω^- and Ω^+. Then the solution u of problem (1.1)-(1.4) admits the representation*

$$u(x) = v(x) + w(x),$$

where the regular part $v(x)$ and the singular part $w(x)$ satisfy for all natural k, $0 \le k \le 4$ the estimates

$$\|v^{(k)}\|_{L_\infty(\overline{\Omega})} \le C, \tag{2.6}$$

and

$$|w^{(k)}(x)| \le C\varepsilon^{-k+1/2} e^-(x,\varepsilon),\ x \in \overline{\Omega}^-, \tag{2.7}$$
$$|w^{(k)}(x)| \le C\varepsilon^{-k/2} e^+(x,\varepsilon),\ x \in \overline{\Omega}^+ \tag{2.8}$$

for some positive constant C independent of the small parameter ε.

Proof. The regular part $v(x)$ is sought in the form

$$v(x) = \sum_{i=0}^{3} \varepsilon^i v_i(x) + \varepsilon^4 R_4(x),\ v_i(x) = \begin{cases} v_i^-(x), & x \in \Omega^-, \\ v_i^+(x), & x \in \Omega^+, \end{cases}$$

where the functions $v_i^-(x)$ are solutions to the problems

$$r(x)(v_i^-)' + q(x)v_i^- = F_i^-(x),\ x \in \Omega^-,\ v_i^-(0) = \Psi_i^-,\ i = 0,1,2,3,$$
$$F_0^-(x) = f(x),\ \Psi_0^- = \psi_0,\ F_j^-(x) = (v_{j-1}^-)'',\ \Psi_j^- = 0,\ j = 1,2,3.$$

and the functions $v_i^+(x)$ satisfy

$$v_0^+ = \frac{f(x)}{q(x)},\ v_i^+ = \frac{(v_{i-1}^+)''(x)}{q(x)},\ i = 0,1,2,3.$$

ETNA
Kent State University
etna@mcs.kent.edu

From the presentation it is clear that the regular component has ε-uniformly bounded derivatives.

$$\|(v_i^{\pm})^{(k)}\|_{L_\infty(\overline{\Omega}^{\pm})} \leq M,\ 0 \leq k \leq 4,\ i = 0,1,2,3. \tag{2.9}$$

The singular part is represented as $w(x) = \sum_{i=0}^{3} \varepsilon^i (w_{1,i} + w_{2,i})$, where the functions $w_{1,i}$, $i = 0,1,2,3$ are solutions to the problems

$$\begin{aligned}
&L^+ w_{1,i} = 0, \quad x \in \Omega^+,\\
&w_{1,i}(\xi) = -[v_i]_{x=\xi},\ w_{1,0}(1) = \psi_1 - v_i^+(1), w_{1,i}(1) = -v_i^+(1),\ i \geq 1.
\end{aligned}$$

and $w_{1,i} = 0$ in Ω^-. Similarly as in [11] we can prove that $w_{1,i}$ satisfy the estimates (2.8). The second term $w_{2,i}$ satisfy for $i = 0,1,2,3$ the problems

$$\begin{aligned}
&L^{\pm} w_{2,i} = 0, \quad x \in \Omega^{\pm},\\
&[w_{2,i}]_{x=\xi} = 0, \quad [w'_{2,i}]_{x=\xi} = -[v'_i + w'_{1,i}]_{x=\xi},\\
&w_{2,i}(0) = 0,\ w_{2,i}(1) = 0.
\end{aligned}$$

We shall prove that the functions $w_{2,i}$ satisfy the bounds

$$|w_{2,i}^{(k)}(x)| \leq \left\{ \begin{array}{ll} M\varepsilon^{-(k+1/2)} \exp(-r_0(\xi - x)/\varepsilon), & x \in \Omega^-,\\ M\varepsilon^{-(k+1)/2} \exp(-\sqrt{q_0}(x-\xi)/\sqrt{\varepsilon}), & x \in \Omega^+. \end{array} \right. \tag{2.10}$$

Define the following barrier function for $k = 0$ and $i = 0,1,2,3$:

$$\Psi_{i,\pm}(x) = \left\{ \begin{array}{ll} -M\sqrt{\varepsilon} \exp(-r_0(\xi - x)/\varepsilon) \pm w_{2,i}, & x \in \Omega^-,\\ -M\sqrt{\varepsilon} \exp(-\sqrt{q_0}(x-\xi)/\sqrt{\varepsilon}) \pm w_{2,i}, & x \in \Omega^+. \end{array} \right.$$

Then using the estimates for v_i and $w_{1,i}$ we have

$$\begin{aligned}
&L^- \Psi_{i,\pm} = \frac{M}{\sqrt{\varepsilon}} \exp(-r_0(\xi - x)/\varepsilon) \left(r_0^2 - r_0 r(x) - \sqrt{\varepsilon} q(x)\right) \leq 0,\ x \in \Omega^-,\\
&L^+ \Psi_{i,\pm} = M\sqrt{\varepsilon} \exp(-\sqrt{q_0}(x-\xi)/\sqrt{\varepsilon})\,(q_0 - q(x)) \leq 0,\\
&[\Psi_{i,\pm}]_{x=\xi} = 0,\ \left[\Psi'_{i,\pm}\right]_{x=\xi} = M\sqrt{q_0} + \frac{Mr_0}{\sqrt{\varepsilon}} \mp [v'_i + w'_{1,i}]_{x=\xi} \geq 0,\\
&\Psi_{i,\pm}(0) = -M\sqrt{\varepsilon} \exp(-r_0 \xi/\varepsilon) \leq 0,\\
&\Psi_{i,\pm}(1) = -M\sqrt{\varepsilon} \exp(-\sqrt{q_0}(1-\xi)/\sqrt{\varepsilon}) \leq 0.
\end{aligned}$$

Now estimates (2.10) for $k = 0$ follow by the maximum principle. The estimates for $k \geq 1$ are derived by induction, similarly to [4]. Finally from the estimates for $w_{1,i}$ and bounds (2.10) we obtain (2.7) and (2.8).

The remainder $R_4(x)$ solves the problem

$$L^{\pm} R_4 = (v_3^{\pm})'',\ x \in \Omega^{\pm},\ [R_4]_{x=\xi} = 0,\ [R'_4]_{x=\xi} = 0,\ R_4(0) = 0,\ R_4(1) = 0.$$

Applying estimates (2.4),(2.5) in Lemma 2.3 we obtain

$$\|\varepsilon^4 (R_4^{\pm})^(k)\}_{L_\infty(\overline{\Omega}^{\pm})} \leq M\varepsilon^{4-k} \leq M,\ 0 \leq k \leq 4. \tag{2.11}$$

The bound (2.6) follows from representation (2.8) and the estimates (2.9), (2.11). $\square$

3. Numerical approximation.

3.1. Grid and grid functions. To obtain ε-uniform convergent difference scheme we shall construct a special partially uniform (Shishkin) mesh $\bar{\omega}_h$ condensed near to boundary $x = 1$ and around interface point $x = \xi$

$$\bar{\omega}_h = \{x_i, x_i = x_{i-1} + h_i, i = 1, 2, \ldots, m + n = N, x_0 = 0, x_m = \xi, x_N = 1\},$$

where

$$h_i = \begin{cases} h^1 = \frac{2(\xi-\delta_1)}{m}, & i = 1, \ldots, m/2, \\ h^2 = \frac{2\delta_1}{m}, & i = m/2+1, \ldots, m \\ h^3 = \frac{4\delta_2}{n}, & i = m+1, \ldots, m+n/4, \\ h^4 = \frac{2(1-\xi-2\delta_2)}{n}, & i = m+n/4+1, \ldots, m+3n/4, \\ h^3 = \frac{4\delta_2}{n}, & i = m+3n/4+1, \ldots, N, \end{cases}$$

$$\delta_1 = \min\{\sigma_1 \varepsilon \ln m/r_0, \xi/2\}, \quad \delta_2 = \min\{\sigma_2 \sqrt{\varepsilon} \ln n/\sqrt{q_0}, (1-\xi)/4\}.$$

The structure of the mesh follows the pattern of the boundary and the interior layers. Near the layers it is $O(\varepsilon)$ in Ω^- and $O(\sqrt{\varepsilon})$ in Ω^+. The parameters σ_1 and σ_2 play essential role in the proof of the uniform convergence below. In section 2 we obtained that the singular part is small outside of the layers. More exactly it is $O(N^{-\sigma_k}), k = 1, 2$. Thus the error due to singular part will be $O(N^{-\sigma_k}), k = 1, 2$ at the points where condensed and coarse mesh meet. Therefore to obtain second order of convergence we should have $\sigma_1, \sigma_2 \geq 2$; see [3, 4].

Let $u(x_i)$ and $v(x_i)$ be mesh functions of the discrete argument $x_i \in \bar{\omega}_h$. Let in addition g be a partially continuous function with possible discontinuity at the mesh point $x_m = \xi$. Denote $g_i = g(x_i)$ and $g_m^{\pm} = g(x_m \pm 0)$. We shall also use the following notations for the grid,

$$\hat{h}_i = \frac{h_i + h_{i+1}}{2}, \; \bar{g}_m = \frac{h_{m+1} g_m^+ + h_m g_m^-}{2\hat{h}_m}, \; D^- v_i = \frac{v_i - v_{i-1}}{h_i}, \; \breve{D}^- = \frac{v_i - v_{i-1}}{\hat{h}_i}$$

$$D^+ v_i = D^- v_{i+1}, \; \hat{D}^+ = \frac{v_{i+1} - v_i}{\hat{h}_i}, \; \hat{D}^+ D^- v_i = \frac{D^- v_{i+1} - D^- v_i}{\hat{h}_i}.$$

Further we shall use the discrete maximum norm

$$\|u\|_{\infty, \bar{\omega}_h} = \max_{x_i \in \bar{\omega}_h} |u_i|.$$

3.2. Finite difference scheme. To obtain a numerical approximation we shall use the balance equation for problem (1.1)-(1.4). Denote by $x_{i-1/2} = x_i - h_i/2, u_{i-1/2} = u(x_{i-1/2})$. Integrating equation (1.1) on the interval $(x_{i-1/2}, x_{i+1/2}), i = 1, \ldots, m-1$ and dividing by $\hat{h}_i$ we get

$$-\frac{\varepsilon}{\hat{h}_i}(u'_{i+1/2} - u'_{i-1/2}) = \frac{1}{\hat{h}_i} \int_{x_{i-1/2}}^{x_{i+1/2}} (f(x) - r(x)u'(x) - q(x)u(x))dx. \tag{3.1}$$

After approximation of the integrals in (3.1) we obtain

$$-\frac{\varepsilon}{\hat{h}_i}\left[u'_{i+1/2}(1 - R^h_{i+1}) - u'_{i-1/2}(1 + R^h_i)\right] + q_i u_i = f_i, \tag{3.2}$$

where

$$r^h_i = r(x_i - 0.5h_i), \; R^h_i = \frac{h_i r^h_i}{2\varepsilon}.$$

ETNA
Kent State University
etna@mcs.kent.edu

Now we approximate

$$u'_{i-1/2} \approx D^- U_i,\ u'_{i+1/2} \approx D^+ U_i. \tag{3.3}$$

and use that

$$1 - R_i^h = \frac{1}{1+R_i^h} - \frac{R_i^h}{1+(R_i^h)^{-1}},\ 1 + R_i^h = \frac{1}{1+R_i^h} + 2R_i^h - \frac{R_i^h}{1+(R_i^h)^{-1}}.$$

Ignoring the term $\varepsilon u''$ in (1.1) we replace the first derivative u' by $(f - qu)/r$ to obtain an approximation

$$\hat{D}^+\left(\frac{\varepsilon R_i^h D^- u_i}{1+(R_i^h)^{-1}}\right) = \frac{1}{2}\hat{D}^+\left(\frac{h_i r_i^h D^- u_i}{1+(R_i^h)^{-1}}\right) \approx \frac{1}{2}\hat{D}^+\left(\frac{h_i(f_i - q_i u_i)}{1+(R_i^h)^{-1}}\right) =$$
$$\frac{1}{2}\hat{D}^+\left(\frac{h_i f_i}{1+(R_i^h)^{-1}}\right) - \frac{1}{2}\hat{D}^+\left(\frac{h_i q_i u_i}{1+(R_i^h)^{-1}}\right) = \frac{1}{2}\hat{D}^+\left(\frac{h_i f_i}{1+(R_i^h)^{-1}}\right) -$$
$$\frac{1}{2}\hat{D}^+\left(\frac{h_i q_i}{1+(R_i^h)^{-1}}\right) u_i - \frac{q_{i+1} h_{i+1}^2/\hat{h}_i}{2(1+(R_{i+1}^h)^{-1})} D^+ u_i \approx \frac{1}{2}\hat{D}^+\left(\frac{h_i f_i}{1+(R_i^h)^{-1}}\right) -$$
$$\frac{1}{2}\hat{D}^+\left(\frac{h_i q_i}{1+(R_i^h)^{-1}}\right) u_i - \frac{h_{i+1}^2/\hat{h}_i}{2(1+(R_{i+1}^h)^{-1})}\frac{q_{i+1} f_i}{r_{i+1}} + \frac{h_{i+1}^2/\hat{h}_i}{2(1+(R_{i+1}^h)^{-1})}\frac{q_{i+1} q_i}{r_{i+1}}.$$

Thus from (3.2) we get on the left the so called modified Samarskii scheme, see [2].

$$L^h U_i = -\varepsilon \hat{D}^+(\kappa_i^h D^- U_i) + r_i^h \breve{D}^- U_i + q_i^h U_i = f_i^h, \quad i = 1, \ldots, m-1. \tag{3.4}$$

where

$$\kappa_i^h = (1+R_i^h)^{-1},\ q_i^h = q_i + \frac{1}{2}\hat{D}^+\left(\frac{h_i q_i}{1+(R_i^h)^{-1}}\right) - \frac{h_{i+1}^2/\hat{h}_i}{2(1+(R_{i+1}^h)^{-1})}\frac{q_{i+1} q_i}{r_{i+1}},$$
$$f_i^h = f_i + \frac{1}{2}\hat{D}^+\left(\frac{h_i f_i}{1+(R_i^h)^{-1}}\right) - \frac{h_{i+1}^2/\hat{h}_i}{2(1+(R_{i+1}^h)^{-1})}\frac{q_{i+1} f_i}{r_{i+1}}.$$

Integrating equation (1.2) on the interval $(x_{i-1/2}, x_{i+1/2})$, $i = m+1, \ldots, N-1$ and dividing by $\hat{h}_i$ we get

$$-\frac{\varepsilon}{\hat{h}_i}(u'_{i+1/2} - u'_{i-1/2}) = \frac{1}{\hat{h}_i}\int_{x_{i-1/2}}^{x_{i+1/2}} (f(x) - q(x)u(x))dx. \tag{3.5}$$

After approximation of the integrals in (3.5), applying (3.3), we obtain the following finite difference scheme on the right of the interface

$$L^h U_i = -\varepsilon \hat{D}^+ D^- U_i + q_i U_i = f_i, \quad i = m+1, \ldots, N-1. \tag{3.6}$$

To obtain numerical approximation on the interface we integrate equation (1.1) on the interval $(x_{m-1/2}, x_m - 0)$, equation (1.2) on the interval $(x_m + 0, x_{m+1/2})$, sum the integrands and use the interface conditions (1.3) to get

$$\begin{aligned} -\frac{\varepsilon}{\hat{h}_m}(u'_{m+1/2} - u'_{m-1/2}) &= \frac{1}{\hat{h}_m}\int_{x_{m-1/2}}^{x_m-0} (f(x) - r(x)u'(x) - q(x)u(x))dx \\ &+ \frac{1}{\hat{h}_m}\int_{x_m+0}^{x_{m+1/2}} (f(x) - q(x)u(x))dx. \end{aligned} \tag{3.7}$$

After approximation of the integrals in (3.7) we get

$$-\frac{\varepsilon}{\hat{h}_m}\left[u'_{m+1/2} - u'_{m-1/2}(1+R^h_m)\right] + \bar{q}_m u_m = \bar{f}_m.$$

Now using the equation (1.2) and the interface conditions (1.3) we approximate

$$\frac{\varepsilon}{\hat{h}_m}(1+R^h_m)u'_{m-1/2} = \frac{\varepsilon}{\hat{h}_m}\left(\frac{1}{1+R^h_m} + 2R^h_m - \frac{R^h_m}{1+(R^h_m)^{-1}}\right)u'_{m-1/2} \approx$$
$$\kappa^h_m D^- U_m + r^h_m \breve{D}^- U_m - \frac{h_m}{2\hat{h}_m(1+(R^h_m)^{-1})}(f_{m-0} - q_{m-0}U_m),$$
$$\frac{\varepsilon}{\hat{h}_m}u'_{m+1/2} \approx \frac{\varepsilon}{\hat{h}_m}u'_{m+1/2} - \frac{\varepsilon h^2_{m+1}}{6\hat{h}_m}u^{(3)}_{m+0} =$$
$$\frac{\varepsilon}{\hat{h}_m}u'_{m+1/2} + \frac{h^2_{m+1}}{6\hat{h}_m}(f'_{m+0} - q'_{m+0}u_m - q_{m+0}u'_{m+0}) \approx$$
$$\frac{\varepsilon}{\hat{h}_m}D^+U_m + \frac{h^2_{m+1}}{6\hat{h}_m}\left(D^+ f_m - U_m D^+ q_m - q_{m+0}D^+U_m\right).$$

The term $-\varepsilon h^2_{m+1}u^{(3)}_{m+0}/(6\hat{h}_m)$ is added to ensure the second order of convergence (see the proof of Theorem 5.1 below). Thus we obtain the following difference scheme on the interface

$$L^h U_m = \left(\rho^h_m - \frac{\varepsilon}{\hat{h}_m}\right)D^+U_m + \left(\frac{h_m r^h_m + \varepsilon\kappa^h_m}{\hat{h}_m}\right)D^-U_m + q^h_m U_m = f^h_m, \tag{3.8}$$

where

$$\rho^h_m = \frac{q_{m+0}h^2_{m+1}}{6\hat{h}_m},\ q^h_m = \bar{q}_m + \frac{h_m q_{m-0}}{2\hat{h}_m(1+(R^h_m)^{-1}} + \frac{h^2_{m+1}}{6\hat{h}_m}D^+q_m,$$
$$f^h_m = \bar{f}_m + \frac{h_m f_{m-0}}{2\hat{h}_m(1+(R^h_m)^{-1})} + \frac{h^2_{m+1}}{6\hat{h}_m}D^+ f_m.$$

Setting the boundary conditions

$$U_0 = \psi_0,\ U_N = \psi_1, \tag{3.9}$$

we obtain the discrete problem (P^h): (3.4), (3.6), (3.8), (3.9). By (1.5) the coefficients in (P^h) are such that

$$\kappa^h > 0,\ q^h_i \geq q_0/2,\ r^h_i \geq r_0,\ i = 1,\dots,m,\ q_i \geq q_0,\ i = 1,\dots,N-1. \tag{3.10}$$

Since $h_{m+1} = 0(\sqrt{\varepsilon}n^{-1}\ln n)$ then for sufficiently large n independent of ε holds

$$\frac{\varepsilon}{\hat{h}_m} - \rho^h_m \geq \rho_0 > 0. \tag{3.11}$$

Therefore this difference scheme satisfies the discrete maximum principle.

LEMMA 3.1. *If $U_0 \geq 0, U_N \geq 0$ and $L^h U_i \geq 0$, then $U \geq 0$ on $\bar{\omega}_h$.*

An immediate consequence of this discrete maximum principle is the following inequality,

LEMMA 3.2. *Let U be a solution to the discrete problem (P^h) satisfying zero boundary conditions. Then*

$$\|U\|_{\infty,\bar{\omega}_h} \leq \frac{2\|L^h U\|_{\infty,\omega_h}}{q_0}.$$

4. A priori estimates. Let V^- be a solution to the discrete problem $(P^{h,-})$,

$$L^{h,-}V_i^- = -\varepsilon\hat{D}^+(\kappa_i^h D^- V_i^-) + r_i^h \breve{D}^- V_i^- + q_i^h V_i^- = f_i^h,\ i = 1 : m-1,$$
$$V_0^- = 0, \quad L^{h,-}V_m^- = \frac{2\varepsilon}{h_m}\kappa_m^h D^- V_m^- + 2r_m^h D^- V_m^- + q_m^{h,-}V_m^- = f_m^{h,-},$$

where

$$q_m^{h,-} = q_{m-0} + \frac{q_{m-0}}{1+(R_m^h)^{-1}},\ f_m^{h,-} = f_{m-0} + \frac{f_{m-0}}{1+(R_m^h)^{-1}}.$$

For the grid functions defined on the mesh $\bar{\omega}_h^- = \{x_0, \ldots, x_m\}$ and vanishing for x_0, define the scalar product

$$(y, v]_{0,\omega_h^-} = \sum_{i=1}^{m-1} \hat{h}_i y_i v_i + \frac{h_m}{2} y_m v_m \tag{4.1}$$

and the norms

$$\|v]|_{1,\omega_h^-} = (|v|, 1]_{0,\omega_h^-}, \quad \|v\|_{\infty,\bar{\omega}_h^-} = \max_{x_i \in \bar{\omega}_h^-} |v_i|.$$

The following problem $(P^{h*,-})$ is adjoint to $(P^{h,-})$ in the sense of the scalar product (4.1),

$$L^{h*,-}W_i^- = -\varepsilon\hat{D}^+(\kappa_i^h D^- W_i^-) - \hat{D}^+(r_i^h W_i^-) + q_i^h W_i^- = f_i^h,\ i = 1 : m-1,$$
$$W_0^- = 0, \quad L^{h*,-}W_m^- = \frac{2\varepsilon}{h_m}\kappa_m^h D^- W_m^- + \frac{2r_m^h}{h_m} W_m^- + q_m^{h,-}W_m^- = f_m^{h,-}.$$

Now, we consider the Green function $G^-(x_i, \eta_j)$ of problem $(P^{h,-})$. As a function of x_i, with η_j held constant, it is defined by the relations

$$L^{h,-}G^-(x_i, \eta_j) = \delta^h(x_i, \eta_j),\ i = 1 : m,\ G^-(0, \eta_j) = 0. \tag{4.2}$$

As a function of η_j, with x_i held constant, Green function $G^-(x_i, \eta_j)$ satisfies the equations

$$L^{h*,-}G^-(x_i, \eta_j) = \delta^h(x_i, \eta_j),\ j = 1 : m,\ G^-(x_i, 0) = 0. \tag{4.3}$$

where

$$\delta^h(x_i, \eta_j) = \begin{cases} \hat{h}_i, & \text{if } x_i = \eta_j, \\ 0, & \text{if } x_i \neq \eta_j. \end{cases}$$

Denote $f^{h,-} = f_i^h$ for $i = 1, \ldots, m-1$. It is obvious that the solution to problem $(P^{h,-})$ is expressed in terms of Green function as

$$V^-(x_i) = (G^-(x_i, \eta_j), f_i^{h,-}]_{0,\omega_h^-}, \tag{4.4}$$

whereas the solution to $(P^{h*,-})$ is written as

$$W^-(x_i) = (G^-(\eta_j, x_i), f_j^h]_{0,\omega_h^-}.$$

LEMMA 4.1. *If conditions (3.10) are fulfilled, the Green function $G^-(x_i, \eta_j)$ is nonnegative and ε-uniformly bounded:*

$$0 \le G^-(x_i, \eta_j) \le r_0^{-1}. \tag{4.5}$$

Moreover, the solution to problem ($P^{h,-}$) satisfies the estimate

$$\|V^-\|_{\infty,\bar\omega_h^-} \le r_0^{-1}\|f^{h,-}]|_{1,\omega_h^-}. \tag{4.6}$$

Proof. Fix $x_i \in \omega_h^-$. The fact that the Green function is nonnegative obviously follows from (4.2) and the validity of the maximum principle for $L^{h,-}$ under conditions (3.10). To derive the estimate in (4.5), consider a point $\eta_{j_0} \in \bar w_h^-$ such that

$$G^-(x_i,\eta_{j_0}) = \max_{\eta_j\in\bar w\omega_h^-} G^-(x_i,\eta_j). \tag{4.7}$$

Multiplying (4.3) by $\hat h_j$ for $j = 1 : m-1$, by $h_m/2$ for $j = 2$ and summing up the results with respect to j from j_0 to m one obtains

$$\begin{aligned}
&\sum_{j=j_0}^{m-1} \hat h_j L^{h*,-}G^-(x_i,\eta_j) + \frac{h_m}{2}L^{h*,-}G^-(x_i,\eta_m) + \\
&r_{j_0}^h G^-(x_i,\eta_{j_0}) + \sum_{j=j_0}^{m-1}\hat h_j q_j^h G^-(x_i,\eta_j) + \frac{h_m}{2}q_m^{h,-}G^-(x_i,\eta_m) = \\
&\sum_{j=j_0}^{m-1}\hat h_j\delta^h(x_i,\eta_j) + \frac{h_m}{2}\delta^h(x_i,\eta_m) \le 1.
\end{aligned} \tag{4.8}$$

Since the Green's function is nonnegative and (4.7) holds, then all terms in (4.8) are non-negative. Thus (3.10) implies (4.5). The estimate (4.6) follows directly from (4.5) and the representation (4.4). □

Let now V^+, $V_i^+ = V_{m+i}^+$ be a solution to the discrete problem ($P^{h,+}$):

$$\begin{aligned}
&L^{h,+}V_i^+ = -\varepsilon\hat D^+D^-V_i^+ + q_iV_i^+ = f_i, \quad i = m+1,\dots,N-1,\\
&V_N^+ = 0, \quad L^{h,+}V_m^+ = \left(\frac{h_{m+1}q_{m+0}}{3} - \frac{2\varepsilon}{h_{m+1}}\right)D^+V_m + q_m^{h,+}V_m^+ = f_m^{h,+}.
\end{aligned}$$

where

$$q_m^{h,+} = q_{m+0} + \frac{h_{m+1}}{3}D^+q_m,\ f_m^{h,+} = f_{m+0} + \frac{h_{m+1}}{3}D^+f_m.$$

Denote $f_i^{h,+} = f_i$ and $q_i^{h,+} = q_i$ for $i = m+1,\dots,N-1$. The problem ($P^{h,+}$) can be written in operator form

$$A^+V_i^+ = f^{h,+},i,\ i = m,\dots,N-1,\ V_N = 0. \tag{4.9}$$

For the grid functions defined on the mesh $\bar\omega_h^+ = \{x_m,\dots,x_N\}$ and vanishing for x_N, define the scalar product

$$[y,v)_{0,\omega_h^+} = \sum_{i=m+1}^{N-1}\hat h_iy_iv_i + \tilde h_my_mv_m,\ (y,v]_{*,\omega_h^+} = \sum_{i=m+1}^{N} h_iy_iv_i. \tag{4.10}$$

where

$$\tilde h_m = \frac{3h_{m+1}\varepsilon}{6\varepsilon - q_{m+0}h_{m+1}^2} \le c_0h_{m+1}.$$

ETNA
Kent State University
etna@mcs.kent.edu

and c_0 is independent of ε constant, given in (3.11).

LEMMA 4.2. *The operator A^+ from (4.9) is selfadjoint and positive definite in the scalar product $[.,.)_{0,\omega_h^+}$ defined in (4.10). For arbitrary discrete functions y, v defined on ω_h^+ and satisfying $y_N = v_N = 0$ holds*

$$[y,v)_{A^+} \equiv [A^+y,v)_{0,\omega_h^+} = \varepsilon(D^-y,D^-v]_{*,\omega_h^+} + [q^{h,+}y,v)_{0,\omega_h^+}. \tag{4.11}$$

Proof. The operator A^+ satisfies

$$[y,v)_{A^+} = -\varepsilon(D^+y_m)v_m - \sum_{i=m+1}^{N-1} \varepsilon\hat{h}_i(\hat{D}^+D^-y_i)v_i + [q^{h,+}y,v)_{0,\omega_h^+} =,$$

$$\sum_{i=1}^{N} \varepsilon h_i D^-y_i D^-v_i + [q^{h,+}y,v)_{0,\omega_h^+} = \varepsilon(D^-y,D^-v]_{*,\omega_h^+} + [q^{h,+}y,v)_{0,\omega_h^+}.$$

Now the statement of the lemma follows from the positivity of $q_i^{h,+}$. ◻

Since A^+ is selfadjoint and positive definite operator, then $(A^+)^{-1}$ is also selfadjoint and positive definite operator. So we can define the energy norms

$$|[v\|_{A^+} = \sqrt{[A^+v,v)_{0,\omega_h^+}}, \quad |[u\|_{(A^+)^{-1}} = \sqrt{[(A^+)^{-1}v,v)_{0,\omega_h^+}}. \tag{4.12}$$

LEMMA 4.3. *Let V^+ be a solution to problem $(P^{h,+})$ then*

$$\|V^+\|_{\infty,\bar{\omega}_h^+} \le \frac{C}{\sqrt{\varepsilon}}|[f^{h,+}\|_{(A^+)^{-1}}. \tag{4.13}$$

for some positive constant C independent of the small parameter ε.

Proof. Since $A^+V^+ = f^{h,+}$ then $V^+ = (A^+)^{-1}f^{h,+}$. Using the embedding inequality (see [16])

$$\|V^+\|_{\infty,\bar{\omega}_h^+} \le C(D^-V^+,D^-V^+]_{*,\omega_h^+}$$

and estimate (4.11) we have

$$\|V^+\|_{\infty,\bar{\omega}_h^+} \le C(D^-V^+,D^-V^+]_{*,\omega_h^+} \le \frac{C}{\sqrt{\varepsilon}}[A^+V^+,V^+)_{0,\omega_h^+}$$
$$= \frac{C}{\sqrt{\varepsilon}}[f^{h,+},(A^+)^{-1}f^{h,+})_{0,\omega_h^+} = \frac{C}{\sqrt{\varepsilon}}|[f^{h,+}\|_{(A^+)^{-1}}. \quad ◻$$

5. Uniform convergence. Suppose that $m \approx n \approx N/2$. Let $Z = U - u$ be the error of the discrete problem (P^h). Then Z satisfies

$$L^hZ = (f_i^h - f_i) - (L^hu_i - Lu_i) \equiv \psi_i, \quad x_i \in \omega_h, \quad Z_0 = Z_N = 0.$$

The next Theorem gives the main result in this paper.

THEOREM 5.1. *Let the conditions in Theorem 2.4 be fulfilled. If the parameters of the mesh satisfy $\sigma_1, \sigma_2 \ge 2$, $n \approx m \approx N/2$, and N is sufficiently large (3.11) to hold, then the solution U of the discrete problem (P^h) is ε-uniformly convergent to the solution u to the continuous problem (1.1)-(1.4) in discrete maximum norm and the following estimates hold*

$$\|U - u\|_{\infty,\omega_h} \le CN^{-2}\ln^2 N, \tag{5.1}$$

ETNA
Kent State University
etna@mcs.kent.edu

for some positive constant C independent of the parameters of the mesh and the small parameter ε.

Proof. Using the fact that the solution u can be decomposed into regular part v and singular part w, that satisfy the estimates (2.6)-(2.8) in Theorem 2.4 we can write the approximation error ψ_i in the form

$$\psi_i = \psi_{1,i} + \psi_{2,i}$$

where

$$\psi_{1,i} = f_i^h - f_i - (L^h v_i - Lv_i), \quad \psi_{2,i} = L^h w_i - Lw_i.$$

Denote $\Delta_i = [x_{i-1}, x_{i+1}]$, $\Delta_i^- = [x_{i-1}, x_{i-0}]$ and $\Delta_i^+ = [x_{i+0}, x_{i+1}]$. We begin with the internal points in ω_h^-. The approximation error has the form, see [4]

$$\begin{aligned}\psi_i &= \varepsilon \hat{D}^+ D^- u_i - \varepsilon u_i'' - \left[\frac{1}{2}(r_{i+1}^h \hat{D}^+ u_i + r_i^h \breve{D}^- u_i) - r_i u_i'\right] \\ &+ \hat{D}\left(\frac{h_i}{2(1+(R_i^h)^{-1})}(r_i^h D^- u_i + q_i u_i - f_i)\right) \\ &+ \frac{h_{i+1}^2/\hat{h}_i}{2(1+(R_i^h)^{-1})}\frac{q_{i+1}}{r_{i+1}}(-r_{i+1} D^+ u_i - q_i u_i + f_i)\end{aligned}$$

Similarly to [2, 4], using that $\varepsilon / \left(\hat{h}_i(1+(R_i^h)^{-1})\right)$ is ε-uniformly bounded, we can prove

$$\begin{aligned}|\psi_i| \le C \Bigg(&(\varepsilon|h_{i+1} - h_i| + \hat{h}_i^2) \sum_{j=1}^{3} \max_{x\in\Delta_i} |u^{(j)}(x)| + \varepsilon \hat{h}_i^2 \sum_{j=1}^{4} \max_{x\in\Delta_i} |u^{(j)}(x)| \\ &+ (|h_{i+1} - h_i| + \hat{h}_i^2) \sum_{j=1}^{2} \max_{x\in\Delta_i} |u^{(j)}(x)| \Bigg). \end{aligned} \tag{5.2}$$

On the interface we present the approximation error in the form

$$\psi_m = \frac{h_m}{2\hat{h}_m}(\psi_{1,m} + \psi_{2,m}) + \frac{\varepsilon h_{m+1}}{2\hat{h}_m \tilde{h}_m}(\eta_{1,m+1} + \mu_m) + \frac{h_{m+1}}{2\hat{h}_m}(\xi_{1,m} + \xi_{2,m}).$$

where

$$\begin{aligned}|\psi_{1,m}| &= \Bigg|\frac{2\varepsilon}{h_m}\left[v_{m-0}' - \frac{h_m}{2} v_{m-0}'' - D^- v_{m-0}\right] + \left[r_{m-0} v_{m-0}' - r_m^h D^- v_{m-0}\right] \\ &+ \frac{1}{1+(R_m^h)^{-1}}\left[-\varepsilon v_{m-0}'' - r_m^h D^- v_{m-0} + r_{m-0} v_{m-0}'\right]\Bigg| \\ &\le C\left(\varepsilon h_m \max_{x\in\Delta_m^-} |v^{(3)}(x)| + h_m \sum_{j=1}^{2} \max_{x\in\Delta_m^-} |v^{(j)}(x)|\right) \end{aligned} \tag{5.3}$$

$$\begin{aligned}|\psi_{2,m}| &= \Bigg|\frac{2\varepsilon}{h_m}\left[w_{m-0}' - \frac{h_m}{2} w_{m-0}'' - D^- w_{m-0}\right] + \left[r_{m-0} w_{m-0}' - r_m^h D^- w_{m-0}\right] \\ &+ \frac{1}{1+(R_m^h)^{-1}}\left[-\varepsilon w_{m-0}'' - r_m^h D^- w_{m-0} + r_{m-0} w_{m-0}'\right]\Bigg|\end{aligned}$$

ETNA
Kent State University
etna@mcs.kent.edu

$$\leq C\left(\varepsilon h_m \max_{x\in\Delta_m^-}|w^{(3)}(x)| + h_m\sum_{j=1}^{2}\max_{x\in\Delta_m^-}|w^{(j)}(x)|\right) \tag{5.4}$$

$$|\eta_{1,m+1}| = \frac{2\tilde{h}_m}{h_{m+1}}\left|-D^+v_{m+0} + v'_{m+1/2} - \frac{h_{m+1}^2}{8}v^{(3)}_{m+1/2}\right|$$

$$\leq Ch_{m+1}^2\max_{x\in\Delta_m^+}|v^{(3)}(x)| \tag{5.5}$$

$$|\mu_m| = \frac{2\tilde{h}_m}{h_{m+1}}\left|v'_{m+0} + \frac{h_{m+1}}{2}v''_{m+0} + \frac{h_{m+1}^2}{6}v^{(3)}_{m+0} - v'_{m+1/2} + \frac{h_{m+1}^2}{8}v^{(3)}_{m+1/2}\right|$$

$$\leq Ch_{m+1}^2\max_{x\in\Delta_m^+}|v^{(3)}(x)| \tag{5.6}$$

$$|\xi_{1,m}| = \frac{h_{m+1}}{3}\left|f'_{m+0} - q'_{m+0}u_m - q_{m+0}u'_{m-0} - D^+f_{m+0} + u_mD^+q_{m+0}\right.$$

$$\left. - q_{m+0}D^-u_{m-0}\right| \leq C\left(h_{m+1}^2 + h_{m+1}h_m\max_{x\in\Delta_m^-}|u^{(2)}(x)|\right) \tag{5.7}$$

$$|\xi_{2,m}| = \frac{2\varepsilon}{h_{m+1}}\left|-D^+w_{m+0} + w'_{m+0} + \frac{h_{m+1}}{2}w''_{m+0} + \frac{h_{m+1}^2}{6}w^{(3)}_{m+0}\right|$$

$$\leq C\varepsilon h_{m+1}^2\max_{x\in\Delta_m^+}|w^{(4)}(x)| \tag{5.8}$$

At the interior points on the interval $(\xi, 1)$ we present the approximation error in the form

$$\psi_i = \varepsilon\hat{D}^+\eta_{1,i} + \xi_{1,i} + \xi_{2,i},$$

where

$$|\eta_{1,i}| = \left|-D^-v_i + v'_{i-0.5} - \frac{h_i^2}{8}v^{(3)}_{i-0.5}\right| \leq Ch_i^2\max_{x\in\Delta_i^-}|v^{(3)}(x)| \tag{5.9}$$

$$|\xi_{1,i}| = \varepsilon\left|v''_i - \frac{1}{\hat{h}_i}\left(v'_{i+0.5} - v'_{i-0.5} - \frac{h_{i+1}^2}{8}v^{(3)}_{i+0.5} + \frac{h_i^2}{8}v^{(3)}_{i-0.5}\right)\right|$$

$$\leq C\varepsilon h_i^2\max_{x\in\Delta_i}|v^{(4)}(x)| \tag{5.10}$$

$$|\xi_{2,i}| = \left|\varepsilon\hat{D}^+D^-w_i - \varepsilon w''_i\right| \leq C\varepsilon\min\left\{\max_{x\in\Delta_i}|w^{(2)}(x)|,\right.$$

$$\left.|h_{i+1} - h_i|\max_{x\in\Delta_i}|w^{(3)}(x)| + h_i^2\max_{x\in\Delta_i}|w^{(4)}(x)|\right\}. \tag{5.11}$$

Now we decompose the error Z in the form $Z = Z^1 + Z^2 + Z^3$. The first term Z^1 is a solution to the problem

$$L^{h,-}Z_i^1 = \psi_{1,i} + \psi_{2,i},\ i = 1,\ldots,m,\ Z_0^1 = 0.$$

Using Theorem 2.4 and estimates (5.2)-(5.4), similarly to [2, 4] we obtain

$$\|\psi_1]|_{1,\omega_h^-} = O(N^{-2}\ln^2 N),\ \|\psi_2]|_{1,\omega_h^-} = O(\sqrt{\varepsilon}N^{-2}\ln^2 N).$$

Applying the a priori estimates (4.6) we get

$$\|Z^1\|_{\infty,\omega_h^-} \leq C\left(\|\psi_1]|_{1,\omega_h^-} + \|\psi_2]|_{1,\omega_h^-}\right) \leq CN^{-2}\ln^2 N. \tag{5.12}$$

ETNA
Kent State University
etna@mcs.kent.edu

The second term Z^2 is a solution to the problem

$$L^{h,+}Z_i^2 = \varepsilon\left(\hat{D}^+\eta_{1,i} + \hat{D}^+\eta_{2,i}\right) = \theta_i, i = m+1,\ldots,N-1,$$
$$L^{h,+}Z_m^2 = \frac{\varepsilon}{\tilde{h}_m}(\eta_{1,m+1} + \eta_{2,m+1}) = \theta_m,\; Z_N^+ = 0.$$

where

$$\eta_{2,i} = \mu_m,\; i = m+1,\ldots,N.$$

Using Theorem 2.4 and estimates (5.5),(5.6), (5.9) we obtain

$$\eta_{1,i} = O(N^{-2}),\; \eta_{2,i} = O(N^{-2}),\; i = m+1,\ldots,N.$$

Let $\zeta = (A^+)^{-1}\theta$ and $\eta_i = \eta_{1,i} + \eta_{2,i}$ then

$$|[\theta\|^2_{(A^+)^{-1}} = |[\zeta\|^2_{A^+} = |[\zeta,\theta)_{0,\omega_h^+} = \varepsilon\zeta_m\eta_{m+1} + \varepsilon\sum_{i=m+1}^{N-1}\hat{h}_i\zeta_i\hat{D}^+\eta_i$$
$$\le -\varepsilon(D^-\zeta,\eta]_{*,\omega_h^+} \le C\sqrt{\varepsilon}N^{-2}|[\zeta\|_{A^+}.$$

Therefore

$$|[\theta\|^2_{(A^+)^{-1}} \le C\sqrt{\varepsilon}N^{-2}.$$

Applying the a priori estimates (4.13) we get

$$\|Z^2\|_{\infty,\omega_h^+} \le \frac{C}{\sqrt{\varepsilon}}|[\theta\|_{(A^+)^{-1}} \le CN^{-2}. \tag{5.13}$$

The last term Z^3 is a solution to the problem

$$L^{h,+}Z_i^3 = \xi_{1,i} + \xi_{2,i},\; i = m,\ldots,N-1,\; Z_N^3 = 0.$$

Using Theorem 2.4 and estimates (5.7),(5.8), (5.10), (5.11), similarly to [2, 4] we obtain

$$\xi_{1,m} = O(N^{-2}\ln^2 N),\; \xi_{1,i} = O(\varepsilon N^{-2}\ln^2 N),\, i = m+1,\ldots,N-1,$$
$$\xi_{2,i} = O(N^{-2}\ln^2 N),\; i = m,\ldots,N.$$

Now from the comparison principle for operator $L^{h,+}$ we find

$$\|Z^3\|_{\infty,\omega_h^+} \le C\|\xi_1 + \xi_2\|_{\infty,\omega_h^+} \le CN^{-2}\ln^2 N. \tag{5.14}$$

The theorem follows now from estimates (5.12)-(5.14) □

REMARK 5.2. The requirements $n \approx m \approx N/2$ are a technicality. In the general case it is clear from the analysis above that the order of convergence will be $O(m^{-2}\ln^2 m + n^{-2}\ln^2 n + mn\ln m\ln n)$.

6. Numerical results. Consider the problem

$$-\varepsilon u'' + (1+\cos(\pi x))\,u' + (1+\sin(\pi x/2))\,u = 1+\sin(\pi x)\cos(\pi x)), \quad x \in (0, 0.5),$$
$$-\varepsilon u'' + (4+cos(\pi x/2))\,u = 3+2\sin(\pi x/2)\cos(\pi x/2), \quad x \in (0.5, 1),$$
$$[u]_{x=0.5} = 0,\ [u']_{x=0.5} = 0,\ u(0) = u(1) = 0.$$

The solution of this problem exhibits typical boundary and interface layer behavior (see Fig. 6.1). For our tests we take $m = n = N/2$. The exact solution of this problem is not known. To investigate the convergence rate we compare our numerical results with the linear interpolation of the solution for $N = 16384$. Table 6.1 displays the results of our numerical experiments. For large N we observe almost second-order ε-uniform convergence. The convergence rate is taken to be

$$\rho_N = \log_2\left(\|E_N\|_{\infty,w}/\|E_{2N}\|_{\infty,w}\right),$$

where $\|E_N\|_{\infty,w}$ is the maximum error norm error for the corresponding value of N. Figure 6.1 shows the approximate solution and the maximal error for $N = 128$ and $\varepsilon = 2^{-10}$. It illustrates very well the boundary and interior layers behavior of the solution. Thus the numerical results support the theoretical ones and show the effectiveness of the special meshes.

TABLE 6.1
Error of the solution on Shishkin's meshes

$\varepsilon\backslash N$	32	64	128	256	512	1024	2048
$\varepsilon = 1$	6.53e-5	1.62e-5	4.04e-6	1.01e-6	2.52e-7	6.28e-8	1.56e-8
ρ_N	2.01	2.01	2.00	2.00	2.00	2.01	2.04
$\varepsilon = 2^{-2}$	2.34e-4	5.46e-5	1.31e-5	3.19e-6	7.87e-7	1.95e-7	4.80e-8
ρ_N	2.10	2.06	2.03	2.02	2.01	2.02	2.07
$\varepsilon = 2^{-4}$	2.01e-3	5.67e-4	1.49e-4	3.81e-5	9.65e-6	2.42e-6	6.00e-7
ρ_N	1.83	1.93	1.96	1.98	1.99	2.01	2.07
$\varepsilon = 2^{-6}$ 2	2.74e-3	9.34e-4	3.84e-4	1.42e-4	4.93e-5	1.61e-5	5.03e-6
ρ_N	1.55	1.28	1.43	1.53	1.61	1.68	1.78
$\varepsilon = 2^{-8}$	1.04e-2	2.77e-3	7.03e-4	1.77e-4	4.41e-5	1.10e-5	2.72e-6
ρ_N	1.91	1.98	1.99	2.00	2.00	2.01	1.95
$\varepsilon = 2^{-10}$	1.80e-2	7.94e-3	2.79e-3	7.09e-4	1.78e-4	4.44e-5	1.10e-5
ρ_N	1.18	1.51	1.98	1.99	2.00	2.01	2.07
$\varepsilon = 2^{-12}$	1.80e-2	7.96e-3	3.02e-3	1.04e-3	3.41e-4	1.06e-4	3.19e-5
ρ_N	1.18	1.40	1.54	1.60	1.68	1.74	1.77
$\varepsilon = 2^{-14}$	1.80e-2	7.98e-3	3.02e-3	1.04e-3	3.41e-4	1.06e-4	3.20e-5
ρ_N	1.18	1.40	1.54	1.61	1.68	1.73	1.80
$\varepsilon = 2^{-16}$	1.80e-2	7.98e-3	3.03e-3	1.04e-3	3.41e-4	1.07e-4	3.20e-5
ρ_N	1.17	1.40	1.54	1.61	1.68	1.73	1.80
$\varepsilon = 2^{-18}$	1.80e-2	7.99e-3	3.03e-3	1.04e-3	3.42e-4	1.07e-4	3.20e-5
ρ_N	1.17	1.40	1.54	1.61	1.68	1.73	1.80

REFERENCES

[1] V. B. ANDREEV, *Green's function and uniform convergence of monotone difference schemes for a singularly perturbed convection-diffusion equation on Shishkin's mesh in one and two dimensions*, DCU Maths. Dept. Preprint Series, MS-01-15, 2001.
[2] V. B. ANDREEV AND I. A. SAVIN, *On the uniform convergence of the monotone Samarski's scheme and its modification*, Comput. Math. Math. Phys., 35 (1995), pp. 581–591.
[3] I. A. BRAIANOV AND L. G. VULKOV, *Grid approximation for the solution of the singularly perturbed heat equation with concentrated capacity*, J. Math. Anal. Appl., 237 (1999), pp. 672–697.
[4] I. A. BRAIANOV AND L. G. VULKOV, *Uniform in a small parameter convergence of Samarskii's monotone scheme and its modification for the convection-diffusion equation with a concentrated source*, Comput. Math. Math. Phys., 40 (2000), pp. 534–550.

ETNA
Kent State University
etna@mcs.kent.edu

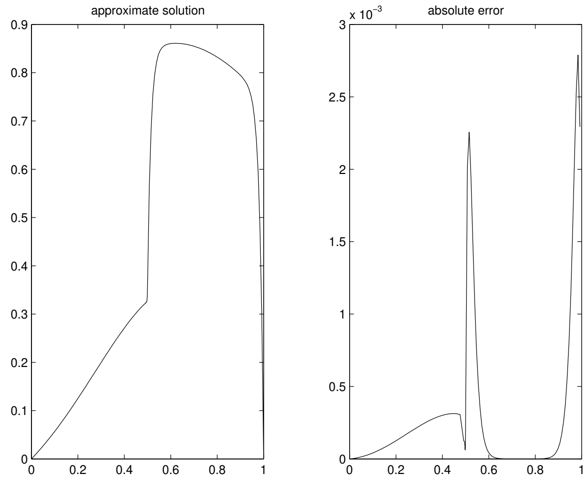

FIG. 6.1. *Approximate solution and error on Shishkin mesh,* $\varepsilon = 2^{-10}, N = 128$

[5] I. A. BRAIANOV AND L. G. VULKOV, *Numerical solution of a reaction-diffusion elliptic interface problem with strong anisotropy*, Computing, 71 (2003), pp. 153–173.

[6] P. A. FARRELL, A. F. HEGARTY, J. J. H. MILLER, E. O'RIORDAN, AND G. I. SHISHKIN, *Singularly perturbed differential equations with discontinuous source terms*, in Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems, J.J.H. Miller, G.I. Shishkin and L. Vulkov, eds., Nova Science Publishers, Inc., New York, USA, 2000, pp. 23–32.

[7] P. A. FARRELL, A. F. HEGARTY, J. J. H. MILLER, E. O'RIORDAN, AND G. I. SHISHKIN, *Singularly perturbed convection-diffusion problems with boundary and week interior layers*, J. Comput. Appl. Math., 166 (2004), pp. 133-151.

[8] P. A. FARRELL, A. F. HEGARTY, J. J. H. MILLER, E. O'RIORDAN, AND G. I. SHISHKIN, *Global maximum norm parameter - uniform numerical method for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient*, Math. Comput. Modelling, 40 (2004), pp. 1375-1392.

[9] T. LINSS, *Finite difference schemes for convection-diffusion problems with a concentrated source and discontinuous convection field*, Comput. Methods Appl. Math., 2 (2002), pp. 41–49.

[10] T. LINSS, *Layer-adapted meshes for convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 1061-1105.

[11] J. J. H. MILLER, E. O'RIORDAN, AND G. I. SHISHKIN, *Fitted Numerical Methods for Singular Perturbation Problems*, World scientific, Singapore, 1996.

[12] J. J. H. MILLER, E. O'RIORDAN, G. I. SHISHKIN, AND S. WANG, *A parameter-uniform Schwarz method for a singularly perturbed reaction-diffusion problem with an interior layer*, Appl. Numer. Math., 35 (2000), pp. 323–337.

[13] H.-G. ROOS, M. STYNES, AND L. TOBISKA, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer-Verlag, Berlin, 1996.

[14] H.-G. ROOS AND H. ZARIN, *A second order scheme for singularly perturbed differential equations with discontinuous source term*, J. Numer. Math., 10 (2002), pp. 275–289.

[15] H.-G. ROOS AND H. ZARIN, *The streamline-diffusion method for a convection-diffusion problem with a point source*, J. Comput. Appl. Math., 150 (2003), pp. 109–128 (2003).

[16] A. A. SAMARSKII, *The Theory of Difference Schemes*, Third ed. Nauka, Moscow, 1989 (in Russian), English translation: Dekker, New York, 2001.

[17] G. I. SHISHKIN, *Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations*, Ural Branch, Russian Academy of Sciences, Ekaterinburg, 1992 (in Russian).