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LOCAL APPROXIMATION ESTIMATORS FOR ALGEBRAIC MULTIGRID*

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Abstract. In Smoothed Aggregation Algebraic Multigrid, the prolongator is defined by smoothing of the output of a simpler tentative prolongator. The weak approximation property for the tentative prolongator is known to give a bound on the convergence factor of the two-level and even multilevel method. It is known how to bound the constants in the weak approximation property when the system matrix is given as the sum of positive semidefinite local matrices. In practice, however, the local matrices are often not known to the solver, or the problem is given in terms of local matrices and additional constraints. We characterize the matrices that can be decomposed into a sum of local positive semidefinite matrices with only given rows and columns allowed to be nonzero, and we show that such a decomposition may not always exist. We then propose a construction of approximate local matrices that may be used for local estimates. Finally, we show how eliminating the constraints from the local matrices can be used to obtain rigorous bounds.

Key words. Adaptive Algebraic Multigrid, robust iterative methods, local element matrices, decomposition of global matrix, apriori convergence estimates, weak approximation property

AMS subject classifications. 65N55, 65N22, 65F10, 65N30

1. Introduction. We are concerned with the development of Algebraic Multigrid (AMG) for symmetric, positive definite linear systems arizing from finite element discretization of elliptic partial differential equations. AMG methods attempt to create coarse levels from the algebraic system automatically, using no or only a minimum of additional information. The basic idea of the multigrid algorithm is that fine level error on which the smoothing process is not effective should be reduced by the coarse correction. Therefore, a-priori estimates of the approximation of fine level functions by coarse level function are important to guide the design of robust AMG methods, cf., [2, 3], and references therein.

A number of coarsening schemes will work well on "reasonable" scalar problems which typically result in M-matrices, such as the Laplace equation discretized by linear elements on a non-degenerate unstructured grid [7, 16]. Methods that incorporate rigid body modes, such as the Smoothed Aggregation AMG [19], work also very well for elasticity. Realistic problems, however, typically include elements violating shape limits, large jumps of coefficients, special kinds of elements, and additional constraints on the values of degrees of freedom, enforced by large penalties ("stiff spring" or "contact" elements in engineering parlance), or even arbitrary additional equations that are eliminated before the matrix is passed to the solver ("multiple point constraints"). Such problems are hard to solve by AMG even if they are symmetric and positive semidefinite. Without a-priori numerical estimates of the rate of convergence, with a rigorous foundation, an AMG algorithm is based simply on the hope that the problem will not have anything unexpected and things will work out in the end.

One common estimate that can be computed a-priori is the weak approximation property, which bounds the error of the best approximation in Euclidean norm of a fine grid vector by the prolongation of a coarse grid vector in terms of the energy norm of the fine grid vector. The weak approximation property for the prolongator is known to imply a bound on the two-level convergence factor [8, 13, 1], albeit a fairly pessimistic one [12]. The weak approximation property is used as an a-priori indicator in element based AMG [2, 3]. In the

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smoothed aggregation method [17, 19], two-level convergence bounds [20, 21] as well as multilevel convergence bounds [18] can be obtained from the weak approximation property for the so-called tentative prolongator, which is similar to piecewise constant interpolation. Hence, the constants in the weak approximation property are relatively easy to compute and estimate. The actual prolongator used in the multigrid algorithm is then obtained by smoothing the output of the tentative prolongator. For more details and further developments of Smoothed Aggregation AMG, see [4, 5, 6, 9, 14, 19]

In the absence of multiple point constraints, the constant in the weak approximation property can be bounded rigorously from the solution of eigenvalue problems based on local element matrices. In [3], it was proposed to select the columns of the tentative prolongator as the eigenvectors of the local problems and to control the convergence of algebraic multigrid by choosing the number of the eigenvectors and by selecting the amount of smoothing of the prolongator.

In practice, the problem to be solved is most conveniently given in terms of a single global stiffness matrix with all constraints incorporated. Then the information contained in the local stiffness matrices is lost, and to bound the constant in the weak approximation property rigorously by the solution of local eigenvalue problems, one would need to decompose the global matrix into the sum of positive semidefinite local matrices. We show that, in general, such a decomposition does not exist. To estimate the contribution of a single aggregate (or, equivalently, of a block of coarse basis functions) to the constant in the weak approximation property, we decompose the global matrix into local matrices corresponding to the decomposition of the set of all nodes into the given aggregate and its complement. We also present further approximation techniques to reduce the cost of the estimation. The resulting estimates are not rigorous but they are still practically useful.

2. Matrix notation. We will work with real matrices denoted by $A = (a_{ij})$, with entries denoted by $a_{ij} = (A)_{ij}$, and column vectors ($n \times 1$ matrices) denoted by x, y, etc. The notation $A \geq 0$ means that A is symmetric positive semidefinite, $A > 0$ means that A is symmetric positive definite, and $A \leq B$ means $A - B \geq 0$. If A is an $m \times n$ matrix and $I \subset \{1, \ldots, m\}$, $J \subset \{1, \ldots, n\}$, then $A(I, J)$ is the submatrix of A consisting of rows $i \in I$ and columns $j \in J$. The notation $A(:, J)$ means the submatrix of all rows $j \in J$. Let Π_I be the $n \times n$ diagonal matrix with $(\Pi_I)_{ii} = 1$ if $i \in I$ and $(\Pi_I)_{ii} = 0$ otherwise. The support of a matrix is defined as the set of all indices of nonzero rows, denoted by $\operatorname{supp} A = \{i \mid \exists\, j : a_{ij} \neq 0\}$. If A and B are of the same size $m \times n$, their inner product is defined by

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}.$$

If $A \geq 0$, the energy (semi)norm associated with A is denoted by $\|u\|_A = \sqrt{u^T A u}$. The Euclidean norm of a vector is $\|u\| = \sqrt{u^T u}$. Finally, $\rho(A)$ is the spectral radius of A, ^{t} denotes the transpose, Ker is the nullspace, and Im is the range.

3. Multilevel algorithm and the weak approximation property. We consider the basic two-level variational scheme for the solution of a linear system $Ax = b$, where $A > 0$ is an $n \times n$ matrix.

ALGORITHM 3.1.

1. *pre-smoothing:* $x \leftarrow x - M(Ax - b)$,
2. *coarse correction:* $P^T(A(x + Py) - b) = 0, \quad x \leftarrow x + Py$,
3. *post-smoothing:* $x \leftarrow x - M^T(Ax - b)$.

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The multilevel algorithm is obtained by recursion, solving the correction problem approximately by one or two steps of the same method starting with $y = 0$. The matrix P is called the *prolongator*. The matrix M is the *smoother preconditioner*, obtained, e.g., from the Gauss-Seidel or Jacobi methods, or simply is a multiple of I.

It is well known [1, 8, 1] that, under suitable conditions on A, if the *weak approximation property* holds, which is given by

$$\forall u\ \exists v: \quad \|u - Pv\|^2 \leq \frac{c_1}{\rho(A)} \|u\|_A^2, \tag{3.1}$$

the convergence factor of Algorithm 3.1 in the $\|.\|_A$ norm is at least $(1 - c(M)/c_1)$. Here, $c(M)$ is a (small) constant, depending on the details of the form of M but not on the condition number of A. This estimate, however, does not generalize to guarantee multilevel convergence independent (or weakly dependent) on the number of levels.

In Smoothed Aggregation AMG, the prolongator is defined as $P = S\hat{P}$, where $S = I - (\omega/\rho(A))$, $\omega \leq 3/2$, is a *prolongator smoother* and $\hat{P}$ is a *tentative* prolongator. If the tentative prolongator $\hat{P}$ satisfies the weak approximation property

$$\forall u\ \exists v: \quad \|u - \hat{P}v\|^2 \leq \frac{c_1}{\rho(A)} \|u\|_A^2, \tag{3.2}$$

then, under suitable assumptions on M, the convergence factor of Algorithm 3.1 in the $\|.\|_{A_S}$ norm can be bounded by $(1 - c(M)/c_1)$. Here, again, $c(M)$ is a small constant that does not depend on the condition number of A. The estimate can be generalized to convergence in the $\|.\|_A$ norm and the multilevel methods, cf., [18, 21].

The tentative prolongator is constructed from an aggregation of variables as follows. The index set is split into disjoint aggregates,

$$\{1, \dots, n\} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_m,$$

and $\hat{P}$ is required to be of the form

$$\hat{P} = [P_1, \dots, P_m], \quad \operatorname{supp} P_i \subset \mathcal{A}_i. \tag{3.3}$$

In [19], the nonzero rows in P_i are constructed by orthogonalizing a restriction of the vectors of rigid body modes onto $\mathcal{A}_i$.

Let the matrix A be given as an assembly of local matrices, as usual in the finite element method,

$$A = A_1 + \dots + A_k, \quad A_j \geq 0. \tag{3.4}$$

We find it convenient to consider local element matrices embedded in a global zero matrix; $\operatorname{supp} A_j$ then plays the role of the indices of the degrees of freedom associated with element j. It is well known that a computational verification of the weak approximation property is then possible by a simple summation argument. We are concerned here with the special case of weak approximation propery for the tentative prologation of the form (3.3), and we proceed by covering each aggregate by a set of elements J_i.

LEMMA 3.2. *Suppose that* $J_1, \dots, J_m \subset \{1, \dots, n\}$ *satisfy*

$$\mathcal{A}_i \subset \bigcup_{j \in J_i} \operatorname{supp} A_j, \quad i = 1, \dots, m,$$

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let $\tilde{A}_i = \sum_{j\in J_i} A_j$ be the subassembled matrices corresponding to J_i, and assume that the local approximation property holds,

$$\forall i \in \{1,\dots,m\}\ \forall u\ \exists v_i: \quad \|\Pi_{\mathcal{A}_j}(u - \hat{P}_j v_j)\|^2 \le \frac{C_j}{\rho(A)} u^T \tilde{A}_j u. \tag{3.5}$$

Then the weak approximation property (3.2) for the tentative prolongator holds with $c_1 = N\max_j C_j$, where $N = \max_j\{i \mid j \in J_i\}$ is the maximum number of overlapping sets J_i.

Proof. Let u be given, let v_j be from (3.5), and define

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}.$$

Then

$$\|u - \hat{P}v\|^2 = \sum_{j=1}^m \|\Pi_j(u - \hat{P}_j v_j)\|^2 \le \sum_{j=1}^m \frac{C_j}{\rho(A)} u^T \tilde{A}_j u \le N \max_j C_j\, u^T A u,$$

because the aggregates are disjoint. □

The best constant C_j in the local weak approximation property (3.5) can be computed as the largest eigenvalue of the generalized eigenvalue problem

$$(\Pi_{\mathcal{A}_j} - \hat{P}_j(\hat{P}_j^T\hat{P}_j)^{-1}\hat{P}_j^T)(\operatorname{supp}\tilde{A}_j, \operatorname{supp}\tilde{A}_j)u = \lambda \frac{\tilde{A}_j(\operatorname{supp}\tilde{A}_j, \operatorname{supp}\tilde{A}_j)}{\rho(A)} u. \tag{3.6}$$

The matrices on the right-hand and left-hand sides of the generalized eigenvalue problem (3.6) are singular. However, the usual construction of the tentative prolongator $\hat{P}$ from rigid body modes [19] implies that if $\tilde{A}_j(\operatorname{supp}\tilde{A}_j, \operatorname{supp}\tilde{A}_j)u = 0$ then u is the restriction of a rigid body mode vector, $(\Pi_{\mathcal{A}_j} - \hat{P}_j(\hat{P}_j^T\hat{P}_j)^{-1}\hat{P}_j^T)(\operatorname{supp}\tilde{A}_j, \operatorname{supp}\tilde{A}_j)u = 0$. Hence, the generalized eigenvalue problem (3.6) reduces modulo $\operatorname{Ker}\tilde{A}_j(\operatorname{supp}\tilde{A}_j, \operatorname{supp}\tilde{A}_j)$ to a problem with the matrix on the right-hand side positive definite.

To guarantee (3.2) with a given maximal C_j and with $N = 1$, it was proposed in [3] to choose the aggregates $\mathcal{A}_k$ as the variables in disjoint clusters of elements, with the variables shared between the clusters assigned to one of the aggregates only, and columns of $\hat{P}_j$ as eigenvectors of $\tilde{A}_j(\operatorname{supp}\tilde{A}_j, \operatorname{supp}\tilde{A}_j)$, restricted to $\mathcal{A}_j$.

4. Decomposition into local matrices. Let V be the space of all $n \times n$ real symmetric matrices, $V = \{A \in \mathbb{R}^{n\times n} \mid A = A^T\}$. It is easy to see that the cone of positive semidefinite matrices in V is self-dual:

LEMMA 4.1. *Let $A \in V$. Then $A \ge 0$ if and only if $\langle A, B\rangle \ge 0$ for all $B \in V$, $B \ge 0$.*

Proof. Let $A \in V$. The matrix A has spectral decomposition $A = \sum_{i=1}^n \lambda_i u_i u_i'$, where $\{u_i \mid i = 1,\dots,n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A. Then

$$\langle A, B\rangle = \sum_{i=1}^n \lambda_i \langle u_i u_i', B\rangle = \sum_{i=1}^n \lambda_i u_i' B u_i.$$

Now $A \ge 0$ if and only if all eigenvalues $\lambda_i \ge 0$. If $A \ge 0$ and $B \ge 0$, then $u_i' B u_i \ge 0$, hence $\langle A, B\rangle \ge 0$. On the other hand, if $\lambda_i < 0$ for some i, the choice $B = u_i u_i'$ gives $B \ge 0$ and $\langle A, B\rangle = \lambda_i < 0$. □

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The following theorem provides a characterization of matrices that can be decomposed into sums of positive semidefinite matrices with given supports.

THEOREM 4.2. *Let $A \in V$, $A \geq 0$, and $I_i \subset \{1, \ldots, n\}$, $i = 1, \ldots, m$. Then there exist matrices $A_i \in V$, $A_i \geq 0$, such that* $\operatorname{supp} A_i \subset I_i$, *$i = 1, \ldots, m$, and $A = \sum_{i=1}^{m} A_i$, if and only if there is no matrix $B \in V$ such that $B|_{I_i} \geq 0$, $k = 1, \ldots, m$, and $\langle A, B\rangle < 0$.*

Proof. It is easy to see that the condition is necessary. Namely, if $A = \sum_{i=1}^{n} A_i$, $A_i \geq 0$, and $B|_{I_i} \geq 0$, then using Lemma 4.1, $\langle A, B\rangle = \sum_{i=1}^{n}\langle A_i, B\rangle \geq 0$. On the other hand, suppose that no decomposition of A into the sum of the matrices A_i such as in the theorem exists. This means that A is not an element of the convex cone $\mathcal{C}$, defined by

$$\mathcal{C} = \left\{ \sum_{i=1}^{m} A_i \mid A_i \in V, A_i \geq 0, \operatorname{supp} A_k \subset I_i \right\}.$$

From the separation theorem for convex sets, there exists a hyperplane separating the cone $\mathcal{C}$ and the point A in the space V. That is, there exist $B \in V$ and $b \in \mathbb{R}$ such that $\langle X, B\rangle \geq b$ for all $X \in \mathcal{C}$ and $\langle A, B\rangle < b$. Because $0 \in \mathcal{C}$, it follows that $b \leq 0$, hence $\langle A, B\rangle < 0$. Let $X \in \mathcal{C}$. Then, for all $t > 0$, $tX \in \mathcal{C}$, hence $t\langle X, B\rangle \geq b$. Letting $t \to \infty$, we obtain $\langle X, B\rangle \geq 0$. □

The main result of this section is that *a decomposition of a positive semidefinite matrix into local positive semidefinite matrices does not always exists*, as demonstrated by the following counterexample.

EXAMPLE 4.3. *Let $n = 3$ and $I_1 = \{1, 2\}$, $I_2 = \{2, 3\}$, $I_3 = \{1, 3\}$ and*

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The eigenvalues of B are $\lambda_1 \approx -0.23$, $\lambda_2 \approx -0.82$, $\lambda_3 \approx 5.41$, and $B|_{I_i} \geq 0$, $i = 1, 2, 3$. Let u_1 be an eigenvector of B corresponding to the eigenvalue λ_1. Then, according to Theorem 4.2, the matrix $A = u_1 u_1' \geq 0$ has no decomposition $A = A_1 + A_2 + A_3$, $A_i \geq 0$, $\operatorname{supp} A_i \subset I_i$*, since $\langle B, A\rangle < 0$.*

5. Approximate decomposition into local matrices. Let A be $n \times n$ matrix, $A > 0$, and $\mathcal{A} \subset \{1, \ldots, n\}$. We want to construct a matrix that would play the role of the local matrix for $\mathcal{A}$, i.e., what the local stiffness matrix might be if $\mathcal{A}$ were the set of degrees of freedom of an element. Because exact decomposition into local matrices may not exist, we can only get approximate local matrices. We choose to construct one approximate local matrix at a time. For this purpose, let

$$\begin{aligned} S_1 &= \{i \in \mathcal{A} \mid \forall j \in \{1, \ldots, n\} \setminus \mathcal{A} : a_{ij} = 0\}, \\ S_2 &= \{i \in \mathcal{A} \mid \exists j \in \{1, \ldots, n\} \setminus \mathcal{A} : a_{ij} \neq 0\}, \\ S_3 &\subset \{1, \ldots, n\} \setminus \mathcal{A}. \end{aligned}$$

The index sets S_1 and S_2 can be interpreted as the interior and the boundary of $\mathcal{A}$, respectively. The choice of S_3 will be specified later. Let

$$S = S_1 \cup S_2 \cup S_3.$$

For systems that arise in elasticity, the numbers $1, \ldots, n$ are indices of nodes and a_{ij} are blocks rather than scalars.

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In the block form corresponding to the index sets S_1, S_2, S_3, we have

$$A(S,S)=\left[\begin{array}{ccc} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{array}\right].$$

It is assumed that the diagonal blocks A_{11}, A_{22}, A_{33} are positive definite, which is the case when A is the stiffness matrix from the Finite Element method.

We wish to decompose the matrix $A(S,S)$ as

$$A(S,S)=A^{(1)}+A^{(2)}, \tag{5.1}$$

where

$$A^{(1)}=\left[\begin{array}{ccc} A_{11} & A_{12} & 0 \\ A_{21} & A_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{array}\right]\geq 0,$$
$$A^{(2)}=\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & A_{22}^{(2)} & A_{23} \\ 0 & A_{32} & A_{33} \end{array}\right]\geq 0.$$

We then take the *pseudolocal* matrix

$$A_{\mathcal{A}}^{\mathrm{pseudolocal}}=\left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22}^{(1)} \end{array}\right] \tag{5.2}$$

as an approximate local matrix for $\mathcal{A}$.

The following lemma is well known [10].

LEMMA 5.1. *Let*

$$B=\left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array}\right]$$

be symmetric and $B_{11}>0$*. Then* $B\geq 0$ *if and only if* $B_{22}\geq B_{21}B_{11}^{-1}B_{12}$.

From Lemma 5.1, we get that $A_{22}^{(1)}$ and $A_{22}^{(2)}$ have to at least satisfy

$$A_{22}^{(1)}\geq A_{21}A_{11}^{-1}A_{12}, \quad A_{22}^{(2)}\geq A_{23}A_{33}^{-1}A_{32}.$$

Splitting the difference evenly so that $A_{22}=A_{22}^{(1)}+A_{22}^{(2)}$, we construct

$$\begin{aligned} A_{22}^{(1)} &= A_{21}A_{11}^{-1}A_{12}+(A_{22}-A_{21}A_{11}^{-1}A_{12}-A_{23}A_{33}^{-1}A_{32})/2 \\ &= (A_{22}+A_{21}A_{11}^{-1}A_{12}-A_{23}A_{33}^{-1}A_{32})/2, \\ A_{22}^{(2)} &= A_{23}A_{33}^{-1}A_{32}+(A_{22}-A_{21}A_{11}^{-1}A_{12}-A_{23}A_{33}^{-1}A_{32})/2 \\ &= (A_{22}-A_{21}A_{11}^{-1}A_{12}+A_{23}A_{33}^{-1}A_{32})/2. \end{aligned}$$

It remains to specify the choice of the index set S_3. If $S_3=\{1,\ldots,n\}\setminus\mathcal{A}$, the decomposition (5.1) gives a decomposition of A into two local matrices, one for $\mathcal{A}$ and one for its complement. However, the the usefulness of such a decomposition is limited, because both local matrices are dense. Moreover, operating with A_{33}^{-1} is expensive. We propose to

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Problem	Matrix (MB)	Equations	Iterations	CPU sec
1	54	118401	20	35
2	102	110211	34	118
3	693	1698525	no convergence	n/a

TABLE 7.1
Test problems and AMG results

take S_3, to be a maximal independent set of neighbors of $\mathcal{A}$, that is, a maximal set S_3 such that

$$S_3 \subset \{i \in \{1, \ldots, n\} \setminus \mathcal{A} \mid \exists j \in \mathcal{A} : a_{ij} \neq 0\},$$
$$\forall i, j \in S_3 : a_{ij} = 0.$$

Such a maximal set is easily found by the greedy algorithm (i.e., add indices to the set as long as it is possible). The matrix A_{33} is then diagonal (or block diagonal in case of systems).

The constant in the weak approximation property is then estimated by using the pseudolocal matrices (5.2) on the right-hand side of (3.6) in place of $\tilde{A}_i(\operatorname{supp} \tilde{A}_i, \operatorname{supp} \tilde{A}_i)$.

6. Local matrices and constraints. In practice, the Finite Element model has additional constraints, usually written in the form

$$u = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix}, \; u^{(2)} = Cu^{(1)}. \tag{6.1}$$

The variables in $u^{(1)}$ are called masters and the variables in $u^{(2)}$ are called slaves. The problem to be solved is then to minimize the energy functional $\frac{1}{2}u^T Au - b^T u$ subject to the constraints (6.1), which is equivalent to solving the system of linear equations for the masters,

$$A^c u^{(1)} = \begin{bmatrix} I \\ C \end{bmatrix}^T b, \quad A^c = \begin{bmatrix} I \\ C \end{bmatrix}^T A \begin{bmatrix} I \\ C \end{bmatrix}^T u^{(1)}.$$

If A is given by assembly of local matrices (3.4), we have

$$A^c = \sum_{i=1}^{n} A_i^c, \qquad A_i^c = \begin{bmatrix} I \\ C \end{bmatrix}^T A_i \begin{bmatrix} I \\ C \end{bmatrix}. \tag{6.2}$$

The estimates developed in Section 3 can be then used with the matrices A_i^c in place of the matrices A_i. The support of the matrices A_i^c, however, depends on the sparsity pattern of the constraint matrix C.

7. Computational results. We have evaluated the estimate from (3.6) for three test problems, with the pseudolocal matrices from (5.2) in place of $\tilde{A}_i$. The aggregates were generated by a version of the greedy algorithm from [19]. The matrix A was scaled in advance to have unit diagonal and $\rho(A)$ was estimated by the 1-norm. An approximation of the largest eigenvalue of the generalized eigenvalue problems was obtained by few iterations of the LOBPCG code of Knyazev [11], preconditioned by the block diagonal matrix $\operatorname{diag}([A_{11}, A_{22}])$. These maximal eigenvalues have the interpretation of the reciprocal of the smallest nonzero eigenvalues of the pseudolocal matrices. Because the matrix A is scaled to have unit diagonal, the spectral radii of the pseudolocal matrices are $O(1)$, hence the approximate maximal eigenvalues are referred to as condition estimates in the figures.

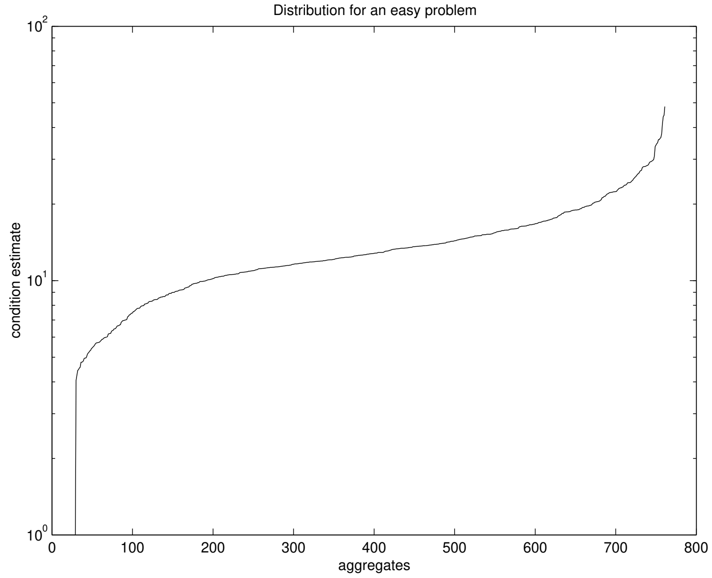

FIG. 7.1. *Estimates for Problem 1*

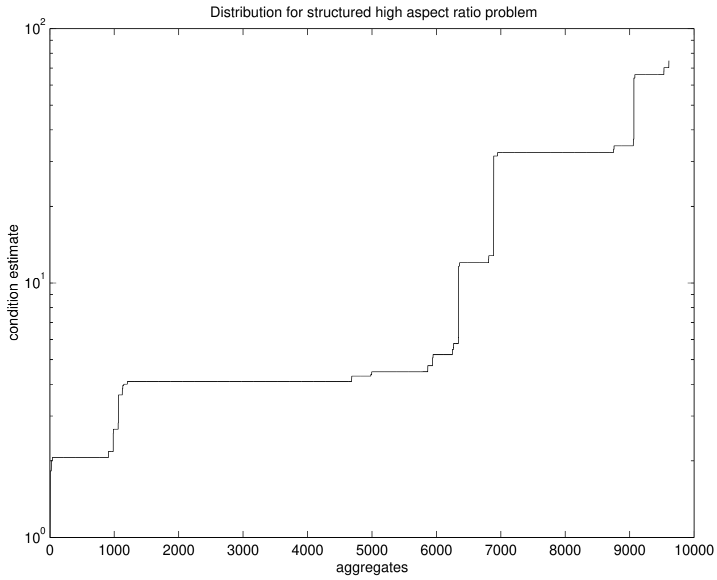

FIG. 7.2. *Estimates for Problem 2*

We have tested the estimates on three isotropic elasticity problems, summarized in Table 7.1. Problem 1 is a block discretized by a reasonable nonuniform 20-noded tetra-mesh. Problem 2 is a block discretized by a regular mesh of elements with very high aspect ratios. Problem 3 is a difficult industrial problem with irregular geometry and numerous multiple point constraints. Iteration counts and CPU times are reported for relative ℓ^2-residual of 10^{-4} and the AMG algorithm implemented in ANSYS 5.7 [15], running on one processor of 500MHz Compaq AlphaServer DS20. The algorithm converges fast for easy problems and

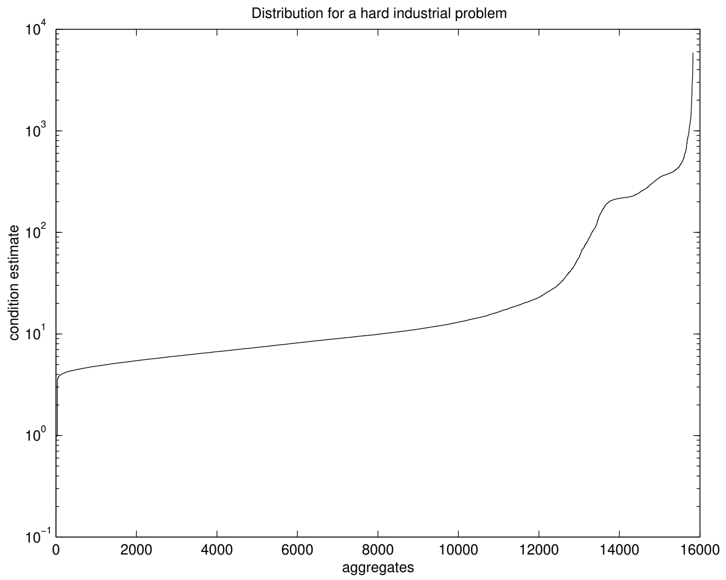

FIG. 7.3. *Estimates for Problem 3*

makes many valiant attempts to salvage convergence for hard problems before giving up. For details and more performance results, see [15].

It should be noted that the current implementation of AMG in ANSYS computes the eigenvalues of the submatrices of the stiffness matrix to assess the "difficulty" of the aggregates. This calculation takes less than 3% of the overall solver time, which is dominated by sparse matrix and vector multiplications. An estimate based on the weak approximation property for pseudolocal matrices should be no more than twice as expensive. The computations reported here were done in MATLAB, using a dump of the problem data and the aggregates from AMG in ANSYS. A practical implementation and the effect of the new estimates on the convergence of the AMG algorithm will be reported elsewhere.

The distribution of the estimates of c_1 from the weak approximation property is in Figs. 7.1, 7.2, 7.3, respectively. The estimates roughly correspond to the difficulty of solving these problems by AMG. Problem 1 is relatively easy to solve and the basic AMG algorithm from [19] works well. The steps in the estimate distribution for Problem 2 are caused by the fact that the mesh is regular, and certain configurations of nodes in the aggregates repeat often. This is a much more difficult problem and the method of [15] employs automatic semicoarsening and other strategies to deal with the high aspect ratios. Problem 3 is extremely difficult, as evidenced by high values of the estimates.

We have experimented with optimization of the aggregates based on estimates using pseudolocal matrices, but the optimization algorithm often found an aggregate where the pseudolocal estimate was low but the aggregate was in fact not good at all, resulting in no convergence of the multigrid method.

8. Conclusion. Our estimates based on the weak approximation property and pseudolocal matrices are useful to assess the difficulty of the problem. However, they are not suitable for optimization of method components. Apparently rigorous local estimates have to be used for that purpose. This will be studied elsewhere.

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