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ON THE ACCURACY OF MULTIGRID TRUNCATION ERROR ESTIMATES*

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Abstract. In solving boundary-value problems, multigrid methods can provide computable estimates of the truncation error by comparing discretizations on grids of different mesh sizes. In the standard formulation, such estimates are contaminated by errors larger than the truncation error itself unless the residual transfer operator satisfies a restrictive condition (typically valid for injection but not for full weighting) or is itself high-order accurate. This paper proves that a simple generalization based on the work of Schaffer leads to accurate truncation error estimates without these restrictions. Numerical results for several model problems illustrate the analysis.

Key words. multigrid, truncation error, tau extrapolation

AMS subject classifications. 65N15, 65N50, 65N55

1. Introduction.

Multigrid methods use approximations on grids of different mesh sizes to obtain fast solvers for boundary value problems [2, 4]. Comparing the approximations on different grids also provides computable estimates of the truncation error, which can then be used in adaptive grid refinement algorithms or in extrapolation to higher-order accuracy (τ-extrapolation). The basic procedures have been presented in many places, e.g., [3, 4, 9]. Bernert [1] gave a detailed analysis of τ-extrapolation. However, the standard formulation includes an assumption (not always explicitly stated) on the residual transfer operator; when this assumption is violated, the truncation error estimates are inaccurate unless a high-order residual transfer is used.

The purpose of this paper is to present a more general formulation based on the approach of [7] which gives accurate truncation error estimates in all cases without the need for high-order residual transfers. Our focus here is on *pointwise* asymptotic estimates (rather than bounds on error norms), with a view toward using these estimates in adaptive multigrid algorithms such as [6]. The presentation is organized as follows. Section 2 defines the problem and notation. In section 3 we state and prove the main result relating the accuracy of the truncation error estimate to the accuracy of the discretization and grid transfers. Alternative approaches are reviewed and compared in section 4. Section 5 applies this analysis to several model problems and includes numerical results illustrating the accuracy. Our conclusions are summarized in section 6.

2. Problem formulation.

Consider a partial differential equation

$$Lu = f \tag{2.1}$$

on a domain $\Omega \subset \mathbb{R}^d$, where $L : C^n(\Omega) \to C(\Omega)$ is a linear differential operator of order n, $f \in C(\Omega)$ is specified, and $u \in C^n(\Omega)$ is the solution being sought. In practice there will be boundary conditions associated with (2.1); however, they will have no impact on the analysis presented here and thus will be ignored.

We consider a discretization of (2.1) on a grid Ω^h indexed by a mesh size parameter h of the form

$$L^h u^h = f^h := I^h f, \tag{2.2}$$

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where L^h is the discrete form of the operator on grid Ω^h, I^h represents a linear continuum-to-grid h transfer for f (e.g., pointwise restriction), and u^h is the corresponding (exact) solution of the discrete equation. While we have in mind finite-difference discretizations, this formulation could also describe finite element or other discretizations. The corresponding (local) truncation error is

$$\tau^h = \tau^h(u) := L^h \hat{I}^h u - I^h L u, \tag{2.3}$$

where $\hat{I}^h$ represents *pointwise restriction* to grid Ω^h (not necessarily the same as I^h). Note that while the above formulation could easily be extended to systems of equations, our use of pointwise restriction for $\hat{I}^h$ limits the present analysis to nonstaggered discretizations.

In addition to the discrete equation (2.2), a multigrid method will also use a corresponding discrete equation on a coarser grid Ω^H, with mesh ratio $\rho = h/H < 1$ (usually $\rho = 1/2$). In the Full Approximation Scheme (FAS) [2], the coarse-grid equation may be written as

$$L^H \hat{u}^H = \hat{f}^H := L^H(\hat{I}_h^H \tilde{u}^h) + I_h^H(f^h - L^h \tilde{u}^h). \tag{2.4}$$

Here $\tilde{u}^h$ is the current approximation to the true (discrete) solution u^h of the fine-grid equation (2.2), $\hat{I}_h^H$ represents the fine-to-coarse transfer of the solution by *injection* (i.e., pointwise restriction), and I_h^H is the fine-to-coarse residual transfer operator (e.g., full weighting), which we assume to be linear. The corresponding truncation error τ^H, defined by (2.3) with h replaced by H, satisfies

$$L^H \hat{I}^H u = f^H + \tau^H, \tag{2.5}$$

showing that τ^H is the correction to the grid Ω^H equation needed to make its solution coincide (on that grid) with the continuous solution u. Similarly, (2.4) may also be written as

$$L^H \hat{u}^H = f^H + \tau_h^H, \tag{2.6}$$

where

$$\tau_h^H := \hat{f}^H - f^H = (L^H \hat{I}_h^H \tilde{u}^h - f^H) - I_h^H(L^h \tilde{u}^h - f^h) \tag{2.7}$$

is known variously as the *relative local truncation error* [3] or *tau correction* [4].[1] Our goal is to use τ_h^H to estimate τ^H. If this can be done with sufficient accuracy, then we can use this estimate to approximate (2.5) and thus obtain a higher-order approximation, a procedure known as τ-extrapolation. As we will show below, the definition (2.7) leads to this desired accuracy, while the standard definition commonly appearing in the literature [c.f. (4.2)] in general does not.

3. Analysis. Our main result is the following.

THEOREM 3.1 (Truncation Error Estimate). *Assume that there exists $p \geq 1$ and $q \geq 1$ such that if $u \in C^{n+p+q}(\Omega)$, the truncation error (2.3) satisfies*
(A1) $\tau^h = h^p \hat{I}^h v + O(h^{p+q})$ *with* $v \in C^q(\Omega)$,
that the approximate solution $\tilde{u}^h$ of the discrete problem (2.2) satisfies
(A2) $\tilde{u}^h = \hat{I}^h(u + w)$, $w = O(h^p)$,
and that there exists $r \geq 1$ such that for any $\phi \in C^r(\Omega)$,
(A3) $I_h^H \hat{I}^h \phi = \hat{I}^H \phi + O(h^r)$ *and*
(A4) $I_h^H I^h \phi = I^H \phi + O(h^r)$.

[1] This is the same as the residual difference r_h^H of Schaffer [7] up to a multiplicative constant.

Then

$$\tau^H - \alpha\tau_h^H = O(h^{p+m}), \tag{3.1}$$

where $\alpha = (1-\rho^p)^{-1} = H^p/(H^p - h^p)$ *and* $m = \min(q, r)$.

Proof. Following [7] we first use (A1) and (A3) to estimate the truncation error difference[2] between grids Ω^h and Ω^H as

$$\begin{aligned}
(\Delta\tau)_h^H &:= \tau^H - I_h^H \tau^h \\
&= H^p \hat{I}^H v - h^p I_h^H \hat{I}^h v + O(h^{p+q}) \\
&= H^p (1-\rho^p) \hat{I}^H v + O(h^{p+r}) + O(h^{p+q}) \\
&= (1-\rho^p) \tau^H + O(h^{p+m}).
\end{aligned} \tag{3.2}$$

We then use (A2) to relate $(\Delta\tau)_h^H$ to τ_h^H via

$$\begin{aligned}
\tau_h^H - (\Delta\tau)_h^H &= \left(L^H \hat{I}_h^H \tilde{u}^h - I_h^H L^h \tilde{u}^h\right) - \left(L^H \hat{I}^H u - I_h^H L^h \hat{I}^h u\right) \\
&= \left(L^H \hat{I}^H - I_h^H L^h\right) \hat{I}^h w \\
&= \left[\tau^H(w) - I_h^H \tau^h(w)\right] + \left(I^H - I_h^H I^h\right) Lw \\
&= O(h^{p+m}) + O(h^{p+r}) = O(h^{p+m}),
\end{aligned} \tag{3.3}$$

where the last line uses estimates from (3.2) and (A4). Combining (3.2) and (3.3) yields the desired result. ∎

We note that in a well-formulated discretization of order p, the solution error $\hat{I}^h u - u^h$ will also be $O(h^p)$, so (A2) will be satisfied at (or near) convergence, i.e., as $\tilde{u}^h \rightarrow u^h$. With an accurate truncation error estimate in hand, extrapolation to higher-order accuracy is immediate:

COROLLARY 3.2 (τ-Extrapolation). *Under the assumptions of Theorem 3.1, the extrapolated discretization*

$$L^H \bar{u}^H = \bar{f}^H := f^H + \alpha\tau_h^H \tag{3.4}$$

has accuracy $O(h^{p+m})$.

Proof. Using (3.1) and (2.5), the associated truncation error is

$$\bar{\tau}^H := L^H \hat{I}^H u - \bar{f}^H = \tau^H - \alpha\tau_h^H = O(h^{p+m}). \tag{3.5}$$ ∎

As noted by Schaffer [7], at convergence on grid Ω^h (i.e., as $\tilde{u}^h \rightarrow u^h$), the extrapolated equation (3.4) gives the same result as Richardson extrapolation for u^h (provided the latter is justified).

4. Alternatives. The definition (2.7) of τ_h^H above is a simple generalization of that presented in most of the multigrid literature. For example, Brandt [3] uses the same FAS equation (2.4) but splits the right-hand side differently, yielding

$$L^H \hat{u}^H = \hat{f}^H = I_h^H f^h + \tilde{\tau}_h^H \tag{4.1}$$

with

$$\tilde{\tau}_h^H := L^H \hat{I}_h^H \tilde{u}^h - I_h^H L^h \tilde{u}^h. \tag{4.2}$$

[2] In [7], the truncation error difference $(\Delta\tau)_h^H$ is denoted by τ_h^H.

We will refer to (4.2) as the "simple tau" approximation. If the residual transfer operator I_h^H satisfies

$$I^H = I_h^H I^h \tag{4.3}$$

(e.g., when I_h^H is an injection and I^h and I^H are pointwise restrictions) then $\tilde{\tau}_h^H = \tau_h^H$ and the analysis of section 3 holds. This is the standard formulation of τ_h^H in the literature, with the assumption (4.3) either explicit [3, 9] or implicit [4].

However, in general (4.3) does not hold (e.g., in the common situation where the residual transfer I_h^H is full weighting). When it does not, truncation error estimates based on (4.2) are less accurate than those based on (2.7). Specifically, using (2.7) and (4.2) in (3.1), we obtain

$$\tau^H - \alpha\tilde{\tau}_h^H = E^H + O(h^{p+m}), \tag{4.4}$$

where

$$E^H := \alpha \left(I_h^H I^h - I^H \right) f. \tag{4.5}$$

In general we expect only $E^H = O(h^r)$ [c.f. (A4)]. Indeed, obtaining $E^H = O(h^{p+m})$ requires a high-order residual transfer which matches with any averaging used for the forcing in the discretization (see examples below).

In a slightly different formulation, Bernert [1] also uses (4.2), but replaces (4.1) with

$$L^H \hat{u}^H = f^H + \tilde{\tau}_h^H, \tag{4.6}$$

which matches the FAS equation (2.4) only when (4.3) holds—which it does not in general. Since the same simple tau approximation is used, truncation error estimates based on this formulation suffer the same loss of accuracy as in (4.4). For this reason, Bernert derives unnecessarily stringent conditions on the grid transfer operators in order to achieve accurate results (see examples below).

5. Examples. To illustrate the above analysis we consider three examples. Each is formulated on a rectangular domain Ω in $\mathbb{R}^2$, with uniform grids Ω^h and Ω^H of mesh sizes h and $H = 2h$, respectively, in both x and y. The residual transfer in each case is standard full weighting, given in stencil notation as

$$I_h^H := \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}. \tag{5.1}$$

Using Taylor expansions, we can show that for any function $\phi \in C^2(\Omega)$,

$$I_h^H \hat{I}^H \phi = \hat{I}^H \phi + \frac{h^2}{4} \hat{I}^H \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) + o(h^2), \tag{5.2}$$

so (A3) holds with $r = 2$.

5.1. Example 1: Poisson problem ($p = 2$). Consider the two-dimensional Poisson problem

$$Lu := -\Delta u = f \qquad \text{in } \Omega \subset \mathbb{R}^2 \tag{5.3}$$

with Dirichlet boundary conditions. The standard second-order finite difference discretization of (5.3) on Ω^h is given by

$$L_2^h u^h = f^h := I^h f, \tag{5.4}$$

where the discrete operator is given in stencil notation as

$$L_2^h := \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} \tag{5.5}$$

and $I^h = \hat{I}^h$ is pointwise restriction for the forcing. If $u \in C^6(\Omega)$ then the corresponding truncation error is

$$\tau^h = -\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) + O(h^4), \tag{5.6}$$

so (A1) holds with $p = 2$ and $q = 2$. Given that I^h (and thus I^H) for this discretization is pointwise restriction, the estimate (5.2) shows that we can take $r = 2$ in (A4). Thus, since $p = q = r = 2$ we have $m = 2$ also, so from (3.1) we see that τ_h^H provides a fourth-order estimate of the truncation error. Consequently, according to (3.5), the τ-extrapolation (3.4) produces an approximation which is $O(h^4)$. This same result holds if we replace full weighting by injection for the residual transfer I_h^H.

This conclusion is illustrated by the numerical results in Fig. 5.1. Here, the problem domain is taken to be the unit square $\Omega = [0,1] \times [0,1]$ with $h = 1/N$, the forcing f and boundary data are chosen to match the analytical solution $u = \cos(4x + 6y)$, and the discrete problem is solved by enough multigrid V-cycles that the solution has effectively converged on the fine grid, thus satisfying assumption (A2). Various measures of the error in the truncation error estimate, i.e., the left-hand side of (3.1), are plotted as functions of N. While the

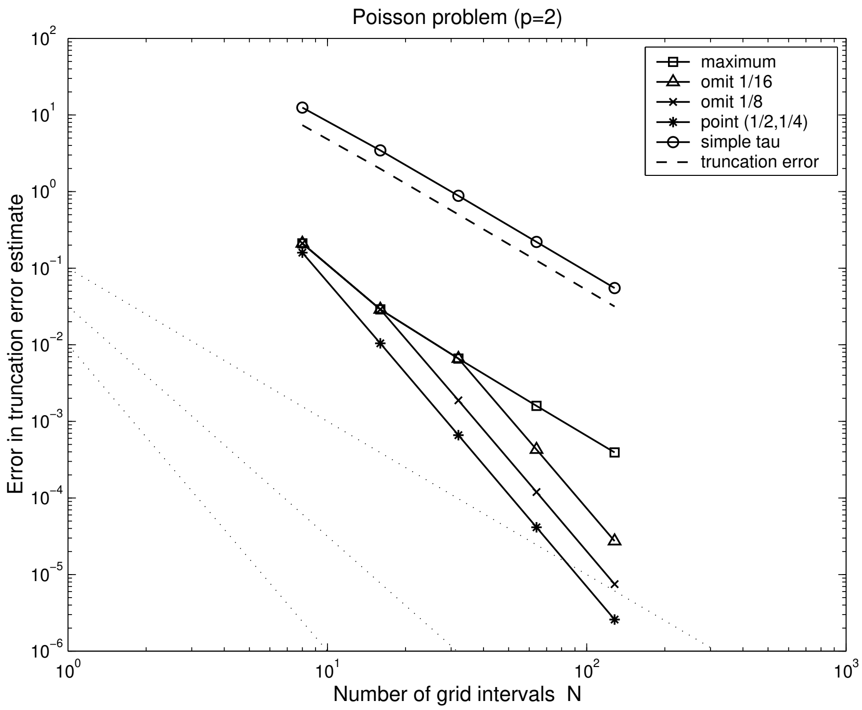

FIG. 5.1. *Error in the truncation error estimate (solid lines) and true truncation error (dashed line) for the second-order Poisson problem (5.4). Dotted lines give slopes for orders $p = 2$, 3, and 4.*

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maximum error over the domain is only $O(h^2)$, if a narrow boundary strip (width $1/8$ or $1/16$) is omitted, the maximum error is asymptotically $O(h^4)$—as it is at a typical point $(x, y) = (\frac{1}{2}, \frac{1}{4})$ as well. This behavior is due to large errors at the four points immediately adjacent to the corners of the domain as shown in Fig. 5.2. This relatively poor performance near the corners does not invalidate the analysis of section 3: pointwise, the errors are $O(h^4)$ as expected. It is interesting to note that even at the worst points, the error in the truncation error estimate is nearly two orders of magnitude smaller than the truncation error itself.

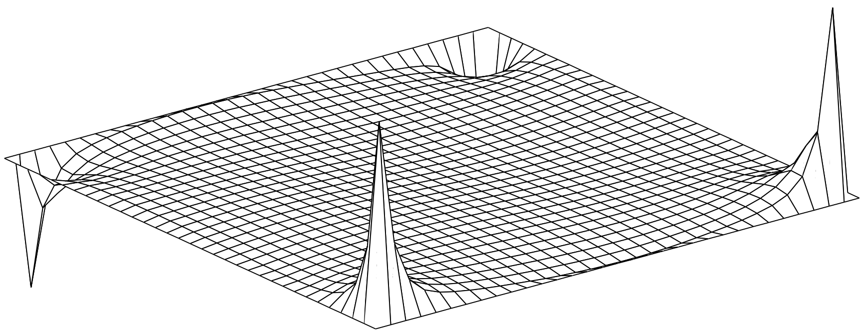

FIG. 5.2. *Error in the truncation error estimate for the second-order Poisson problem (5.4) with $N = 64$ fine-grid intervals as a function of x and y.*

In contrast, if we use the simple tau approximation (4.2), then the leading term in the error in approximating the truncation error via (4.4) is seen from (5.2) to be $E^H = O(h^2)$. Consequently, using $\tilde{\tau}_h^H$ provides only a second-order estimate of the truncation error. The analysis of [1] predicts the same result: to achieve a higher-order approximation of τ^H one must use a higher-order residual transfer I_h^H, precisely because that formulation uses the simple tau approximation. Figure 5.1 illustrates the low accuracy obtained in this way. Indeed, the error in this approximation to τ^H is approximately twice as large as τ^H itself, so the approximation is useless.

5.2. Example 2: Poisson problem ($p = 4$). A compact fourth-order finite difference discretization of (5.3) on Ω^h is given by the so-called "Mehrstellen" [8] discretization

$$L_4^h u^h = f^h, \tag{5.7}$$

where the discrete operator is given in stencil notation as

$$L_4^h := \frac{1}{6h^2}\begin{bmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{bmatrix} \tag{5.8}$$

and the forcing term uses the averaging operator

$$I^h := \frac{1}{12}\begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix}. \tag{5.9}$$

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If $u \in C^8(\Omega)$ and $f \in C^6(\Omega)$ then it can be shown that

$$L_4^h u = -\Delta u - \frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 y^2} + \frac{\partial^4 u}{\partial y^4}\right) - \frac{h^4}{360}\left[\frac{\partial^6 u}{\partial x^6} + 5\left(\frac{\partial^6 u}{\partial x^4 y^2} + \frac{\partial^6 u}{\partial x^2 y^4}\right) + \frac{\partial^6 u}{\partial y^6}\right] + O(h^6) \tag{5.10}$$

and

$$I^h f = f + \frac{h^2}{12}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{h^4}{144}\left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4}\right) + O(h^6), \tag{5.11}$$

so the truncation error satisfies

$$\tau^h = \frac{h^4}{720}\left[3\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right) - 5\left(\frac{\partial^6 u}{\partial x^4 y^2} + \frac{\partial^6 u}{\partial x^2 y^4}\right)\right] + O(h^6). \tag{5.12}$$

Thus, in (A1) we have $p = 4$ and $q = 2$. Also, from (5.2) and (5.11) we can show that (A4) holds with $r = 2$. Thus, since $p = 4$ and $q = r = 2$ we have $m = 2$ also, so from (3.1) we see that τ_h^H provides a sixth-order estimate of the truncation error. Consequently, according to (3.5), the τ-extrapolation (3.4) produces an approximation which is $O(h^6)$. This same result holds if we replace full weighting by injection for the residual transfer I_h^H. These results are illustrated by the numerical results in Fig. 5.3, where the details of the test case are the same as in the second-order case presented in Fig. 5.1. Here there are no anomalously large errors near the corners, so the maximum error on the domain is $O(h^6)$.

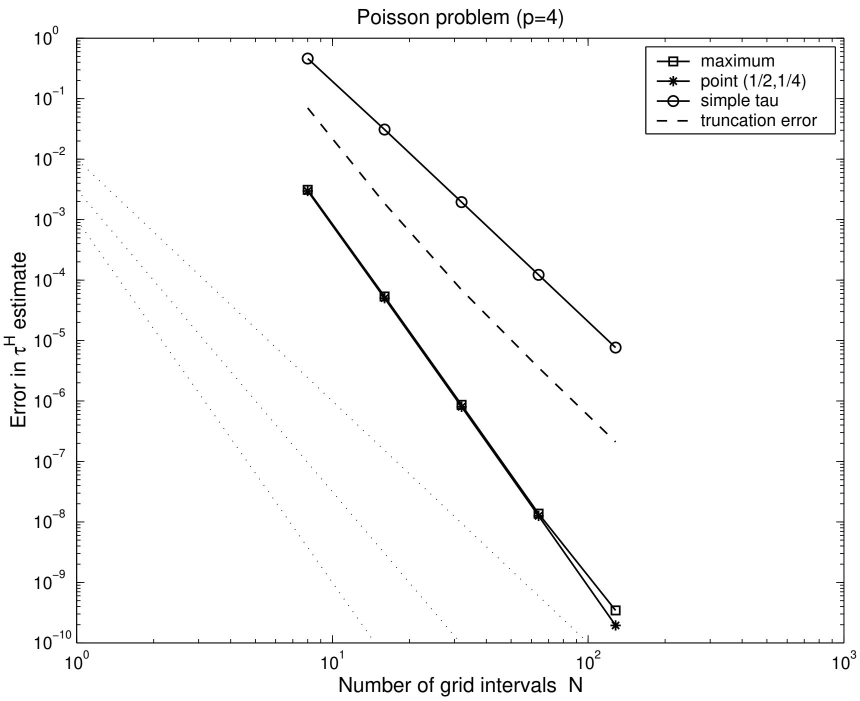

FIG. 5.3. *Error in the truncation error estimate (solid lines) and true truncation error (dashed line) for the fourth-order Poisson problem (5.7). Dotted lines give slopes for orders $p = 4$, 5, and 6.*

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In contrast, if we use the "simple tau" approximation (4.2), then the leading term in the error in approximating the truncation error [cf. (4.5)] can be shown from (5.2) and (5.11) to be $E^H = O(h^4)$, with the second-order terms fortuitously dropping out. Consequently, using $\tilde{\tau}_h^H$ provides only a fourth-order estimate of the truncation error, as shown in Fig. 5.3. In this case, the error in this approximation to τ^H is more than an order of magnitude larger than τ^H itself, so again the approximation is useless.

5.3. Example 3: Monge-Ampere equation. While the analysis of section 3 is valid for a linear operator L, in practice similar results may be obtained for some nonlinear problems. As an example, consider the Monge-Ampere equation

$$Lu = \left(1 - \frac{\partial^2 u}{\partial x^2}\right)\left(1 - \frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = f \tag{5.13}$$

on a rectangle $\Omega \subset \mathbb{R}^2$. This equation can be discretized by second-order centered finite differences and solved by a multigrid method; the details appear in [5]. As for the second-order discretization of the Poisson equation we obtain $p = q = 2$ and (with full weighting for the residual transfer) $r = 2$, so the truncation error estimate has accuracy $O(h^4)$. This is illustrated by the numerical results in Fig. 5.4, where the details of the test case are as given in [5]. Indeed, the analysis of section 3 explains the need for injection rather than full weighting in equation (4.4) of [5], corresponding to defining τ_h^H by (2.7) rather than using the simple-tau approximation (4.2).

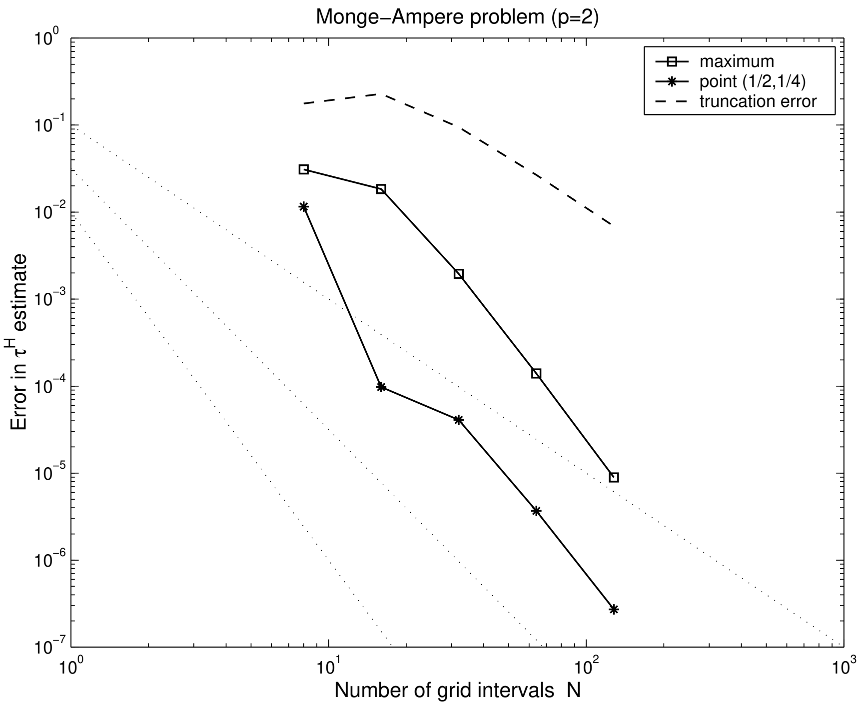

FIG. 5.4. *Error in the truncation error estimate (solid lines) and true truncation error (dashed line) for the second-order Monge-Ampere problem (5.13). Dotted lines give slopes for orders $p = 2$, 3, and 4.*

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6. Conclusions. Truncation error estimates can be computed easily during multigrid processing. For these to be accurate they must in general be based on the proper definition of the relative local truncation error τ_h^H given by (2.7). The "simple tau" approximation $\tilde{\tau}_h^H$ given by (4.2) commonly presented in the literature gives this accuracy only when the discretization and residual transfer satisfy $I^H f = I_h^H I^h f$ (e.g., when I_h^H is injection). Otherwise (e.g., when I_h^H is full weighting) this approximate truncation error estimate is inaccurate, typically with error larger than the truncation error itself. Fortunately, the proper formulation is no more complicated; indeed, in an FAS multigrid method it usually requires only one subtraction per coarse-grid point.

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