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RECENT TRENDS ON ANALYTIC PROPERTIES OF MATRIX ORTHONORMAL POLYNOMIALS *

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Abstract. In this paper we give an overview of recent results on analytic properties of matrix orthonormal polynomials. We focus our attention on the distribution of their zeros as well as on the asymptotic behavior of such polynomials under some restrictions about the measure of orthogonality.

Key words. matrix orthogonal polynomials, zeros, asymptotic behavior.

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1. Introduction. Consider a $p \times p$ positive definite matrix of measures $W(x) = (W_{i,j}(x))_{i,j=1}^{p}$ supported on Ω ($\Omega = \mathbb{R}$ or $\Omega = \mathbb{T}$, the unit circle), i.e., for every Borel set $A \subset \Omega$ the numerical matrix $W(A) = (W_{i,j}(A))_{i,j=1}^{p}$ is positive semi-definite. Notice that the diagonal entries of W are positive measures and the non-diagonal entries are complex measures with $W_{i,j} = \overline{W_{j,i}}$.

For a positive definite matrix of measures W the support of W is the support of the trace measure $\tau(W) = \sum_{i=1}^{p} W_{i,i}$. A matrix polynomial of degree m is a mapping $P : \mathbb{C} \to \mathbb{C}^{(p,p)}$ such that

$$P(x) = M_m x^m + M_{m-1}x^{m-1} + \cdots + M_1 x + M_0,$$

where $(M_k)_{k=0}^{m} \in \mathbb{C}^{(p,p)}$ and M_m is different of the zero matrix. We denote $P^*(x) = M_m^*\overline{x}^m + M_{m-1}^*\overline{x}^{m-1} + \cdots + M_1^*\overline{x} + M_0^*$.

Assuming that $\int_\Omega P(x)dW(x)P^*(x)$ is nonsingular for every matrix polynomial P with nonsingular leading coefficient, we introduce an inner product in the linear space of matrix polynomials $\mathbb{C}^{(p,p)}[x]$ in the following way

$$\langle P, Q\rangle_L \stackrel{\text{def}}{=} \int_\Omega P(x)dW(t)Q^*(x). \tag{1.1}$$

Using the Gram-Schmidt orthonormalization process for the canonical sequence $\{x^n I_p\}_{n=0}^{\infty}$, we will obtain many sequences of matrix polynomials which are orthonormal with respect to (1.1). Indeed, if (P_n) is a sequence of matrix polynomials such that

$$\langle P_n, P_m\rangle_L = \delta_{n,m} I_p \tag{1.2}$$

then for every sequence (U_n) of unitary matrices, the sequence (R_n) such that $R_n = U_n P_n$ satisfies

$$\langle R_n, R_m\rangle_L = \delta_{n,m} I_p.$$

Such orthonormal polynomials are interesting not only from a theoretical point of view but by their applications in many scientific domains.

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Orthogonal matrix polynomials on the real line appear in the Lanczos method for block matrices [11, 12], in the spectral theory of doubly infinite Jacobi matrices [18], in the analysis of sequences of polynomials satisfying higher order recurrence relations [9], in rational approximation and in system theory [10].

Orthogonal matrix polynomials on the unit circle are used in the inversion of finite block Toeplitz matrices which arise naturally in linear estimation theory. The matrix to be inverted is the covariance matrix of a multivariate stationary stochastic process [14]. Furthermore, they appear in the analysis of sequences of polynomials orthogonal with respect to scalar measure supported on equipotential curves in the complex plane [13]. Finally, another application in time series analysis consists in the frequency estimation of a stationary harmonic process (X_n), i.e.,

$$X_n = \sum_{k=1}^{n} [A_k \cos nw_k + B_k \sin nw_k] + Z_n,$$

where (A_k), (B_k) are matrices of dimension p and Z_n is a white noise. The frequencies $(w_k)_{k=1}^n$ are unknown and need to be estimated from the data. They can be given in terms of zeros of matrix orthogonal polynomials associated with some purely discrete measure supported on the unit circle [18].

The aim of the present contribution is to give a framework of the subject and summarize some recent contributions focused in two aspects:

1. The asymptotic behavior of sequences of matrix orthonormal polynomials in several cases (real and unit circle, respectively).
2. The distribution of the zeros of such polynomials as well as their connection with matrix quadrature formulas.

These questions have attracted during the last decade the interest of several research groups. A big effort was done in the analytic theory by A. J. Durán and coworkers in Universidad of Sevilla, and W. Van Assche in Katholieke Universiteit Leuven among others. We hope that our work will be a useful approach for beginners, following the nice surveys [15, 18].

The structure of the paper is the following. In section 2 we introduce matrix orthogonal polynomials on the real line, and we consider the three-term recurrence relation which characterizes them. In section 3 we give some basic results about the zeros of such polynomials, and we explain the analog of the gaussian quadrature formulas in the matrix case. In section 4, the matrix Nevai class is studied, and thus relative asymptotics for the corresponding sequences of matrix orthonormal polynomials are discussed. Furthermore, the analysis of perturbations in the Nevai class by the addition of a discrete measure supported in a singleton is presented. In section 5, we analyze matrix orthonormal polynomials on the unit circle. We focus our attention on the study of their zeros and, as an application, we find quadrature formulas extending the well known results of the scalar case. Finally, in section 6 we present the connection between matrix of measures supported on the interval $[-1, 1]$ and matrix of measures supported on the unit circle.

2. Orthogonal matrix polynomials on the real line. Let W be a matrix of measures supported on the real line. As in the scalar case, the shift operator

$$\mathbf{H} : \mathbb{C}^{(p,p)}[x] \to \mathbb{C}^{(p,p)}[x], \quad \mathbf{H}[P](x) = xP(x)$$

is symmetric with respect to the inner product (1.1). Thus, for every sequence (P_n) of matrix orthonormal polynomials with respect to (1.1), we get a three-term recurrence relation

$$\begin{aligned} z\, P_n(z;W) = D_{n+1}(W)P_{n+1}(z;W) \\ + E_n(W)P_n(z;W) + D_n^*(W)P_{n-1}(z;W), \quad n \geq 0, \end{aligned} \tag{2.1}$$

where

$$E_n = \langle xP_n, P_n\rangle_L = \langle P_n, xP_n\rangle_L = E_n^*, \quad n \geq 0$$

is a Hermitian matrix, and

$$D_n = \langle xP_{n-1}, P_n\rangle_L, \;\; n \geq 1.$$

Notice that if the leading coefficient A_n of P_n is nonsingular, then $D_n = A_{n-1}A_n^{-1}$, i.e., D_n is a nonsingular matrix. On the other hand, since (U_nP_n) is another sequence of matrix orthonormal polynomials when (U_n) are unitary matrices, in such a case the corresponding coefficients in the three-term recurrence relation are $\tilde{E}_n = U_nE_nU_n^*$ and $\tilde{D}_n = U_{n-1}D_nU_n^*$.

Conversely, given two sequences of matrices (D_n) and (E_n) of dimension p, such that (D_n) are nonsingular matrices and (E_n) are Hermitian matrices, then there exists a positive definite matrix of measures W such that the matrix polynomials defined by the recurrence relation

$$xY_n = D_{n+1}Y_{n+1} + E_nY_n + D_n^*Y_{n-1}, \;\; n \geq 0, \tag{2.2}$$

with the initial conditions $Y_{-1} = 0$ and $Y_0 = I_p$ constitute a sequence of matrix polynomials orthonormal with respect to the inner product (1.1) associated with W. In fact, W is related with the spectral resolution of the identity for the operator $\mathbf{H}$ defined as above (cf. [1]). This result constitutes the matrix analog of the Favard´s theorem in the scalar case (cf. [2, 4]).

A second polynomial solution of (2.2) is associated with initial conditions $Y_1 = D_1^{-1}$ and $Y_0 = 0$. Then if we denote it by (Q_n), we get $deg\, Q_n = n - 1$. In fact

$$Q_n(z;W) = \int_{\mathbb{R}} \frac{P_n(z;W) - P_n(s;W)}{z-s} dW(s). \tag{2.3}$$

Such a sequence of matrix polynomials is called the sequence of matrix orthonormal polynomials of the second kind with respect to the matrix of measures W, where we assume $\int_{\mathbb{R}} dW(s) = I_p$.

¿From (2.3) we get

$$Q_n(z;W) = P_n(z;W)\int_{\mathbb{R}} \frac{dW(s)}{z-s} - \int_{\mathbb{R}} \frac{P_n(s;W)}{z-s} dW(s). \tag{2.4}$$

If the operator $\mathbf{H}$ is bounded, then the function $\mathrm{F}(z;W) \overset{\text{def}}{=} \int_{\mathbb{R}} \frac{dW(s)}{z-s}$ is analytic outside the spectrum of $\mathbf{H}$. In a neighborhood of the infinity we get $\mathrm{F}(z;W) = \sum_{k=0}^{\infty} \frac{S_k}{z^{k+1}}$, where $S_k = \int_{\mathbb{R}} s^k dW(s)$ are the moments for the matrix of measures W. Thus (2.4) yields

$$P_n(z;W)\mathrm{F}(z;W) - Q_n(z;W) = \frac{A_{n,1}}{z^{n+1}} + \cdots .$$

The regular rational matrix function $\pi_n(z) = P_n^{-1}(z;W)Q_n(z;W)$ is said to be the nth Padé fraction for $\mathrm{F}(z;W)$. This constitutes one of the main applications of matrix orthonormal polynomials in approximation theory. The connection with rational matrix approximation and matrix continued fractions follows immediately [1].

As in the scalar case, we introduce the nth kernel polynomial associated with the matrix of measures W.

DEFINITION 2.1. *The matrix polynomial* $K_n(x,y;W) \overset{def}{=} \sum_{j=0}^{n} P_j^*(y;W)P_j(x;W)$ *is said to be the* n*th kernel polynomial associated with* W.

PROPOSITION 2.2 (Reproducing property).

$$\langle Q(x), K_n(x,y;W)\rangle_L = Q(y)$$

for every matrix polynomials Q *of degree less than or equal to* n.
Notice that the nth kernel is the same for every sequence of matrix orthonormal polynomials associated with W. In fact, if $R_n = U_nP_n$ with (U_n) unitary matrices, then $K_n(x,y;W) = \sum_{j=0}^{n} R_j^*(y;W)R_j(x;W) = \sum_{j=0}^{n} P_j^*(y;W)P_j(x;W)$.

3. Zeros and quadrature formulas. A point x_0 is said to be a zero of the matrix polynomial $P(x)$ if $\det P(x_0) = 0$. If $P(x)$ is a $p\times p$ matrix polynomial and x_0 is a zero of $P(x)$, we write

$$\mathcal{N}(x_0,P) = \left\{ v \in \mathbb{C}^{(p,1)} : \;\; P(x_0)v = 0 \right\},$$

i.e., the null space for the singular matrix $P(x_0)$.

LEMMA 3.1 ([7]). *If* $dim\,\mathcal{N}(x_0,P) = n$*, then* $[Adj\ P(x)]^{(l)}(x_0) = 0$ *for* $l = 0,1,\cdots,n-2$*, and* x_0 *is a zero of* $P(x)$ *of multiplicity at least* n*. Here* $Adj\ P(x)$ *is the matrix such that* $P(x)\ Adj\ P(x) = Adj\ P(x)\ P(x) = (\det P(x))\ I_p$.

LEMMA 3.2 ([7]). *For* $n\in\mathbb{N}$*, the zeros of the matrix polynomial* $P_n(x)$ *are the same as those of the polynomial* $\det(xI_{np} - \mathbb{J}_{np})$ *(with the same multiplicity) where* $\mathbb{J}_{np}$ *is the Jacobi matrix of dimension* np*, i.e., the eigenvalues of the matrix*

$$\mathbb{J}_{np} = \begin{pmatrix} E_0(.) & D_1(.) & 0 & \cdots & 0 \\ D_1^*(.) & E_1(.) & D_2(.) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & D_{n-1}(.) \\ 0 & \cdots & 0 & D_{n-1}^*(.) & E_{n-1}(.) \end{pmatrix} \in \mathbb{C}^{(np,np)}.$$

Notice that if v is an eigenvector corresponding to the eigenvalue x_0, writing v as a block column vector

$$v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \in \mathbb{C}^{(np,1)},$$

where $v_i \in \mathbb{C}^{(p,1)}$, the equation $\mathbb{J}_{np}v = x_0 v$ reads

$$\begin{array}{rcl} v_1 &=& P_1(x_0)v_0 \\ v_2 &=& P_2(x_0)v_0 \\ \vdots &=& \vdots \\ v_{n-1} &=& P_{n-1}(x_0)v_0, \end{array} \qquad\qquad 0 = P_n(x_0)v_0,$$

or, equivalently,

$$v = \begin{pmatrix} P_0(x_0) \\ P_1(x_0) \\ \vdots \\ P_{n-1}(x_0) \end{pmatrix} v_0.$$

In other words, there exists a bijection between $\mathcal{N}(x_0, P_n)$ and the subspace of eigenvectors of the matrix $\mathbb{J}_{np}$ associated with the eigenvalue x_0.

THEOREM 3.3 ([7]).

1. *The zeros of P_n have a multiplicity less than or equal to p. All the zeros are real.*
2. *If x_0 is a zero of P_n with multiplicity m then $rank\, P_n(x_0) = p - m$.*
3. *If we write $x_{n,k}$ $(k = 1, \cdots, np)$ for the zeros of P_n in increasing order and taking into account their multiplicities, then the following interlacing property holds,*

$$x_{n+1,k} \leq x_{n,k} \leq x_{n+1,k+p}$$

for $k = 1, 2, \cdots, np$.

4. *If x_0 is both a zero of P_n and P_{n+1}, then $\mathcal{N}(x_0, P_n) \cap \mathcal{N}(x_0, P_{n+1}) = \{0\}$.*
5. *If $x_{n,k}$ is a zero of P_n with multiplicity p, then it can not be a zero of P_{n+1}.*

We will denote $\mathcal{Z}_n(W) \stackrel{\text{def}}{=} \{x_{n,k},\ k = 1, \cdots, np,\ \det\ P_n(x_{n,k}; W) = 0\}$ the set of zeros of the orthonormal matrix polynomial $P_n(.\,; W)$. Let

$$\mathrm{M}_N = \overline{\cup_{n \geq N} \mathcal{Z}_n(W)} \ \text{ and } \ \Gamma = \cap_{N \geq 1} \mathrm{M}_N,$$

then $supp(dW) \subset \Gamma$. We will denote by $\hat{\Gamma}$ the smallest closed interval which contains the support of dW.

As in the scalar case, we can deduce quadrature formulas for matrix polynomials. The next theorem shows how to compute the quadrature coefficients (the matrix Christoffel constants) by means of the eigensystem of the Jacobi matrix $\mathbb{J}_{np}$.
Let $v_{i,j}$ $(j = 1, 2, \cdots, m_i)$ be the eigenvectors of the matrix $\mathbb{J}_{np}$ associated with the eigenvalue x_i $(i = 1, \cdots, k)$ (with multiplicity m_i). Let

$$\Lambda_i = \underbrace{\begin{pmatrix} v_{i,1}^{(0)}\ v_{i,2}^{(0)}\ \cdots\ v_{i,m_i}^{(0)} \end{pmatrix}}_{v_i^{(0)}} \mathrm{M}_i^{-1} \underbrace{\begin{pmatrix} v_{i,1}^{(0)^*} \\ v_{i,2}^{(0)^*} \\ \vdots \\ v_{i,m_i}^{(0)^*} \end{pmatrix}}_{v_i^{(0)^*}} \in \mathbb{C}^{(p,p)},$$

where $v_{i,s}^{(0)} \in \mathbb{C}^{(p,1)}$ $(s = 1, \cdots, m_i)$ is the vector consisting of the first p components of $v_{i,s}$ and $\mathrm{M}_i = \left(v_i^{(0)}\right)^* K_{n-1}(x_i, x_i) v_i^{(0)}$. Then

THEOREM 3.4 ([17]). *The quadrature formula*

$$\mathrm{I}_n(P, Q) \stackrel{def}{=} \int_a^b P(x) dW(x) Q^*(x) = \sum_{i=1}^{k} P(x_i)\, \Lambda_i\, Q^*(x_i)$$

is exact for matrix polynomials P and Q with $deg\, P + deg\, Q \leq 2n - 1$. Here k denotes the number of different zeros of P_n.

This quadrature formula yields, as in the scalar case,

$$\mathrm{I}_n(P, I_p) = \sum_{i=1}^{k} P(x_i)\,\Lambda_i, \quad deg\, P \leq 2n-1.$$

An alternative approach is given in the following sense.

THEOREM 3.5 ([5]). *Let $x_{n,i}$ $(i = 1, \cdots, k)$ be the different zeros of the matrix polynomial P_n with multiplicities m_i respectively. Let*

$$\Gamma_{n,i} \stackrel{def}{=} \frac{1}{[\det\, P_n(x)]^{(m_i)}\,(x_{n,i})}\,[Adj\, P_n(x)]^{(m_i-1)}\,(x_{n,i})\; Q_n(x_{n,i}).$$

Then

1. *For every polynomial P with $deg\, P \leq 2n-1$ we get*

$$\int_{\mathbb{R}} P(x) dW(x) = \sum_{i=1}^{k} P(x_{n,i})\,\Gamma_{n,i}.$$

2. *$\{\Gamma_{n,i}\}_{i=1}^{k}$ are positive semi-definite matrices of rank m_i.*

Using the above quadrature formula, we get the following matrix analog of the Markov´s theorem

THEOREM 3.6 ([5]). *Assume the positive definite matrix of measures W is determinate, i.e., no other positive definite matrix of measures has the same moments as those of W. Then*

$$\lim_{n\to\infty} P_n^{-1}(z;W)\,Q_n(z;W) = \int_{\mathbb{R}} \frac{dW(t)}{z-t} \stackrel{def}{=} \mathrm{F}(z;W)$$

locally uniformly in $\mathbb{C}\setminus\Gamma$.

$\mathrm{F}(z;W)$ is called the Stieltjes (Markov) function associated with the matrix of measures W.

The above result means that if W is determinate, the nth Padé fraction of $\mathrm{F}(z;W)$ converges locally uniformly to $\mathrm{F}(z;W)$ in $\mathbb{C}\setminus\Gamma$.

4. The Nevai class. We will introduce an analog of the so-called Nevai class for matrix orthonormal polynomials

DEFINITION 4.1. *Given two matrices D and E, where E is Hermitian, a sequence of matrix orthonormal polynomials P_n satisfying (2.1) belongs to the matrix Nevai class $\mathrm{M}(D,E)$ if $\lim_{n\to\infty} D_n(W) = D$ and $\lim_{n\to\infty} E_n(W) = E$, respectively.*
A positive definite matrix of measures W belongs to the Nevai class $\mathrm{M}(D,E)$ if some of the corresponding sequences of matrix orthonormal polynomials belongs to $\mathrm{M}(D,E)$. Notice that a positive definite matrix of measures can belong to several Nevai classes because of the non-uniqueness of the corresponding sequences of matrix orthonormal polynomials. If D is a nonsingular matrix, we can introduce the sequence of matrix polynomials $\{\mathrm{U}_n(z;D,E)\}$ defined by the recurrence formula

$$z\,\mathrm{U}_n(z) = D^*\mathrm{U}_{n+1}(z) + E\mathrm{U}_n(z) + D\mathrm{U}_{n-1}(z) \;\; n \geq 0$$

with the initial conditions $\mathrm{U}_0(z) = I_p,\;\; \mathrm{U}_{-1}(z) = 0$. According to Favard´s theorem, this sequence is orthonormal with respect to a positive definite matrix of measures $W_{D,E}$. Notice that they are the matrix analogs of Chebyshev polynomials of second kind.

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REMARK 4.2. *The Jacobi matrix associated with a sequence of matrix orthonormal polynomials in the Nevai class is a compact perturbation of the Jacobi matrix*

$$\mathbb{J}(.) \stackrel{def}{=} \begin{pmatrix} E(.) & D(.) & 0 & \\ D^*(.) & E(.) & D(.) & \ddots \\ 0 & D^*(.) & E(.) & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

If $\mathrm{F}(z; D, E)$ is the Stieltjes function associated with the matrix of measures $W_{D,E}$ and D is nonsingular matrix, we get

THEOREM 4.3 ([6]). *Let $(P_n) \in \mathrm{M}(D, E)$ with D a nonsingular matrix. Then*

$$\lim_{n\to\infty} P_{n-1}(z)P_n^{-1}(z)D_n^{-1} = \mathrm{F}(z; D, E)$$

locally uniformly in $\mathbb{C} \setminus \Gamma$. Furthermore, $\mathrm{F}(z; D, E)$ is a solution of the quadratic matrix equation

$$D^*XDX + (E - zI_p)X + I_p = 0.$$

In particular, if D is a positive definite matrix, we can give the explicit expression for $\mathrm{F}(z; D, E)$. Let $S(z) = \frac{1}{2}D^{-1/2}(zI_p - E)D^{-1/2}$, then

$$\mathrm{F}(z; D, E) = D^{-1/2}\left[S(z) - \left(S^2(z) - I_p\right)^{1/2}\right] D^{-1/2}.$$

Furthermore, the matrix $S(z)$ is diagonalizable up to a finite set of complex numbers z and also so is $D^{1/2}\mathrm{F}(z; D, E)D^{1/2}$. If a is an eigenvalue of $S(z)$, then $a - (a^2 - 1)^{\frac{1}{2}}$ is an eigenvalue of $D^{1/2}\mathrm{F}(z; D, E)D^{1/2}$ assuming that $\mid a - (a^2 - 1)^{\frac{1}{2}} \mid < 1$ for $a \in \mathbb{C} \setminus [-1, 1]$ which guarantees the existence of an appropriate square root. Since for $x \in \mathbb{R}$, $I_p - S^2(x)$ is Hermitian, then

$$I_p - S^2(x) = U(x)N(x)U^*(x),$$

where $N(x)$ is a diagonal matrix with entries $\{d_{i,i}\}_{i=1}^p$ and $U(x)$ is an unitary matrix. Then the matrix weight $W_{D,E}(x)$, $x \in \mathbb{R}$, is

$$dW_{D,E}(x) = \frac{1}{\pi}D^{-1/2}\, U(x) \left[N^+(x)\right]^{\frac{1}{2}} U^*(x)\, D^{-1/2}dx,$$

where $N^+(x)$ is the diagonal matrix with entries $d^+_{i,i}(x) \stackrel{\text{def}}{=} \max\{d_{i,i}(x), 0\}$.

The support of $W_{D,E}$ is then the set of real numbers

$$\{y \in \mathbb{R} :\ S(y)\,\text{has an eigenvalue in}\,[-1, 1]\}\,.$$

In fact, $W_{D,E}$ is absolutely continuous with respect to the Lebesgue measure multiplied by the identity matrix, and the support is the finite union of at most p disjoint and bounded intervals.

If D is Hermitian, in [6] an example where $W_{D,E}$ is absolutely continuous with respect to the Lebesgue measure times the identity matrix but with an unbounded Radon-Nikodym derivative is presented. In the general case of non-singularity of D, nothing is known about the support of $W_{D,E}$. Furthermore, nothing is known about the absolute continuity for the entries with respect to the Lebesgue measure as well as Dirac deltas can appear. This last case is one of the reasons for the analysis of perturbation of matrix of measures on the Nevai class by the addition of Dirac deltas.

Let W be a matrix of measures supported on the real line, M a positive definite matrix of dimension p, and $c \in \mathbb{R} \setminus \Gamma$. Consider the matrix of measures $\tilde{W}$ such that

$$d\tilde{W}(x) = dW(x) \;+\; \mathrm{M}\delta(x-c).$$

If $(P_n(x;W))$ and $(P_n(x;\tilde{W}))$ are two sequences of matrix orthonormal polynomials with respect to W and $\tilde{W}$ respectively, satisfying a three-term recurrence relation such that $D = \lim_{n\to\infty} D_n(W)$ is nonsingular, then

THEOREM 4.4 ([22],[21]). *There exists a sequence of matrix orthonormal polynomials $(P_n(x;\tilde{W}))$ such that*

1. $$\lim_{n\to\infty}\left[A_n(\tilde{W})A_n(W)^{-1}\right]^*\left[A_n(\tilde{W})A_n(W)^{-1}\right] = I_p + \mathrm{F}(c;D,E)\;[\mathrm{F}'(c;D,E)]^{-1}\;\mathrm{F}(c;D,E).$$
2. *If $\Lambda(c)\Lambda(c)^*$ is the Cholesky factorization of the positive definite matrix given in the right hand side of the above expression, then*

$$\lim_{n\longrightarrow\infty} P_n(x,\tilde{W})P_n^{-1}(x,W) = \Lambda(c)^{-1} + \tfrac{1}{c-x}\left\{\Lambda(c)^* - \Lambda(c)^{-1}\right\}\left\{\mathrm{F}^{-*}(c;D,E) - \mathrm{F}^{-1}(c;D,E)\right\}$$

locally uniformly in $\mathbb{R}\setminus\left\{\hat{\Gamma}\cup\{c\}\right\}$.

3. *$(P_n(x;\tilde{W}))$ belongs to the matrix Nevai class $\mathrm{M}(\tilde{D},\tilde{E})$ with*

$$\begin{aligned}
\tilde{D} &= \Lambda^*(c)D\Lambda^{-*}(c)\\
\tilde{E} &= \Lambda^*(c)E\Lambda^{-*}(c) + \Lambda^*(c)\;\Big\{D\left[\Lambda^{-*}(c).\Lambda^{-1}(c)-I_p\right]D^*\mathrm{F}(c;D,E)\\
&\qquad - \left[\Lambda^{-*}(c).\Lambda^{-1}(c)-I_p\right]D^*\mathrm{F}(c;D,E)D\Big\}\;\Lambda^{-*}(c).
\end{aligned}$$

When D is Hermitian and nonsingular, we associate with the Nevai class $\mathrm{M}(D,E)$ the sequence $\{\mathrm{T}_n(z;D,E)\}$ of matrix orthonormal polynomials defined by the recurrence formula

$$z\,\mathrm{T}_n(z) = D\mathrm{T}_{n+1}(z) + E\mathrm{T}_n(z) + D\mathrm{T}_{n-1}(z),\;\; n\geq 2,$$

$$z\,\mathrm{T}_1(z) = D\mathrm{T}_2(z) + E\mathrm{T}_1(z) + \sqrt{2}\,D\mathrm{T}_0(z)$$

with the initial conditions

$$\mathrm{T}_0(z) = I_p, \quad \mathrm{T}_1(z) = (1/\sqrt{2})D^{-1}(z\,I_p - E).$$

Notice that $\{\mathrm{T}_n(z;D,E)\}$ is orthonormal with respect to a positive definite matrix of measures that we denote $V_{D,E}$. They are the matrix analogue of the orthonormal Chebyshev polynomials of the first kind and, in fact, as in the scalar case, the sequence of associated polynomials of the first kind for our sequence $(\mathrm{T}_n(z))$ is $(1/\sqrt{2})\,\mathrm{U}_n(z;D,E)\,D^{-1},\;\; n\geq 1$ (cf. [8]).

If we denote

$$\sigma_n(x) = \frac{1}{np}\sum_{j=1}^{k} m_j\delta(x-x_{n,j}),$$

where $x_{n,j}$ $(j = 1, \cdots, k)$ denotes as in Theorem 3.3 the set of zeros of a sequence (P_n) of matrix orthonormal polynomials with respect to a matrix of measures W, m_k is the multiplicity of $x_{n,k}$, then the Nevai class $\mathrm{M}(D, E)$ has the zero asymptotic behavior.

THEOREM 4.5 ([8]). *Let (P_n) be a sequence of matrix orthonormal polynomials in the Nevai class $\mathrm{M}(D, E)$. Then there exists a positive definite matrix of measures μ such that the sequence of discrete matrices of measures (μ_n)*

$$\mu_n(x) = \frac{1}{np} \sum_{j=1}^{k} \left(\sum_{i=0}^{n-1} P_i(x_{n,k}) \Gamma_{n,k} P_i^*(x_{n,k}) \right) \delta(x - x_{n,k})$$

converges in the $*$-weak topology to μ. Furthermore, σ_n converges to $\tau(\mu)$ in the same topology.

Notice that if D is a Hermitian and nonsingular matrix, then μ can be explicitly given (cf. [8]) by

$$\mu = \frac{1}{p} V_{D,E}.$$

5. Orthogonal matrix polynomials on the unit circle. Let W be a matrix of measures supported on the unit circle $\mathbb{T}$. As in the scalar case the shift operator $\mathbf{H} : \mathbb{C}^{(p,p)}[x] \to \mathbb{C}^{(p,p)}[x]$, $\mathbf{H}[P](x) = xP(x)$ is a unitary operator with respect to the inner product

$$\langle P, Q \rangle_L \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} P(e^{i\theta}) dW(\theta) Q^*(e^{i\theta}). \tag{5.1}$$

Let $(\Phi_n(.; W))$ be a sequence of matrix orthonormal polynomials with respect to (5.1), i.e., $\langle \Phi_n, \Phi_m \rangle_L = \delta_{n,m} I_p$. Notice that $(U_n \Phi_n(.; W))$ is a sequence of matrix polynomials orthonormal with respect to (5.1) if we assume (U_n) is a sequence of unitary matrices. Furthermore, taking into account the polar decomposition for the leading coefficient of (Φ_n), we can assume that such a matrix coefficient is a positive definite matrix, and thus, we can choose this normalization in order to have uniqueness for our sequence of matrix orthonormal polynomials [1, page 333]. For a sake of simplicity we will assume such a condition. On the other hand, in (5.1) we consider the measure $\frac{1}{2\pi} dW(\theta)$ so as to have a probability measure, i.e., $\frac{1}{2\pi} \int_0^{2\pi} dW(\theta) = I_p$.

We also introduce the reversed polynomial $\tilde{P}(z) = z^n P^*(1/\overline{z})$ for every polynomial $P \in \mathbb{C}^{(p,p)}[x]$ with $deg\, P = n$. This means that for $P(z) = \sum_{k=0}^n D_{n,k} z^k$, $\tilde{P}(z) = \sum_{k=0}^n D_{n,n-k}^* z^k$. Reversed polynomials will play an important role in our presentation.

As in the case of the real line (cf. Section 2, Definition 2.1) we can consider the sequence of matrix polynomials (K_n),

$$K_n(x, y; W) = \sum_{j=0}^{n} \Phi_j^*(y; W)\, \Phi_j(x; W),$$

the so called nth kernel polynomial associated with the matrix of measures W. Next, we introduce a sequence $(\Psi_n(.; W))$ of matrix polynomials such that

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi_n^*(e^{i\theta}) dW(\theta) \Psi_m(e^{i\theta}) = \delta_{n,m} I_p.$$

The sequence $(\Psi_n(.; W))$ is said to be a right orthonormal sequence of matrix polynomials with respect to W. Notice that in the real case, right orthonormal polynomials are related with the "left" or standard sequence by the transposed coefficients.

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As an analog of the backward and forward recurrence relations for the scalar case [19], we can deduce two mixed recurrence relations where "left" and "right" matrix orthonormal polynomials are involved.
Let

$$\begin{array}{c}\Phi_n(z) = \sum_{j=0}^{n} A_{n,j} z^j, \textit{and} \\ \Psi_n(z) = \sum_{j=0}^{n} B_{n,j} z^j.\end{array} \tag{5.2}$$

Since

$$\int_0^{2\pi} \tilde{\Psi}_n(z) dW(\theta) \Phi_n^*(z) = \int_0^{2\pi} \Psi_n^*(z) dW(\theta) \tilde{\Phi}_n(z) = \sum_{j,k=0}^{n} B_{n,n-j}^* C_{j-k} A_{n,k}^*, \tag{5.3}$$

where $z = e^{i\theta}$ and $C_l = \frac{1}{2\pi}\int_0^{2\pi} e^{i\,l\theta} dW(\theta)$, then taking into account that the leading coefficients of the polynomials Φ_n and Ψ_n are nonsingular matrices, as well as the orthonormality conditions, we get

$$B_{n,n}^{-1} A_{n,0}^* = B_{n,0}^* A_{n,n}^{-1}.$$

We can introduce the reflection coefficients

$$\mathrm{H}_n = A_{n,n}^{-*} B_{n,0} = A_{n,0} B_{n,n}^{-*}$$

in such a way that

$$\begin{array}{c}(I_p - \mathrm{H}_n^* \mathrm{H}_n)^{1/2} = B_{n,n}^{-1} B_{n-1,n-1} = B_{n-1,n-1}^* B_{n,n}^{-*}, \\ (I_p - \mathrm{H}_n \mathrm{H}_n^*)^{1/2} = A_{n,n}^{-*} A_{n-1,n-1}^* = A_{n-1,n-1} A_{n,n}^{-1}.\end{array}$$

Since we have assumed the leading coefficients are positive definite matrices, we have that $\| \mathrm{H}_n \|_2 < 1$. Thus, the recurrence relations can be written using the matrices H_n (as in the scalar case):

$$\Phi_n(z;W) = (I_p - \mathrm{H}_n \mathrm{H}_n^*)^{\frac{1}{2}}\, z \Phi_{n-1}(z;W) + \mathrm{H}_n \tilde{\Psi}_n(z;W), \tag{5.4a}$$

$$\Psi_n(z;W) = z \Psi_{n-1}(z;W)(I_p - \mathrm{H}_n^* \mathrm{H}_n)^{\frac{1}{2}} + \tilde{\Phi}_n(z;W) \mathrm{H}_n \tag{5.4b}$$

(backward recurrence relation).

¿From (5.4b) we get

$$\tilde{\Psi}_n(z;W) = (I_p - \mathrm{H}_n^* \mathrm{H}_n)^{\frac{1}{2}} \tilde{\Psi}_{n-1}(z;W) + \mathrm{H}_n^* \Phi_n(z;W),$$

and from (5.4a), we deduce that

$$\tilde{\Phi}_n(z;W) = \tilde{\Phi}_{n-1}(z;W)(I_p - \mathrm{H}_n \mathrm{H}_n^*)^{\frac{1}{2}} + \Psi_n(z;W) \mathrm{H}_n^*.$$

Finally, by substitution in (5.4) we obtain the so-called forward recurrence relations

$$(I_p - \mathrm{H}_n \mathrm{H}_n^*)^{\frac{1}{2}} \Phi_n(z;W), = z\, \Phi_{n-1}(z;W) + \mathrm{H}_n \tilde{\Psi}_{n-1}(z;W) \tag{5.5a}$$

$$\Psi_n(z;W)(I_p - \mathrm{H}_n^* \mathrm{H}_n)^{\frac{1}{2}} = z \Psi_{n-1}(z;W) + \tilde{\Phi}_{n-1}(z;W) \mathrm{H}_n. \tag{5.5b}$$

THEOREM 5.1 ((Christoffel-Darboux formula) [3]).

$$(1-\overline{y}z)\sum_{j=0}^{n}\Phi_j^*(y)\Phi_j(z)=\tilde{\Psi}_n^*(y)\tilde{\Psi}_n(z)-\overline{y}z\,\Phi_n^*(y)\Phi_n(z). \tag{5.6}$$

As a consequence, writing $y=z$ in (5.6), we get

$$\mid z\mid^2\ \Phi_n^*(z)\Phi_n(z)=\tilde{\Psi}_n^*(z)\tilde{\Psi}_n(z)+(\mid z\mid^2-1)\sum_{j=0}^{n}\Phi_j^*(z)\Phi_j(z). \tag{5.7}$$

Since the right hand side of (5.7) is a positive definite matrix for $\mid z\mid>1$, the matrix $\Phi_n(z)$ is nonsingular for $\mid z\mid>1$. Assume that $\Phi_n(z_0)$ is a singular matrix for $\mid z_0\mid=1$. Let u be an eigenvector of $\Phi_n(z_0)$. Then, from (5.7) it follows that $\tilde{\Psi}_n(z_0)u=0$. Thus from (5.4a) we get $\Phi_{n-1}(z_0)u=0$. By induction, $\Phi_0(z_0)$ is a singular matrix in contradiction with the non-singularity of the matrix $\Phi_0(z_0)=A_{0,0}$.

COROLLARY 5.2. *The zeros of Φ_n belong to the unit disk.*
In order to compute such zeros, writing $y=0$ in (5.6), we get

$$\tilde{\Psi}_n^*(0)\tilde{\Psi}_n(z)=K_n(z,0).$$

But $\tilde{\Psi}_n^*(0)=B_{n,n}$, and so $\tilde{\Psi}_n(z)=B_{n,n}^{-1}K_n(z,0)$. From (5.5a)

$$z\,\Phi_{n-1}(z)=(I_p-\mathrm{H}_n\mathrm{H}_n^*)^{\frac{1}{2}}\Phi_n(z)-\mathrm{H}_n\,B_{n-1,n-1}^{-1}\sum_{k=0}^{n-1}\Phi_k^*(0)\Phi_k(z).$$

Taking into account $\Phi_j^*(0)=\tilde{\Psi}_j^*(0)\mathrm{H}_j^*=B_{j,j}\mathrm{H}_j^*$, we get

$$\begin{aligned} z\,\Phi_{n-1}(z) &= (I_p-\mathrm{H}_n\mathrm{H}_n^*)^{\frac{1}{2}}\Phi_n(z)-\mathrm{H}_n\,\textstyle\sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1}(I_p-\mathrm{H}_j^*\mathrm{H}_j)^{\frac{1}{2}}\right)\mathrm{H}_k^*\Phi_k(z)\\ &= \textstyle\sum_{k=0}^{n} M_{n-1,k}\Phi_k(z).\end{aligned}$$

Thus, if we denote

$$\mathbb{M}_n=\begin{pmatrix} M_{0,0} & M_{0,1} & & & & 0\\ M_{1,0} & M_{1,1} & M_{1,2} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & & \\ M_{n-2,0} & M_{n-2,1} & \cdots & \cdots & M_{n-2,n-2} & M_{n-2,n-1}\\ M_{n-1,0} & M_{n-1,1} & \cdots & \cdots & M_{n-1,n-2} & M_{n-1,n-1}\end{pmatrix},$$

then

$$z\begin{pmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{pmatrix}=\mathbb{M}_n\begin{pmatrix}\Phi_0(z)\\ \Phi_1(z)\\ \vdots\\ \Phi_{n-1}(z)\end{pmatrix}+\begin{pmatrix}0\\ \vdots\\ 0\\ (I_p-\mathrm{H}_n\mathrm{H}_n^*)^{\frac{1}{2}}\Phi_n(z)\end{pmatrix}.$$

THEOREM 5.3. *For $n \in \mathbb{R}$, the zeros of the matrix polynomials $\Phi_n(z)$ are the eigenvalues of the block Hessenberg matrix*

$$\mathbb{M}_n \in \mathbb{C}^{(np,np)}$$

with the same multiplicity.
Notice that if v is an eigenvalue corresponding to the eigenvalue z_0 of $\mathbb{M}_n$, writing v as a block column vector

$$v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \in \mathbb{C}^{(np,1)},$$

where $v_i \in \mathbb{C}^{(p,1)}$, then the equation $\mathbb{M}_n v = z_0 v$ becomes

$$\begin{array}{rcl} v_1 & = & \Phi_1(z_0)v_0 \\ v_2 & = & \Phi_2(z_0)v_0 \\ \cdots & & \cdots \\ v_{n-1} & = & \Phi_{n-1}(z_0)v_0, \end{array} \qquad\qquad 0 = \Phi_n(z_0)v_0.$$

In other words, there exists a bijection between $\mathcal{N}(z_0, \Phi_n)$ and the subspace of eigenvectors of the matrix $\mathbb{M}_n$ associated with the eigenvalue z_0. From (5.7), if $\mid z \mid= 1$, then $\Phi_n^*(z)\Phi_n(z) = \Psi_n(z)\Psi_n^*(z)$ is nonsingular matrix. As in the scalar case, we will analyze two kinds of quadrature formulas.
Let us now define the weight matrix function $W_n(\theta) = \left[\Phi_n^*(e^{i\,\theta})\,\Phi_n(e^{i\,\theta})\right]^{-1}$ as well as $\Omega_n(\theta) = \int_0^\theta W_n(s)\,ds$. Because the rational matrix function $[\Phi_n^*(z)]^{-1}$ is analytic in the closed disk, one has

THEOREM 5.4 ([3]). *For $0 \le j, k \le n$*

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi_j(e^{i\,\theta})d\Omega_n(\theta)\Phi_k^*(e^{i\,\theta}) = \delta_{j,k}I_p.$$

This means that the sequences of matrix orthonormal polynomials corresponding to the matrices of measures Ω_n and W have the same $n+1$ first elements. Furthermore, if P and Q are polynomials of degree less than or equal to n, we get

$$\frac{1}{2\pi}\int_0^{2\pi} P(e^{i\,\theta})dW(\theta)Q^*(e^{i\,\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(e^{i\,\theta})d\Omega_n(\theta)Q^*(e^{i\,\theta}). \tag{5.8}$$

¿From the above result a straightforward proof of the Favard´s theorem on the unit circle follows ([3]). In fact, given a sequence of matrices (H_n) of dimension p with $\| \mathrm{H}_n \|_2 < 1$, there exists a unique matrix of measures W such that the sequence of matrix polynomials given by (5.5) is orthonormal with respect to W.

Notice that zeros of Φ_n which lie in the unit disk are involved in (5.8). In order to obtain a quadrature formula with knots on the unit circle, we need to introduce the concept of para-orthogonality.

DEFINITION 5.5 ([16], [18]). *Let U_n be a unitary matrix. The matrix polynomial*

$$\mathrm{B}_n(z; U_n) = \Phi_n(z) + U_n\tilde{\Psi}_n(z)$$

is said to be para-orthogonal with respect to the matrix of measures W.

THEOREM 5.6 ([16]).

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1. *The zeros of* $\mathrm{B}_n(z;U_n)$ *are the eigenvalues of a unitary block Hessenberg matrix* $\mathbb{N}_n$ *whose multiplicities are less than or equal to* p.
   The blocks of the matrix $\mathbb{N}_n$ *are the same as those of* $\mathbb{M}_n$ *up to the corresponding to the last row. In fact,*

$$\mathbb{N}_{n-1,k} = \mathbb{M}_{n-1,k} - (I_p - H_n H_n^*)^{1/2}(U_n^* + H_n^*)^{-1} \prod_{j=k+1}^{n} (I_p - H_j^* H_j)^{1/2} H_k^*.$$

2. *If* $(m_i)_{i=1}^k$ *are the multiplicities of the eigenvalues* $(z_i)_{i=1}^k$ *and* $(v_{i,j})_{i=1}^k$ $(j = 1, 2, \cdots, m_i)$ *the corresponding eigenvectors, then*

$$\langle P, Q\rangle_L = \sum_{i=1}^{k} P(z_i)\Lambda_i Q^*(z_i),$$

where P, Q *are Laurent matrix polynomials* $P \in \mathbf{L}_{-s,t},\ \ Q \in \mathbf{L}_{-(n-1-t),(n-1-s)}$ $(\mathbf{L}_{r,s} = \left\{\sum_{k=r}^{s} A_k z^k,\ A_k \in \mathbb{C}^{(p,p)},\ r \leq s\right\})$,

$$\Lambda_i = \left(v_{i,1}^{(0)}, v_{i,2}^{(0)}, \cdots, v_{i,m_i}^{(0)}\right) \mathrm{G}_i^{-1} \begin{pmatrix} v_{i,1}^{(0)^*} \\ \vdots \\ v_{i,m_i}^{(0)^*} \end{pmatrix},$$

$v_{i,s}^{(0)} \in \mathbb{C}^{(p,1)}$ $(s = 1, \cdots, m_i)$ *is the vector consisting of the first* p *component of* $v_{i,s}$, *and* $(\mathrm{G}_i)_{k,l} = v_{i,k}^* v_{i,l}$.

6. Orthogonality in the unit circle from orthogonality in $[-1, 1]$**.** Let W be a matrix of measures supported on $[-1, 1]$. We can introduce a matrix of measures $\tilde{W}$ on the unit circle $\mathbb{T}$ in the following way

$$\tilde{W}(\theta) = \begin{cases} -W(\cos\theta) & 0 \leq \theta \leq \pi \\ \ \ W(\cos\theta) & \pi \leq \theta \leq 2\pi. \end{cases} \tag{6.1}$$

Taking into account the symmetry of the above measure, the matrix coefficients in (5.2) are related in the following way

$$B_{n,j} = A_{n,j}^* \quad j = 0, 1, \cdots, n.$$

In this case, it is easy to prove that the reflection matrix parameters (H_n) associated with the matrix of measures given in (6.1) are Hermitian. The first question to solve is the connection between these reflection parameters and the parameters of some sequence of matrix orthonormal polynomials with respect to W.

PROPOSITION 6.1 ([20]). *Let* (Φ_n) *and* (Ψ_n) *be sequences of left and right matrix orthonormal polynomials on the unit circle with respect to* $\tilde{W}(\theta)$ *with positive definite matrices as leading coefficients. The sequence of matrix polynomials*

$$P_n(x;W) = \frac{1}{\sqrt{2\pi}}(I_p + \mathrm{H}_{2n})^{-1/2}\left[\Phi_{2n}(z;\tilde{W}) + \tilde{\Psi}_{2n}(z;\tilde{W})\right] z^{-n},$$

where $x = \frac{1}{2}(z + \frac{1}{z})$, *is orthonormal with respect to* W. *The sequence of the matrix orthonormal polynomials* (P_n) *satisfies (2.1) with*

$$\begin{aligned} D_n(W) &= \tfrac{1}{2}(I_p + \mathrm{H}_{2n-2})^{\frac{1}{2}}(I_p - \mathrm{H}_{2n-1}^2)^{\frac{1}{2}}(I_p - \mathrm{H}_{2n})^{\frac{1}{2}} \\ E_n(W) &= \tfrac{1}{2}(I_p - \mathrm{H}_{2n})^{\frac{1}{2}}\mathrm{H}_{2n-1}(I_p - \mathrm{H}_{2n})^{\frac{1}{2}} \\ &\quad -\tfrac{1}{2}(I_p + \mathrm{H}_{2n})^{\frac{1}{2}}\mathrm{H}_{2n+1}(I_p + \mathrm{H}_{2n})^{\frac{1}{2}} \end{aligned} \tag{6.2}$$

for $n \geq 1$.
Notice that if $\mathrm{H}_n = \mathrm{H}$ for every n, we get

$$D = D_n = \frac{1}{2}(I_p - \mathrm{H}^2) > 0 \quad \text{and} \quad E = E_n = -\mathrm{H}^2.$$

Thus, the sequence (P_n) is a sequence analyzed in §4, i.e., it belongs to the matrix Nevai class $\mathrm{M}(D, E)$. In such a case, the corresponding matrix of measures is given by

$$dW = \frac{1}{\pi} D^{-1} \left(I_p - \left[(I_p - \mathrm{H}^2)^{-1} (xI_p + \mathrm{H}^2)^2 \right] \right)^{1/2} dx.$$

On the other hand, the support of the matrix of measures W lives in a finite union of at most p disjoint bounded non-degenerate intervals whose end points are some zeros of the scalar polynomial

$$\det \left[(I_p - \mathrm{H}^2) - (xI_p + \mathrm{H}^2) \right]^2 = 0,$$

i.e.,

$$\det \left[I_p + xI_p \right] = 0 \quad \text{or} \quad \det \left[I_p - 2\mathrm{H}^2 - xI_p \right] = 0.$$

This means that the set of ends points is contained in

$$\{-1\} \cup \left\{ 1 - 2\lambda^2, \ \lambda \text{ eigenvalue of } \mathrm{H} \right\}.$$

Since $0 \leq \mathrm{H} < I_p$, the above set is $[-1, 1]$.

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