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A RATIONAL SPECTRAL PROBLEM IN FLUID–SOLID VIBRATION*

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Abstract. In this paper we apply a minmax characterization for nonoverdamped nonlinear eigenvalue problems to a rational eigenproblem governing mechanical vibrations of a tube bundle immersed in an inviscid compressible fluid. This eigenproblem is nonstandard in two respects: it depends rationally on the eigenparameter, and it involves non-local boundary conditions. Comparison results are proved comparing the eigenvalues of the rational problem to those of certain linear problems suggesting a way how to construct ansatz vectors for an efficient projection method.

Key words. nonlinear eigenvalue problem, maxmin principle, fluid structure interaction.

AMS subject classification. 49G05.

1. Introduction. In this paper we study a model which governs the vibrations of a tube bundle immersed in a inviscid compressible fluid under the following simplifying assumptions. The tubes are rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is very long along the generating lines of the tubes. Due to these assumptions three dimensional effects are neglected, and the problem is studied in any transversal section of the cavity.

The mathematical model describing the dynamical behaviour of this system was obtained by Planchard in [9] and was studied in [2], [3], e.g. It is an elliptic eigenvalue problem with non-local conditions on the boundaries of the tubes which depend nonlinearly on the eigenparameter and which can be transformed to a rational eigenvalue problem.

Using methods from linear functional analysis Conca, Planchard and Vanninathan [3] proved that this problem has a countable set of eigenvalues which are positive and real and which converge to infinity. To this end they transformed the rational eigenproblem to a linear compact eigenproblem on a Hilbert space which is nonselfadjoint but can be symmetrized easily.

In this paper we prove that the eigenvalues of the underlying rational eigenproblem can be characterized as minmax values of a Rayleigh functional, from which we immediately obtain the existence of countably many real and positive eigenvalues. Moreover, considering the nonlinear problem as perturbation of suitable linear eigenproblems we obtain inclusion results for the eigenvalues. These comparison theorems at the same time suggest how to determine ansatz spaces for projection methods yielding efficient methods to solve the nonlinear eigenvalue problem numerically.

The paper is organized as follows. Section 2 briefly summarizes variational characterizations of the eigenvalues of symmetric nonlinear eigenvalue problems. In Section 3 we present the mathematical model which describes the problem governing free vibrations of a tube bundle immersed in an inviscid, slightly compressible fluid, and show that the eigenvalues are minmax values of a Rayleigh functional, and in Section 4 the comparison results are derived. Finally, in Section 5 we propose a projection method based on the comparison results of Section 4, and we demonstrate its efficiency by a numerical example.

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2. Characterization of eigenvalues of nonlinear eigenproblems. We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{2.1}$$

where $T(\lambda)$ is a selfadjoint and bounded operator on a real Hilbert space H for every λ in an open real interval J. As in the linear case $\lambda \in J$ is called an eigenvalue of problem (2.1) if equation (2.1) has a nontrivial solution $x \neq 0$. Such an x is called an eigenelement or eigenvector corresponding to λ.

We assume that

$$f : \begin{cases} J \times H & \to \quad \mathbb{R} \\ (\lambda, x) & \mapsto \quad \langle T(\lambda)x, x\rangle \end{cases} \tag{2.2}$$

is continuously differentiable, and that for every fixed $x \in H^0$, $H^0 := H \setminus \{0\}$, the real equation

$$f(\lambda, x) = 0 \tag{2.3}$$

has at most one solution in J. Then equation (2.3) implicitly defines a functional p on some subset D of H^0 which we call the Rayleigh functional.

We assume that

$$\frac{\partial}{\partial\lambda} f(\lambda, x)\big|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D. \tag{2.4}$$

Then it follows from the implicit function theorem that D is an open set and that p is continuously differentiable on D.

For the linear eigenvalue value problem $T(\lambda) := \lambda I - A$ where $A : H \to H$ is selfadjoint and continuous the assumptions above are fulfilled, p is the Rayleigh quotient and $D = H^0$. If A additionally is completely continuous then A has a countable set of eigenvalues which can be characterized as minmax and maxmin values of the Rayleigh quotient by the principles of Poincaré and of Courant, Fischer and Weyl.

For the nonlinear case variational properties using the Rayleigh functional were proved for overdamped systems (i.e. if the Rayleigh functional is defined on H^0) by Duffin [4] and Rogers [10] for the finite dimensional case and by Hadeler [5], [6], Rogers [11] and Werner [15] for the infinite dimensional case, and for nonoverdamped systems by Werner and the author [14], [13]

In this section we assemble the results of [14] concerning the minmax characterization of the nonlinear eigenvalue problem (2.1) corresponding to the Poincaré principle.

We denote by H_j the set of all j-dimensional subspaces of H and by $V_1 := \{v \in V : \|v\| = 1\}$ the unit sphere of the subspace V of H.

We assume that for every fixed $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that the linear operator $T(\lambda) + \nu(\lambda)I$ is completely continuous. Then the essential spectrum of $T(\lambda)$ contains only the point $-\nu(\lambda)$, and every eigenvalue $\mu > -\nu(\lambda)$ of $T(\lambda)$ can be characterized as maxmin value of the Rayleigh quotient of $T(\lambda)$. In particular, if λ is an eigenvalue of the nonlinear problem (2.1), then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $n \in \mathbb{N}$ such that

$$\mu_n(\lambda) := \max_{V \in H_n} \min_{v \in V_1} \langle T(\lambda)v, v\rangle = 0. \tag{2.5}$$

In this case we call λ an n-th eigenvalue of the nonlinear eigenvalue problem (2.1).

The following results were proved in [14]

THEOREM 2.1. *Under the conditions given above the following assertions hold:*

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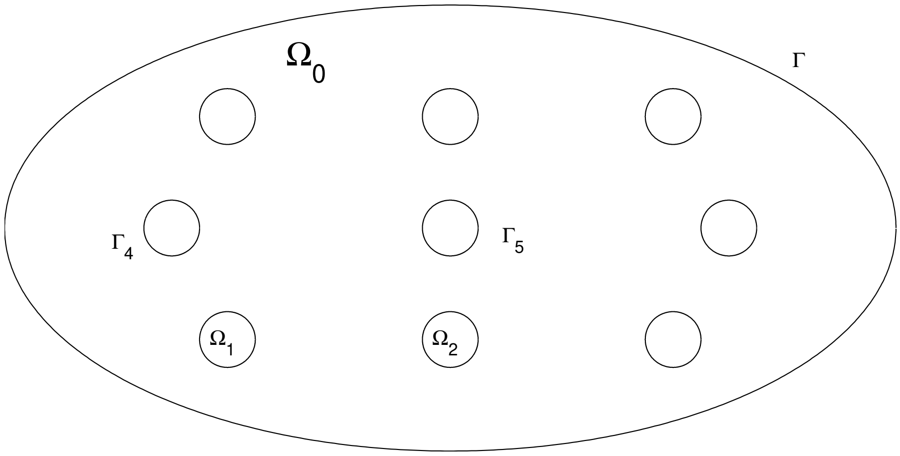

Fig. 1: Domain Ω_0

(i) For every $n \in \mathbb{N}$ there is at most one n-th eigenvalue of problem (2.1) which can be characterized by

$$\lambda_n = \min_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v). \tag{2.6}$$

The minimum is attained by the invariant subspace W of $T(\lambda_n)$ corresponding to the n largest eigenvalues of $T(\lambda_n)$, and $\sup_{v \in W \cap D} p(v)$ is attained by all eigenvectors of (2.1) corresponding to λ_n. The set of eigenvalues of (2.1) is at most countable.

(ii) If

$$\lambda_n = \inf_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v) \in J \tag{2.7}$$

for some $n \in \mathbb{N}$ then λ_n is the n-th eigenvalue of (2.1) and (2.6) holds.

(iii) If there exists the m-th and the n-th eigenvalue λ_m and λ_n in J and $m < n$ then J contains the k-th eigenvalue λ_k for $m < k < n$ and

$$\inf J < \lambda_m \leq \lambda_{m+1} \leq \ldots \leq \lambda_n < \sup J.$$

(iv) If $\lambda_1 \in J$ and $\lambda_n \in J$ for some $n \in \mathbb{N}$ then every $V \in H_j$ with $V \cap D \neq \emptyset$ and $\lambda_j = \sup_{u \in V \cap D} p(u)$ is contained in D, and the characterization (2.6) can be replaced by

$$\lambda_j = \min_{\substack{V \in H_j \\ V_1 \subset D}} \max_{v \in V_1} p(v) \quad j = 1, \ldots, n. \tag{2.8}$$

The characterization of the eigenvalues in Theorem 2.1 is a generalization of the minmax principle of Poincaré for linear eigenvalue problems. In a similar way as in [14] the maxmin characterization of Courant, Fischer and Weyl can be generalized to the nonlinear case (cf. [13]).

3. A rational eigenvalue problem in fluid structure interaction. This section is devoted to the presentation of the mathematical model which describes the problem governing free vibrations of a tube bundle immersed in a slightly compressible fluid under the following simplifying assumptions: The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they can not move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity. Considering small vibrations of the fluid (and the tubes) around the state of rest, it can also be assumed that the fluid is irrotational.

Mathematically this problem can be described in the following way (cf. [9], [3]). Let $\Omega \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1, \dots, K$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega$ for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set $\Omega_0 := \Omega \setminus \bigcup_{j=1}^{K} \Omega_j$. Then the boundary of Ω_0 consists of $K+1$ connected components which are Γ and Γ_j, $j = 1, \dots, K$.

We denote by $H^1(\Omega_0) = \{u \in L^2(\Omega_0) \,:\, \nabla u \in L^2(\Omega_0)^2\}$ the standard Sobolev space equipped with the usual scalar product

$$(u, v) := \int\limits_{\Omega_0} (u(x)v(x) + \nabla u(x) \cdot \nabla v(x))\, dx.$$

Then the eigenfrequencies and the eigenmodes of the fluid-solid structure are governed by the following variational eigenvalue problem (cf. [9], [3])

Find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega_0)$ such that for every $v \in H^1(\Omega_0)$

$$c^2 \int\limits_{\Omega_0} \nabla u \cdot \nabla v\, dx = \lambda \int\limits_{\Omega_0} uv\, dx + \sum_{j=1}^{K} \frac{\lambda \rho_0}{k_j - \lambda m_j} \int\limits_{\Gamma_j} un\, ds \cdot \int\limits_{\Gamma_j} vn\, ds. \tag{3.1}$$

Here u is the potential of the velocity of the fluid, c denotes the speed of sound in the fluid, ρ_0 is the specific density of the fluid, k_j represents the stiffness constant of the spring system supporting tube j, m_j is the mass per unit length of the tube j, and n is the outward unit normal on the boundary of Ω_0.

The eigenvalue problem is non–standard in two respects: The eigenparameter λ appears in a rational way in the boundary conditions, and the boundary conditions are non-local.

In Conca et al. [3] it was shown that the eigenvalues are the characteristic values of a linear compact operator acting on a Hilbert space. The operator associated with this eigenvalue problem is not selfadjoint, but it can be symmetrized in the sense that one can prove the existence of a selfadjoint operator which has the same spectrum as the original operator. Hence, the set of eigenvalues is a countably infinite set of positive real numbers that converge to infinity.

Obviously $\lambda = 0$ is an eigenvalue of (3.1) with eigenfunction $u = \text{const}$. We reduce the eigenproblem (3.1) to the space

$$H := \{u \in H^1(\Omega_0) \,:\, \int\limits_{\Omega_0} u(x)\, dx = 0\}$$

and consider the scalar product

$$\langle u, v \rangle := \int\limits_{\Omega_0} \nabla u(x) \cdot \nabla v(x)\, dx.$$

on H which is known to define a norm on H which is equivalent to the norm induced by $(\cdot,\cdot)$.

By the Lax–Milgram lemma the variational eigenvalue problem (3.1) is equivalent to the nonlinear eigenvalue problem

Determine λ and $u \in H$ such that

$$T(\lambda)u := (-I + \lambda A + \sum_{j=1}^{k} \frac{\rho_0 \lambda}{k_j - \lambda m_j} B_j)u = 0 \tag{3.2}$$

where the linear symmetric operators A and B_j are defined by

$$\langle Au, v\rangle := \int_{\Omega_0} uv\,dx \quad \text{for every } u, v \in H \tag{3.3}$$

$$\langle B_j u, v\rangle := \Big(\int_{\Gamma_j} un\,ds\Big)\cdot\Big(\int_{\Gamma_j} vn\,ds\Big) \quad \text{for every } u, v \in H. \tag{3.4}$$

A is completely continuous by Rellich's embedding theorem and $w := B_j u$, $j = 1,\dots,k$, is the weak solution in H of the elliptic problem

$$\Delta w = 0 \text{ in } \Omega_0, \quad \frac{\partial}{\partial n} w = 0 \text{ on } \partial\Omega_0 \setminus \Gamma_j, \quad \frac{\partial}{\partial n} w = n \cdot \int_{\Gamma_j} un\,ds \text{ on } \Gamma_j.$$

By the continuity of the trace operator B_j is continuous, and since the range of B_j is twodimensional spanned by the solutions $w_i \in H$ of

$$\Delta w_i = 0 \text{ in } \Omega_0, \quad \frac{\partial}{\partial n} w = 0 \text{ on } \partial\Omega_0 \setminus \Gamma_j, \quad \frac{\partial}{\partial n} w = n_i \text{ on } \Gamma_j,\ i = 1, 2,$$

it is even completely continuous. Hence, the general conditions of Section 2 are satisfied.

Rayleigh functionals corresponding to problem (3.2) are defined by the real function

$$\begin{aligned} f(\lambda, u) &:= \langle T(\lambda)u, u\rangle \\ &= -c^2 \int_{\Omega_0} |\nabla u|^2\,dx + \lambda \int_{\Omega_0} u^2\,dx + \sum_{j=1}^{K} \frac{\rho_0\lambda}{k_j - \lambda m_j} \Big|\int_{\Gamma_j} un\,ds\Big|^2. \end{aligned} \tag{3.5}$$

Since

$$\frac{\partial}{\partial\lambda} f(\lambda, u) = \int_{\Omega_0} u^2\,dx + \sum_{j=1}^{K} \frac{\rho_0 k_j}{(k_j - \lambda m_j)^2} \Big|\int_{\Gamma_j} un\,ds\Big|^2 > 0 \quad \text{for } \lambda \neq \frac{k_j}{m_j} \tag{3.6}$$

for every interval $J \subset \mathbb{R}$ such that $\frac{k_j}{m_j} \notin J$ for $j = 1,\dots,k$ there exists a Rayleigh functional corresponding to the eigenvalue problem (3.2) and the results of Section 2 apply: if the open interval $J \subset \mathbb{R}_+$ does not contain $\frac{k_j}{m_j}$, $j = 1,\dots,K$ then all eigenvalues of problem (3.2) in J are minmax values of the Rayleigh functional defined by (3.5).

4. Comparison Results. We now assume that the quotients $\frac{k_j}{m_j}$ are ordered by magnitude

$$0 =: \frac{k_0}{m_0} < \frac{k_1}{m_1} \leq \frac{k_2}{m_2} \leq \dots \leq \frac{k_K}{m_K} < \infty =: \frac{k_{K+1}}{m_{K+1}}.$$

If $\frac{k_{\ell-1}}{m_{\ell-1}} < \frac{k_\ell}{m_\ell}$ for some $\ell \in \{1, \dots, K+1\}$ then problem (3.1) has a Rayleigh functional p_ℓ corresponding to the interval $J_\ell := (\frac{k_{\ell-1}}{m_{\ell-1}}, \frac{k_\ell}{m_\ell})$ which is defined in the domain of definition denoted by D_ℓ.

For $\kappa \in J_\ell$ we consider the linear eigenvalue problem

Find $\lambda \in \mathbb{R}$ and $u \in H^0$ such that for every $v \in H^0$

$$c^2 \int_{\Omega_0} \nabla u \cdot \nabla v \, dx + \sum_{j=1}^{\ell-1} \frac{\rho_0 \kappa}{\kappa m_j - k_j} \int_{\Gamma_j} u n_j \, ds \cdot \int_{\Gamma_j} v n_j \, ds$$

$$= \lambda \Big(\int_{\Omega_0} uv \, dx + \sum_{j=\ell}^{K} \frac{\rho_0}{k_j - \kappa m_j} \int_{\Gamma_j} u n_j \, ds \cdot \int_{\Gamma_j} v n_j \, ds \Big), \tag{4.1}$$

and we denote by

$$R_\kappa(u) := \frac{c^2 \int_{\Omega_0} |\nabla u|^2 \, dx + \sum_{j=1}^{\ell-1} \frac{\rho_0 \kappa}{\kappa m_j - k_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2}{\int_{\Omega_0} u^2 \, dx + \sum_{j=\ell}^{K} \frac{\rho_0}{k_j - \kappa m_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2}$$

the Rayleigh quotient of problem (4.1).

LEMMA 4.1. *Assume that $\kappa \in J_\ell$ and $R_\kappa(u) \in J_\ell$ for some $u \in H^0$. Then $u \in D_\ell$, and*

$$\min(\kappa, R_\kappa(u)) \le p_\ell(u) \le \max(\kappa, R_\kappa(u)). \tag{4.2}$$

Proof. From

$$\begin{aligned}
f(R_\kappa(u), u) = &-c^2 \int_{\Omega_0} |\nabla u|^2 \, dx - \sum_{j=1}^{\ell-1} \frac{R_\kappa(u)\rho_0}{R_\kappa(u) m_j - k_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 \\
&+ \sum_{j=\ell}^{K} \frac{R_\kappa(u)\rho_0}{k_j - R_\kappa(u) m_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 + R_\kappa(u) \int_{\Omega_0} u^2 \, dx \\
= &-c^2 \int_{\Omega_0} |\nabla u|^2 \, dx - \sum_{j=1}^{\ell-1} \frac{\kappa\rho_0}{\kappa m_j - k_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 \\
&+ R_\kappa(u) \Big(\sum_{j=\ell}^{K} \frac{\rho_0}{k_j - \kappa m_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 + \int_{\Omega_0} u^2 \, dx \Big) \\
&+ \sum_{j=1}^{\ell-1} \Big(\frac{\kappa\rho_0}{\kappa m_j - k_j} - \frac{R_\kappa(u)\rho_0}{R_\kappa(u) m_j - k_j} \Big) \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 \\
&+ R_\kappa(u) \sum_{j=\ell}^{K} \Big(\frac{\rho_0}{k_j - R_\kappa(u) m_j} - \frac{\rho_0}{k_j - \kappa m_j} \Big) \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2
\end{aligned}$$

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$$= \rho_0(R_\kappa(u)-\kappa)\sum_{j=1}^{\ell-1}\frac{k_j}{(R_\kappa(u)m_j-k_j)(\kappa m_j-k_j)}\Big|\int_{\Gamma_j} un_j\,ds\Big|^2$$

$$+R_\kappa(u)\rho_0(R_\kappa(u)-\kappa)\sum_{j=\ell}^{K}\frac{m_j}{(k_j-R_\kappa(u)m_j)(k_j-\kappa m_j)}\Big|\int_{\Gamma_j} un_j\,ds\Big|^2$$

$$\begin{cases} \le 0 & \text{for } R_\kappa(u)\le\kappa \\ \ge 0 & \text{for } R_\kappa(u)\ge\kappa \end{cases}$$

and

$$f(\kappa,u) = -c^2\int_{\Omega_0}|\nabla u|^2\,dx - \sum_{j=1}^{\ell-1}\frac{\kappa\rho_0}{\kappa m_j-k_j}\Big|\int_{\Gamma_j} un_j\,ds\Big|^2$$

$$+\sum_{j=\ell}^{K}\frac{\kappa\rho_0}{k_j-\kappa m_j}\Big|\int_{\Gamma_j} un_j\,ds\Big|^2+\kappa\int_{\Omega_0}u^2\,dx$$

$$=(\kappa-R_\kappa(u))\Big(\int_{\Omega_0}u^2\,dx+\sum_{j=\ell}^{K}\frac{\rho_0}{k_j-\kappa m_j}\Big|\int_{\Gamma_j} un_j\,ds\Big|^2\Big)$$

$$\begin{cases} \ge 0 & \text{for } R_\kappa(u)\le\kappa \\ \le 0 & \text{for } R_\kappa(u)\ge\kappa \end{cases}.$$

it follows that in both cases $u\in D_\ell$ and

$$\min(\kappa,R_\kappa(u))\le p_\ell(u)\le\max(\kappa,R_\kappa(u)).$$

□

From Lemma 4.1 we obtain comparison results. First we consider the case $\ell=1$ because in this case $\inf_{u\in D_1}p_1(u)\in J_1$, and even the characterization (2.8) of the eigenvalues in J_1 holds.

LEMMA 4.2.

$$\inf_{u\in D_1}\;p_1(u)>0$$

Proof. The proof is given by contradiction. Assume that $\inf_{u\in D_1}p_1(u)=0$, and let $\{u_\nu\}\subset D_1$ such that

$$c^2\int_{\Omega_0}|\nabla u_\nu|^2\,dx=1\quad\text{and}\quad\lim_{\nu\to\infty}p_1(u_\nu)=0.$$

We consider the comparison problem (4.1) for $\kappa=0.5\min_j\frac{k_j}{m_j}$. Then for the smallest eigenvalue μ_1 and for every $u\in H^0$ Rayleigh's principle yields

$$c^2\int_{\Omega_0}|\nabla u|^2\,dx\ge\mu_1\Big(\int_{\Omega_0}u^2\,dx+\sum_{j=1}^{K}\frac{\rho_0}{k_j-\kappa m_j}\Big|\int_{\Gamma_j} un_j\,ds\Big|^2\Big),$$

and for $p_1(u_\nu) < \kappa$ we obtain the contradiction

$$\begin{aligned}
0 &= -c^2 \int\limits_{\Omega_0} |\nabla u_\nu|^2\,dx + p_1(u_\nu)\Big(\int\limits_{\Omega_0} u_\nu^2\,dx + \sum_{j=1}^{K} \frac{\rho_0}{k_j - p_1(u_\nu)m_j}\Big|\int\limits_{\Gamma_j} u_\nu n_j\,ds\Big|^2\Big) \\
&\le -c^2 \int\limits_{\Omega_0} |\nabla u_\nu|^2\,dx + p_1(u_\nu)\Big(\int\limits_{\Omega_0} u_\nu^2\,dx + \sum_{j=1}^{K} \frac{\rho_0}{k_j - \kappa m_j}\Big|\int\limits_{\Gamma_j} u_\nu n_j\,ds\Big|^2\Big) \\
&\le -c^2 \int\limits_{\Omega_0} |\nabla u_\nu|^2\,dx + p_1(u_\nu)\frac{c^2}{\mu_1}\int\limits_{\Omega_0} |\nabla u_\nu|^2\,dx \\
&= c^2\Big(\frac{p_1(u_\nu)}{\mu_1} - 1\Big)\int\limits_{\Omega_0} |\nabla u_\nu|^2\,dx \longrightarrow -1 \quad \text{for } \nu \to \infty.
\end{aligned}$$

□

THEOREM 4.3. *Let $\kappa \in J_1$. Assume that the comparison problem (4.1) has r eigenvalues*

$$\mu_1 \le \mu_2 \le \ldots \le \mu_r$$

in J_1. Then the rational eigenvalue problem (3.1) has r eigenvalues

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_r$$

in J_1, and for the m-th eigenvalue λ_m of (3.1) the following inequality holds

$$\min(\mu_m, \kappa) \le \lambda_m \le \max(\mu_m, \kappa),\ j = 1, \ldots, r. \tag{4.3}$$

Proof. We first show that there exist r eigenvalues of problem (3.1) in J_1. Since by Lemma 4.2 $\inf_{v\in D_1} p_1(v) \in J_1$ we only have to prove that there exists $W \in H_r$ such that $W_0 \subset D_1$ and $sup_{v\in W_0} p_1(v) \in J_1$.

Let $W \in H_r$ be the invariant subspace of problem (4.1) corresponding to $\mu_1, \ldots, \mu_r$, i.e.

$$\mu_r = \min_{V\in H_r}\max_{v\in V_0} R_\kappa(v) = \max_{v\in W_0} R_\kappa(v) < \min_j \frac{k_j}{m_j}.$$

Then $R_\kappa(v) \le \mu_r < \min_j \frac{k_j}{m_j}$ for every $v \in W_0$, and it follows from Lemma 4.1 $W_0 \subset D_1$, and

$$p_1(v) \le \max(\kappa, R_\kappa(v)) \le \max(\kappa, \mu_r) < \min_j \frac{k_j}{m_j}.$$

Hence, the rational eigenproblem (3.1) has (at least) r eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_r$ in J_1.

We now prove the inequality (4.3). For $m \in \{1, \ldots, r\}$ let $Z \in H_m$ such that

$$\mu_m = \min_{V\in H_m}\max_{v\in V_0} R_\kappa(v) = \max_{v\in Z_0} R_\kappa(v).$$

Then $Z_0 \subset W_0 \subset D_1$, and from $p_1(v) \le \max(\kappa, R_\kappa(v))$ for every $v \in Z_0$ we obtain

$$\lambda_m = \min_{\substack{V\in H_m \\ V_0\subset D_1}}\max_{v\in V_0} p_1(v) \le \max_{v\in Z_0} p_1(v) \le \max(\kappa, \max_{v\in Z_0} R_\kappa(v)) = \max(\kappa, \mu_m),$$

which proves the upper bound of λ_m in (4.3).

To obtain the lower bound let $Y \in H_m$ such that $Y_0 \subset D_1$ and

$$\lambda_m = \min_{\substack{V \in H_m \\ V_0 \subset D_1}} \max_{v \in V_0} p_1(v) = \max_{v \in Y_0} p_1(v).$$

For every $u \in Y_0$

$$0 = f(p_1(u), u) = -c^2 \int_{\Omega_0} |\nabla u|^2 \, dx + p_1(u) \int_{\Omega_0} u^2 \, dx + \sum_{j=1}^{K} \frac{p_1(u)\rho_0}{k_j - p_1(u) m_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2,$$

and for $p_1(u) \le \kappa$ it follows

$$0 \le -c^2 \int_{\Omega_0} |\nabla u|^2 \, dx + p_1(u) \Big(\int_{\Omega_0} u^2 \, dx + \sum_{j=1}^{K} \frac{\rho_0}{k_j - \kappa m_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 \Big),$$

from which we obtain $p_1(u) \ge R_\kappa(u)$.

Hence, if $p_1(u) \le \kappa$ for every $u \in Y_0$, then

$$\lambda_m = \max_{u \in Y_0} p_1(u) \ge \max_{u \in Y_0} R_\kappa(u) \ge \min_{V \in H_m} \max_{u \in V_0} R_\kappa(u) = \mu_m. \tag{4.4}$$

If $\mu_m \le \kappa$ then the first part of the proof implies $p_1(u) \le \lambda_m \le \max(\kappa, \mu_m) = \kappa$ for every $u \in Y_0$, and (4.4) yields $\lambda_m \ge \min(\kappa, \mu_m)$.

For $\mu_m > \kappa$ the lower bound in (4.3) follows by contradiction, since from $\lambda_m < \kappa$ we again would get $\lambda_m \ge \mu_m > \kappa$ from (4.4). □

THEOREM 4.4. *Let $\kappa \in J_\ell$, $\ell = 2, \ldots, K+1$ and assume that the m-th eigenvalue μ_m of the comparison problem (4.1) satisfies $\mu_m \in J_\ell$. Then the rational eigenvalue problem (3.1) has an m-th eigenvalue $\lambda_m \in J_\ell$, and*

$$\min(\mu_m, \kappa) \le \lambda_m \le \max(\mu_m, \kappa). \tag{4.5}$$

Proof. We prove that

(i) there exists $V \in H_m$ such that $V \cap D_\ell \ne \emptyset$ and $\sup_{u \in W \cap D\ell} p_\ell(u) \le \max(\kappa, \mu_m)$

(ii) $\sup_{u \in V \cap D_\ell} p_\ell(u) \ge \min(\kappa, \mu_m)$ for every $V \in H_m$ such that $V \cap D_\ell \ne \emptyset$.

Then

$$\lambda_m := \inf_{\substack{V \in H_m \\ V \cap D_\ell \ne \emptyset}} \sup_{u \in V \cap D_\ell} p_\ell(u) \in J_\ell,$$

i.e. λ_m is an m-th eigenvalue of problem (3.1), and inequality (4.5) holds.

Let $W \in H_m$ and $w \in W_0$ such that

$$\mu_m = \max_{u \in W_0} R_\kappa(u) = R_\kappa(w).$$

Then by Lemma 4.1 $w \in D_\ell$, i.e. $W \cap D_\ell \ne \emptyset$. $R_\kappa(u) \le \mu_m$ for every $u \in W_0$ yields

$$-c^2 \int_{\Omega_0} |\nabla u|^2 \, dx + \sum_{j=1}^{\ell-1} \frac{\kappa \rho_0}{k_j - \kappa m_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 + \sum_{j=\ell}^{K} \frac{\mu_m \rho_0}{k_j - \kappa m_j} \Big| \int_{\Gamma_j} u n_j \, ds \Big|^2 + \mu_m \int_{\Omega_0} u^2 \, dx \ge 0.$$

For $\sigma := \max(\kappa, \mu_m)$ it follows

$$-c^2 \int\limits_{\Omega_0} |\nabla u|^2 \, dx + \sum_{j=1}^{K} \frac{\sigma \rho_0}{k_j - \sigma m_j} \Big| \int\limits_{\Gamma_j} u n_j \, ds \Big|^2 + \sigma \int\limits_{\Omega_0} u^2 \, dx \geq 0,$$

and therefore $p_\ell(u) \leq \sigma$ for every $u \in D_\ell \cap W$. Hence, for $W \in H_m$ it holds

$$W \cap D_\ell \neq \emptyset \quad \text{and} \quad \sup_{u \in W \cap D_\ell} p_\ell(u) \leq \max(\kappa, \mu_m). \tag{4.6}$$

(ii) is shown by contradiction. Assume that there exists $V \in H_m$ such that $V \cap D_\ell \neq \emptyset$ and $\sup_{v \in V \cap D_\ell} p_\ell(v) < \min(\kappa, \mu_m)$. Let $u_V \in V$ such that $R_\kappa(u_V) = \max_{v \in V_0} R_\kappa(v)$. Then $R_\kappa(u_V) \notin J_\ell$, for otherwise it follows from Lemma 4.1 $u_V \in D_\ell$, and $p_\ell(u_V) \geq \min(\kappa, R_\kappa(u_V))$, i.e.

$$\sup_{u \in V \cap D_\ell} p_\ell(u) \geq p_\ell(u_V) \geq \min(\kappa, R_\kappa(u_V)) \geq \min(\kappa, \mu_m).$$

For $\sigma := \min(\kappa, \mu_m)$ $(\leq \min(\kappa, R_\kappa(u_V))$

$$\begin{aligned} f(\sigma, u_V) &= -c^2 \int\limits_{\Omega_0} |\nabla u_V|^2 \, dx + \sum_{j=1}^{\ell-1} \frac{\sigma \rho_0}{k_j - \sigma m_j} \Big| \int\limits_{\Gamma_j} u_V n_j \, ds \Big|^2 \\ &\quad + \sum_{j=\ell}^{K} \frac{\sigma \rho_0}{k_j - \sigma m_j} \Big| \int\limits_{\Gamma_j} u_V n_j \, ds \Big|^2 + \sigma \int\limits_{\Omega_0} u_V^2 \, dx \\ &\leq -c^2 \int\limits_{\Omega_0} |\nabla u_V|^2 \, dx + \sum_{j=1}^{\ell-1} \frac{\kappa \rho_0}{k_j - \kappa m_j} \Big| \int\limits_{\Gamma_j} u_V n_j \, ds \Big|^2 \\ &\quad + \sum_{j=\ell}^{K} \frac{R_\kappa(u_V) \rho_0}{k_j - \kappa m_j} \Big| \int\limits_{\Gamma_j} u_V n_j \, ds \Big|^2 + R_\kappa(u_V) \int\limits_{\Omega_0} u_V^2 \, dx = 0. \end{aligned}$$

For fixed $u \in V \cap D_\ell$ let $w(t) := tu + (1-t)u_V$ and $\phi(t) := f(\sigma, w(t))$. Then ϕ is continuous on $[0, 1]$, and

$$f(\sigma, u_V) = \phi(0) < 0 < \phi(1) = f(\sigma, u)$$

yields the existence of $\hat{t} \in (0, 1)$ such that $f(\sigma, w(\hat{t})) = 0$. Hence, $w(\hat{t}) \in W \cap D_\ell$ and $p_\ell(w(\hat{t})) = \min(\kappa, \mu_m)$. □

5. A projection method. Iterative projection methods (Lanczos, Arnoldi, Jacobi-Davidson, e.g.), where approximations of the wanted eigenvalues and corresponding eigenvectors are obtained from projections to subspaces which are expanded in the course of the algorithm a very efficient for linear sparse eigenproblems.

Generalizations of this approach to the nonlinear eigenvalue problem $T(\lambda)x = 0$ are contained in recent papers by Ruhe [12] and Hager and Wiberg [8], [7] who updated linear eigenvalue problems which approximate the projection of the nonlinear eigenproblem to a Krylov space of $T(\sigma)^{-1}T(\lambda)$ for some shift σ and varying λ, and for symmetric nonlinear problems having a Rayleigh functional by Betcke and the author [1] who constructed ansatz vectors for a projection method by a Jacobi-Davidson type approach.

The comparison results in the last section suggest to derive an ansatz space for a projection method for the numerical solution of problem (3.1) in the following way:

1. Choose a small number of shifts $\kappa_1, \dots, \kappa_r \in J_\ell$.
2. For $j = 1, \dots, r$ determine the eigenvectors u_{jk}, $k = 1, \dots, s_j$, of the linear problem (4.1) with shift κ_j corresponding to eigenvalues in J_ℓ.
3. Let U be the matrix with columns u_{jk}, $j = 1, \dots, r$, $k = 1, \dots, s_j$. Determine the QR factorization with column pivoting which produces the QR factorization of UE where E denotes a permutation matrix such that the absolute values of the diagonal elements of R are monotonely decreasing.
4. For every j with $|r_{jj}| < \tau \cdot |r_{11}|$ drop the j-th column of Q where $\tau \in [0, 1)$ is a given tolerance, and denote by V the space that is spanned by the remaining columns of Q.
5. Project the nonlinear eigenvalue problem (3.1) to V and solve the projected problem by inverse iteration with variable shifts.

As a numerical example we consider the rational eigenvalue problem (3.1) where Ω is the square $(-5, 5) \times (-5, 5)$, and the tubes are defined by $\Omega_1 = (-3, -2) \times (-3, -2)$, $\Omega_2 = (2, 3) \times (-3, -2)$, $\Omega_3 = (-3, -2) \times (2, 3)$, $\Omega_4 = (2, 3) \times (2, 3)$. We assume $c = 1$, $\rho_0 = 1$, $k_j = 1$ for all j, $m_1 = 5$, $m_2 = 5$, $m_3 = 2$ and $m_4 = 1$.

We discretized this eigenvalue problem with linear elements obtaining a matrix eigenvalue problem

$$Ax = \lambda Bx + \frac{\lambda}{1-5\lambda}C_1x + \frac{\lambda}{1-5\lambda}C_2x + \frac{\lambda}{1-2\lambda}C_3x + \frac{\lambda}{1-\lambda}C_4x \tag{5.1}$$

of dimension $n = 23597$. Problem (5.1) has 8 eigenvalues $\lambda_1 \le \dots \le \lambda_8$ in the interval $J_1 = [0, 0.2)$, 6 eigenvalues $\tilde\lambda_5 \le \dots \le \tilde\lambda_{10}$ in $J_2 := (0.2, 0.5)$, 6 eigenvalues $\hat\lambda_9 \le \dots \le \hat\lambda_{14}$ in $J_3 := (0.5, 1.0)$,and 10 eigenvalues $\bar\lambda_{13} \le \dots \le \bar\lambda_{22}$ in $J_4 = (1, 2)$. Notice that (5.1) is not just a small perturbation of the linear eigenproblem $Ax = \lambda Bx$ which has only 4 eigenvalues in each of the intervals J_1, J_2, and J_3.

To approximate the eigenvalues in J_1 we solved the linear eigenvalue problem

$$\left(B + \frac{1}{1-5\kappa}C_1 + \frac{1}{1-5\kappa}C_2 + \frac{1}{1-2\kappa}C_3 + \frac{1}{1-\kappa}C_4\right)x = \mu(A + \sigma B)x$$

by Lanczos' method with complete reorthogonalization for different parameters of κ obtaining approximations to eigenvectors of problem (5.1). We added σB on the right hand side with a small $\sigma > 0$ since A is singular.

With 4 parameters $\kappa_1 = 0.1$, $\kappa_2 = 0.15$, $\kappa_3 = 0.175$ and $\kappa_4 = 0.1875$ and tolerances $\tau_1 = 1e-1$, $\tau_2 = 1e-3$, and $\tau_3 = 0$ we obtained eigenvalue approximations to $\lambda_1, \dots, \lambda_8$ the relative errors of which are displayed in Figure 2 on the left. The dimensions of the projected eigenvalue problems were 13, 20 and 23, respectively.

On an Intel Pentium 4 with 2 GHz and 1 GB RAM it took 19.34 seconds to solve the 4 linear eigenvalue problems, and 0.20 seconds for the QR factorization with column pivoting. To solve the projected nonlinear eigenvalue problems by safeguarded inverse iteration it took 0.61, 0.84 and 0.94 seconds, respectively.

To approximate the eigenvalues in J_2 we solved the linear problem

$$\left(B + \frac{1}{1-2\kappa}C_3 + \frac{1}{1-\kappa}C_4\right)x = \mu\left(A + \frac{\kappa}{5\kappa-1}C_1 + \frac{\kappa}{5\kappa-1}C_2 + \sigma B\right)x$$

for 4 parameters $\kappa_1 = 0.3$, $\kappa_2 = 0.4$, $\kappa_3 = 0.45$, and $\kappa_4 = 0.475$, and with the same tolerances as before we obtained the relative errors in Figure 2 on the right. The dimensions of the projected problems are 17, 23 and 27, respectively. The CPU times in this run were

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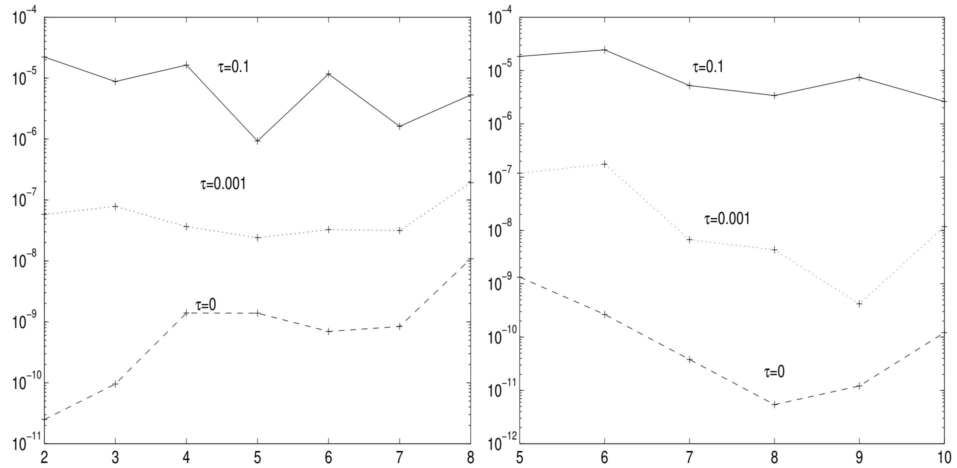

Fig. 2: relative errors; eigenvalues in (0,0.2) and (0.2,0.5)

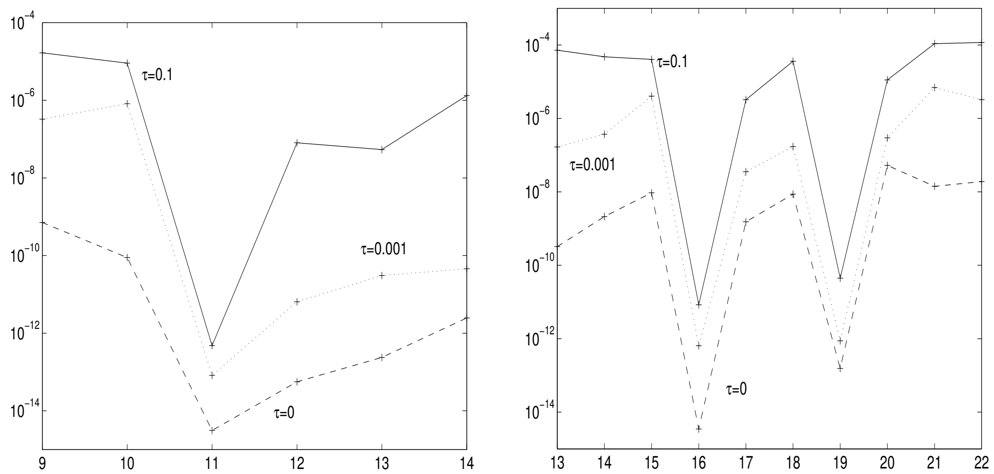

Fig. 3: relative errors; eigenvalues in (0.5,1.0) and (1.0,2.0)

32.34 seconds for the linear eigenproblems, 0.27 seconds for the QR factorization, and 0.69, 0.89, and 1.05 seconds for inverse iteration.

Figure 3 shows the relative errors of eigenvalues $\hat{\lambda}_j$, $j = 9, \dots, 14$, in the interval $J_3 = (0.5, 1)$ which were obtained with shift parameters $\kappa_1 = 0.7$ and $\kappa_2 = 0.9$, $\kappa_3 = 0.95$ and $\kappa_4 = 0.975$, and the relative errors of $\bar{\lambda}_{13}, \dots, \bar{\lambda}_{22}$ in the interval $(1, 2)$ obtained with shifts $\kappa_1 = 1.25$, $\kappa_2 = 1.5$ and $\kappa_3 = 1.75$. For the eigenvalues in J_3 the dimensions of the nonlinear projected problem were 18, 23, and 30, respectively, and the CPU times were 47.05 seconds for the linear eigenproblems, 0.33 second for the QR factorization, and 1.03, 1.23, and 1.80 seconds for inverse iteration. For the eigenvalues in J_4 we needed 67.16 seconds to solve the linear eigenproblems, 0.61 seconds for the QR factorization, and 1.03, 1.23 and 1.80 seconds to solve the projected problems of dimensions 25, 30, and 42, respectively.

The Jacobi-Davidson type method proposed in [1] achieved the same accuracy as our method for $\tau = 0$ for the eigenvalue approximations in $J_1, \dots, J_4$, respectively, projecting problem (5.1) to a rational eigenproblem of dimension 21, 22, 32, and 48, and requiring 114.06, 100.53, 130.95 and 152.13 seconds, respectively. Hence, the method considered here is more efficient than the method from [1] which on the other hand applies to a much wider class of nonlinear eigenproblems, including non-symmetric problems.

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