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PERTURBATION ANALYSIS FOR EIGENSTRUCTURE ASSIGNMENT OF LINEAR MULTI-INPUT SYSTEMS*

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Abstract. The state-feedback pole (or eigenvalue) assignment problem is a fundamental problem in control system design. The term *eigenstructure* denotes the specification of eigenvalues *and* eigenvectors (or certain properties of the latter). Normally, the eigenvectors are calculated as an intermediate solution. In assignment for multi-input systems, the solution (the feedback matrix) is not unique. However, the solution is unique if the eigenvectors are set. Perturbation bounds are given for multi-input eigenstructure assignment of eigenvalues and eigenvectors occurring in complex-conjugate pairs. Numerical results which support the analysis are also provided.

Key words. controllable system, state feedback, eigenstructure assignment, multi-input pole assignment, perturbation analysis.

AMS subject classifications. 15A18, 65F15, 65F35, 93B55.

1. Introduction.

Consider the linear, time-invariant, multi-input control system with dynamic state equation

$$\dot{x} = Ax + Bu. \tag{1.1}$$

Here, $A \in \mathcal{R}^{n\times n}$ is the *state matrix*, $B \in \mathcal{R}^{n\times m}$ is the *input matrix*, $x \in \mathcal{R}^n$ is the *state vector*, $u \in \mathcal{R}^m$ is the *input vector*, and $\cdot$ denotes the differential operator d/dt. In many cases, the control of such dynamical systems is accomplished through the use of linear state feedback. That is, the input vector u is chosen to be a linear function of x, e.g.

$$u = -Kx, \tag{1.2}$$

where $K \in \mathcal{R}^{m\times n}$ is the *feedback gain matrix*. Choosing K so that the *closed-loop system*

$$\dot{x} = (A - BK)x \tag{1.3}$$

has desired characteristics is one of the fundamental topics in linear control theory; see [1]. In particular, in pole assignment, K is determined so that $A - BK$ has a desired set of eigenvalues. Of primary interest in this paper is the multi-input case, i.e. when $1 < m < n$.

The definition of *eigenstructure assignment* varies in the literature. For example, eigenstructure assignment may refer to pole assignment along with setting the eigenvectors; see [13]. A broader meaning is that certain properties of the eigenvector matrix are assigned. For example, in the *robust* eigenstructure assignment problem, the eigenvector matrix is constrained to be as well conditioned as possible; see [10]. For the multi-input problem, the nonuniqueness of the eigenvector matrix can be exploited so that design criteria, such as decoupling, insensitivity of eigenvalues and eigenvectors, and robustness, can be optimized; see [1, 16]. If B is of full rank, then once a nonsingular V has been specified, K is unique; see [10]. Note that K is a real matrix if it is assumed that A and B are real and that complex eigenvalues and eigenvectors occur in conjugate pairs.

The general eigenstructure assignment problem can be defined in the following way.

PROBLEM 1.1. *Given A and B, as in (1.1), a self-conjugate set of eigenvalues $\{\lambda_j\}, j = 1,\ldots,n$, find a self-conjugate set of linearly independent eigenvectors $\{v_j\}, j = 1,\ldots,n$, and $K \in \mathcal{R}^{m\times n}$ such that*

$$(A - BK)V = V\Lambda, \tag{1.4}$$

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where

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n),$$

and where

$$V = [v_1, \ldots, v_n].$$

Necessary conditions for the existence of a solution to PROBLEM 1.1 can be found in [10]. Necessary and sufficient conditions for the existence of a nonsingular V solution to PROBLEM 1.1, for the case where A and B are complex and the eigenvalues are not restricted to occur in complex conjugate pairs, are in [12].

It is assumed that the pair (A, B) is controllable, which is equivalent to the statement, [9],

$$rank[A - \lambda I_n, B] = n, \forall \lambda \in \mathcal{C}. \quad (1.5)$$

As a result of (1.5), and the assumption that $1 < m < n$, it follows that

$$\|A - \lambda_j I\| > 0, \;\; j = 1, \ldots, n. \quad (1.6)$$

Without loss of generality, it is assumed that B is of full rank.

Though there are many solution algorithms, there are fewer references on perturbation analysis of the pole-assignment problem [14]. A perturbation analysis for the single-input case is given in [2]. Perturbation analyses for both single-input and multi-input cases are given in [14], under the assumption that the closed-loop system has no repeated eigenvalues. With the same assumption, perturbation theory for the single-input case is developed in [11]. Note that for the single input case, $A - BK$ is diagonalizable if and only if its eigenvalues are distinct [10]. Characterization of the set of ill-posed problems (A, B), under the assumption that there is no intersection between the set of eigenvalues of A and that of $A - BK$, is considered in [5]. A perturbation analysis for multiple input pole placement can be found in [12], where perturbation results for the feedback gain and the poles of the closed loop system are provided, along with computational results comparing the actual poles with the eigenvalues of the closed loop system formed from (A, B) and the perturbed gain matrix.

The goal of this paper is to present a perturbation analysis of PROBLEM 1.1. The primary result is a perturbation bound for the feedback gain matrix K. An intermediate result, for perturbations in the eigenvector matrix V, is motivated by the dependence of K on V. The bound on perturbations in V applies to all possible eigenvector matrices. These results are independent of the algorithm used for computation. Throughout the paper it is assumed that a nonsingular matrix V of right eigenvectors can be found. Without loss of generality it is assumed that V has been normalized so that each column has unit Euclidean length. Supporting numerical results are provided.

To establish notation and to provide a foundation for the analysis of the next section, some known results are stated here, without proof.

LEMMA 1.1. [10] *Given A, B, Λ, and V, with B full rank, and V nonsingular, there exists a real matrix K, which is the solution to equation* (1.4) *if and only if*

$$U_2^T(AV - V\Lambda) = 0, \quad (1.7)$$

where

$$B = UB^0 := [U_1, U_2] \begin{bmatrix} B_1^0 \\ 0 \end{bmatrix} \tag{1.8}$$

with $U = [U_1, U_2]$ *orthogonal and* B_1^0 *nonsingular. Moreover,* K *is found as the solution of*

$$B_1^0 K V = U_1^T (AV - V\Lambda). \tag{1.9}$$

Without loss of generality it is assumed that B_1^0 is upper triangular with positive diagonal entries.

In the next lemma the condition number $\kappa_2(B)$ is defined as

$$\kappa_2(B) = \|B^+\|_2 \|B\|_2.$$

LEMMA 1.2. [4] *Suppose full rank matrix* $B \in \mathcal{C}^{n\times m}$, $m \le n$ *has* QR *factorization* $B = Q_1 R$ *with* $Q_1 \in \mathcal{C}^{n\times m}$, $R \in \mathcal{C}^{m\times m}$. *If* $\Delta B \in \mathcal{C}^{n\times m}$ *satisfies*

$$\kappa_2(B) \frac{\|Q_1^T \Delta B\|_2}{\|B\|_2} < 1, \tag{1.10}$$

then there is a unique QR *factorization*

$$B + \Delta B = (Q_1 + \Delta Q_1)(R + \Delta R)$$

with

$$\|\Delta Q_1\|_F \le \sqrt{2}\kappa_2(B) \frac{\|\Delta B\|_F}{\|B\|_2} + O(\epsilon^2), \tag{1.11}$$

and

$$\frac{\|\Delta R\|_F}{\|R\|_2} \le \sqrt{2}\kappa_2(B) \frac{\|\Delta B\|_F}{\|B\|_2} + O(\epsilon^2), \tag{1.12}$$

where $\|\Delta B\|_F \equiv \epsilon \|B\|_2$.

LEMMA 1.3. [15] *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be* k*-dimensional subspaces of* $\mathcal{C}^n$ *with* $\mathcal{S}$, $\mathcal{S}^\perp$, $\mathcal{T}$, $\mathcal{T}^\perp$ *having orthonormal bases* $\{p_1, \ldots, p_k\}$, $\{p_{k+1}, \ldots, p_n\}$, $\{q_1, \ldots, q_k\}$ *and* $\{q_{k+1}, \ldots, q_n\}$, *respectively. Defining matrices*

$$P_1 = [p_1, \ldots, p_k], \quad P_2 = [p_{k+1}, \ldots, p_n], \quad Q_1 = [q_1, \ldots, q_k], \quad Q_2 = [q_{k+1}, \ldots, q_n],$$

the distance between subspaces $\mathcal{S}$ *and* $\mathcal{T}$, $d(\mathcal{S}, \mathcal{T})$ *satisfies*

$$d(\mathcal{S}, \mathcal{T}) = \|Q_2^* P_1\|_2 = \|Q_1^* P_2\|_2 = \|P_1^* Q_2\|_2 = \|P_2^* Q_1\|_2. \tag{1.13}$$

LEMMA 1.4. [3] *For* $F \in \mathcal{R}^{n\times n}$ *with* $\|F\| < 1$, *using an operator matrix norm,*

$$\|(I - F)^{-1}\| \le \frac{1}{1 - \|F\|}.$$

The Kronecker product of $A \in \mathcal{C}^{m\times n}$ and $B \in \mathcal{C}^{p\times q}$ is the matrix $D \in \mathcal{C}^{mp\times nq}$ defined as

$$D = A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ & \ddots & \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

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LEMMA 1.5. [8] *For $A \in \mathcal{C}^{m\times n}$, $X \in \mathcal{C}^{n\times p}$, $B \in \mathcal{C}^{p\times q}$ and $C \in \mathcal{C}^{m\times q}$, the equation*

$$AXB = C$$

is equivalent to

$$(B^T \otimes A)\text{vec}(X) = \text{vec}(C),$$

where

$$\text{vec}(X) = [x_{11} \dots x_{n1}\; x_{12} \dots x_{n2} \dots\; \dots x_{1n} \dots x_{nn}]^T.$$

2. Analysis. Let

$$\widetilde{A} = A + \Delta A$$
$$\widetilde{B} = B + \Delta B$$
$$\widetilde{\lambda}_j = \lambda_j + \delta\lambda_j,$$

where

$$\|\Delta B\|_F \le \epsilon \|B\|_2$$
$$(2.1) \qquad \|\Delta A - \delta\lambda_j I\|_F \le \epsilon \|A - \lambda_j I\|_2,\ j = 1, \dots, n.$$

The latter inequality is motivated by (1.6) and subsequent analysis. Separate bounds on ΔA and $\delta\lambda_j$ would lead to results which are similar to, yet more complicated than, the results in this paper.

The goal is to analyze the difference between solution K of (1.4), and $\widetilde{K}$, the solution of the perturbed problem

$$(2.2) \qquad (\widetilde{A} - \widetilde{B}\widetilde{K})\widetilde{V} = \widetilde{V}\widetilde{\Lambda},$$

where

$$\widetilde{\Lambda} = \text{diag}(\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_n),$$

and $\widetilde{V}$ satisfies a perturbed variation of (1.7). A perturbation result for V will also be derived.

The first step is to bound perturbations to B_1^0 and U_2 defined in (1.8).

LEMMA 2.1. *Suppose that B, U_1, U_2, and B_1^0 satisfy* (1.8). *Define perturbation matrices ΔU_1, ΔU_2, and ΔB_1^0 so that $[U_1 + \Delta U_1, U_2 + \Delta U_2]$ is orthogonal, $(B_1^0 + \Delta B_1^0)$ is upper triangular with positive diagonal entries, of full rank, and*

$$B + \Delta B = [U_1 + \Delta U_1, U_2 + \Delta U_2] \begin{bmatrix} B_1^0 + \Delta B_1^0 \\ 0 \end{bmatrix}.$$

Then

$$(2.3) \qquad \|\Delta U_2\|_2 \le \frac{\|\Delta U_1\|_2}{1 - \|\Delta U_1\|_2}.$$

Also, if

$$\epsilon\sqrt{2}\kappa_2(B) < 1, \tag{2.4}$$

then

$$\|\Delta U_2\|_2 \leq \frac{\epsilon\sqrt{2}\kappa_2(B)}{1-\epsilon\sqrt{2}\kappa_2(B)} + O(\epsilon^2), \tag{2.5}$$

and

$$\|\Delta B_1^0\|_F \leq \epsilon\sqrt{2}\|B\|_2\kappa_2(B) + O(\epsilon^2). \tag{2.6}$$

Proof. The equality

$$(U_1+\Delta U_1)^T\left(U_2 - (U_1\Delta U_1^T)(I+U_1\Delta U_1^T)^{-1}U_2\right) = 0 \tag{2.7}$$

motivates the definition

$$\Delta U_2 := -(U_1\Delta U_1^T)(I+U_1\Delta U_1^T)^{-1}U_2$$

so that using Lemma 1.4,

$$\|\Delta U_2\|_2 \leq \|U_1\Delta U_1^T\|_2\|(I+U_1\Delta U_1^T)^{-1}\|_2 \leq \frac{\|\Delta U_1^T\|_2}{1-\|\Delta U_1^T\|_2}, \tag{2.8}$$

and therefore (2.3) is proven. It can be shown that (1.10) follows from (2.4), with Q_1 replaced by U_1, as follows:

$$\begin{aligned} 1 > \epsilon\sqrt{2}\kappa_2(B) > \epsilon\kappa_2(B) &\geq \frac{\|\Delta B\|_F}{\|B\|_2}\kappa_2(B) \\ &\geq \frac{\|\Delta B\|_2}{\|B\|_2}\kappa_2(B) \geq \frac{\|U_1^T\Delta B\|_2}{\|B\|_2}\kappa_2(B). \end{aligned}$$

Using (1.11), it follows that

$$\begin{aligned} \|\Delta U_1\|_2 \leq \|\Delta U_1\|_F &\leq \sqrt{2}\kappa_2(B)\frac{\|\Delta B\|_F}{\|B\|_2} + O(\epsilon^2) \\ &= \sqrt{2}\|B^+\|_2\|\Delta B\|_F + O(\epsilon^2) \leq \epsilon\sqrt{2}\kappa_2(B) + O(\epsilon^2). \end{aligned}$$

Combining this with (2.8) proves (2.5).

Using (1.12), it follows that

$$\frac{\|\Delta B_1^0\|_F}{\|B_1^0\|_2} \leq \sqrt{2}\kappa_2(B)\frac{\|\Delta B\|_F}{\|B\|_2} + O(\epsilon^2) \leq \sqrt{2}\kappa_2(B)\frac{\epsilon\|B\|_2}{\|B\|_2} + O(\epsilon^2). \tag{2.9}$$

and (2.6) follows. ◻

The next two lemmas lead to a theorem for a bound on the perturbation to V.

LEMMA 2.2. *Suppose V and U_2 satisfy (1.7) and ΔV is defined so that the columns of $\widetilde{V} = V + \Delta V$ satisfy*

$$[U_2+\Delta U_2]^T(\widetilde{A}-\widetilde{\lambda}_j I)\widetilde{v}_j = 0,\ j = 1,\ldots,n. \tag{2.10}$$

Define M_j *as* $M_j := U_2^T(A - \lambda_j I)$ *and let* ΔM_j *be defined so that* (2.10) *may be written*

$$(M_j + \Delta M_j)\widetilde{v}_j = 0. \tag{2.11}$$

If $\epsilon\sqrt{2}\kappa_2(B) < 1$, *then*

$$\|\Delta M_j\|_F \le \frac{\epsilon(1+\sqrt{2})\kappa_2(B)\|A - \lambda_j I\|_F}{1 - \epsilon\sqrt{2}\kappa_2(B)} + O(\epsilon^2). \tag{2.12}$$

Proof. Equation (2.10) can be written in the form of (2.11) by defining ΔM_j as

$$\Delta M_j = U_2^T(\Delta A - \delta\lambda_j I) + \Delta U_2^T(A - \lambda_j I) + \Delta U_2^T(\Delta A - \delta\lambda_j I) \tag{2.13}$$

so that, using (2.5),

$$\begin{aligned}\|\Delta M_j\|_F &\le \|U_2^T(\Delta A - \delta\lambda_j I)\|_F + \|\Delta U_2^T(A - \lambda_j I)\|_F + \|\Delta U_2^T(\Delta A - \delta\lambda_j I)\|_F \\ &\le \|\Delta A - \delta\lambda_j I\|_F + \|\Delta U_2^T\|_2\|A - \lambda_j I\|_F + \mathcal{O}(\epsilon^2) \\ &\le \epsilon\|A - \lambda_j I\|_F + \frac{\epsilon\sqrt{2}\kappa_2(B)}{1 - \epsilon\sqrt{2}\kappa_2(B)}\|A - \lambda_j I\|_F + \mathcal{O}(\epsilon^2),\end{aligned}$$

and the result follows noting that $\kappa_2(B) \ge 1$. □

Let $\widetilde{M}_j = M_j + \Delta M_j$. Introduce matrices W_j, V_j, R_j and corresponding perturbed forms $\widetilde{W}_j$, $\widetilde{V}_j$, and $\widetilde{R}_j$ so that $M_j^* = [W_j, V_j]R_j$, and $\widetilde{M}_j^* = [\widetilde{W}_j, \widetilde{V}_j]\widetilde{R}_j$, where $[W_j, V_j]$ and $[\widetilde{W}_j, \widetilde{V}_j] \in \mathcal{C}^{n\times n}$ are unitary, and R_j and $\widetilde{R}_j \in \mathcal{C}^{n\times(n-m)}$ are full-rank and upper triangular with positive diagonal elements.

To simplify notation, define the quantity

$$c_{F2}(M_j) = \|A - \lambda_j I\|_F\|M_j^+\|_2. \tag{2.14}$$

Note that

$$c_{F2}(M_j) \ge \kappa_2(M_j) \ge 1.$$

Then the following lemma holds.

LEMMA 2.3. *Let* $\widetilde{V}_j = V_j + \Delta V_j$. *If*

$$\epsilon(2 + 2\sqrt{2})\kappa_2(B)c_{F2}(M_j) < 1, \tag{2.15}$$

then

$$\|\Delta V_j\|_2 \le \frac{\epsilon(2+\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1 - \epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)} + O(\epsilon^2). \tag{2.16}$$

Proof. From (1.11) and (2.12), and with $\widetilde{W}_j = W_j + \Delta W_j$,

$$\begin{aligned}\|\Delta W_j\|_2 \le \|\Delta W_j\|_F &\le \sqrt{2}\kappa_2(M_j)\frac{\|\Delta M_j\|_F}{\|M_j\|_2} + O(\epsilon^2) \\ &= \sqrt{2}\|M_j^+\|_2\|\Delta M_j\|_F + O(\epsilon^2) \\ &\le \frac{\epsilon\sqrt{2}\|M_j^+\|_2(1+\sqrt{2})\kappa_2(B)\|A - \lambda_j I\|_F}{1 - \epsilon\sqrt{2}\kappa_2(B)} + O(\epsilon^2) \\ &= \frac{\epsilon(2+\sqrt{2})c_{F2}(M_j)\kappa_2(B)}{1 - \epsilon\sqrt{2}\kappa_2(B)} + O(\epsilon^2).\end{aligned}$$

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Using (2.3) with M_j^* in place of B, V_j in place of U_2 and W_j in place of U_1,

$$\begin{aligned}\|\Delta V_j\|_2 &\le \frac{\|\Delta W_j\|_2}{1-\|\Delta W_j\|_2}\\ &\le \frac{\frac{\epsilon(2+\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon\sqrt{2}\kappa_2(B)}}{1-\frac{\epsilon(2+\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon\sqrt{2}\kappa_2(B)}}+O(\epsilon^2),\end{aligned}$$

and the result follows. □

By definition, V_j is a matrix of orthonormal columns such that $M_jV_j = 0$, and $\widetilde{V}_j$ is a matrix of orthonormal columns such that $\widetilde{M}_j\widetilde{V}_j = 0$. The following theorem uses Lemma 2.3 to derive an upper bound on $\|\Delta V\|_F$.

THEOREM 2.4. *Suppose* (2.15) *holds for* $j = 1,\ldots,n$. *Then*

$$\|\Delta V\|_F \le \epsilon(2+\sqrt{2})\kappa_2(B)\sqrt{\sum_{j=1}^{n}(c_{F2}(M_j))^2}+O(\epsilon^2). \tag{2.17}$$

Proof. Let $\mathcal{V}_j = \text{range}(V_j)$ and $\widetilde{\mathcal{V}}_j = \text{range}(\widetilde{V}_j)$. Note that $\mathcal{V}_j$ and $\widetilde{\mathcal{V}}_j$ are subspaces of $\mathcal{C}^n$, and by (1.13), $d(\mathcal{V}_j,\widetilde{\mathcal{V}}_j) = \|\widetilde{V}_j^* W_j\|_2$.

It follows that

$$\begin{aligned}d(\mathcal{V}_j,\widetilde{\mathcal{V}}_j) &= \|\widetilde{V}_j^*W_j\|_2 = \|(V_j+\Delta V_j)^*W_j\|_2 = \|\Delta V_j^*W_j\|_2 \le \|\Delta V_j^*\|_2\\ &\le \frac{\epsilon(2+\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}+O(\epsilon^2).\end{aligned}$$

For $v_j \in \mathcal{V}_j$, there exists a $\widetilde{v}_j \in \widetilde{\mathcal{V}}_j$ such that

$$\|v_j-\widetilde{v}_j\|_2 \le \frac{\epsilon(2+\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}+O(\epsilon^2).$$

Therefore, there exists

$$\begin{aligned}\widetilde{V} &= [\widetilde{v}_1,\ldots,\widetilde{v}_n]\\ &= [v_1,\ldots,v_n]+[\delta v_1,\ldots,\delta v_n]\\ &= V+\Delta V,\end{aligned}$$

with ΔV satisfying

$$\|\Delta V\|_F^2 = \sum_{j=1}^{n}\|\delta v_j\|_2^2 \le \sum_{j=1}^{n}\left(\frac{\epsilon(2+\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}+O(\epsilon^2)\right)^2.$$

Therefore,

$$\|\Delta V\|_F \le \epsilon(2+\sqrt{2})\kappa_2(B)\sqrt{\sum_{j=1}^{n}\left(\frac{c_{F2}(M_j)}{1-\epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}\right)^2}+O(\epsilon^2), \tag{2.18}$$

and the result follows. □

LEMMA 2.5. *Suppose* (2.15) *holds for* $j = 1, \dots, n$. *Then, there exists a solution* $\widetilde{K}$ *to the perturbed problem* (2.2) *which satisfies the equation*

$$(\Phi + \Delta\Phi)\text{vec}(\widetilde{K}) = s + \delta s, \tag{2.19}$$

where

$$\Phi = V^T \otimes B_1^0,$$

$$s = \text{vec}\left(U_1^T[AV - V\Lambda]\right),$$

$$\|\Delta\Phi\|_2 \le \epsilon(2 + 2\sqrt{2})\|V\|_2\|B\|_2\kappa_2(B)\sqrt{\sum_{j=1}^{n}(c_{F2}(M_j))^2} + O(\epsilon^2), \tag{2.20}$$

and

$$\|\delta s\|_2 \le \epsilon(3 + 2\sqrt{2})\kappa_2(B)\sqrt{\sum_{j=1}^{n}[\|A - \lambda_j I\|_2 c_{F2}(M_j)]^2} + O(\epsilon^2). \tag{2.21}$$

Proof. From Lemmas 1.1 and 1.5, K satisfies

$$\Phi\ \text{vec}(K) = s. \tag{2.22}$$

Suppose $\widetilde{K}$ solves the perturbed problem

$$(V + \Delta V)^T \otimes (B_1^0 + \Delta B_1^0)\text{vec}(\widetilde{K}) = \text{vec}\left((U_1 + \Delta U_1)^T[(A + \Delta A)(V + \Delta V) - (V + \Delta V)(\Lambda + \Delta\Lambda)]\right).$$

Expanding both sides of this equation and comparing with (2.19) motivates the definitions

$$\Delta\Phi := (\Delta V)^T \otimes B_1^0 + V^T \otimes (\Delta B_1^0)^T + (\Delta V)^T \otimes (\Delta B_1^0)^T \tag{2.23}$$

and

$$\delta s_j := U_1^T[A - \lambda_j I]\delta v_j + U_1^T[\Delta A - \delta\lambda_j I]v_j + \Delta U_1^T[A - \lambda_j I]v_j + \mathcal{O}(\epsilon^2). \tag{2.24}$$

Recall that the columns of V are normalized so that $\|v_j\|_2 = 1$, $j = 1, \cdots, n$. From (2.6), (2.17) and the inequality

$$\|V\|_2 \ge \frac{\|V\|_F}{\sqrt{n}} = 1,$$

it follows that

$$\begin{aligned}\|\Delta\Phi\|_2 &\le \|V^T \otimes \Delta B_1^0\|_2 + \|\Delta V \otimes B_1^0\|_2 + \mathcal{O}(\epsilon^2)\\ &\le \epsilon\sqrt{2}\|V\|_2\|B\|_2\kappa_2(B) + \epsilon(2 + \sqrt{2})\|B\|_2\kappa_2(B)\sqrt{\sum_{j=1}^{n}(c_{F2}(M_j))^2} + O(\epsilon^2)\\ &\le \epsilon(2 + 2\sqrt{2})\|V\|_2\|B\|_2\kappa_2(B)\sqrt{\sum_{j=1}^{n}(c_{F2}(M_j))^2} + O(\epsilon^2),\end{aligned}$$

and the result (2.20) follows.

Then also

$$\begin{aligned}\|\delta s_j\|_2 &= \|U_1^T[A-\lambda_j I]\delta v_j + U_1^T[\Delta A - \delta\lambda_j I]v_j + \Delta U_1^T[A-\lambda_j I]v_j\|_2 + \mathcal{O}(\epsilon^2)\\ &\le \epsilon\|A-\lambda_j I\|_2\left(\frac{(2+\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)} + 1 + \sqrt{2}\kappa_2(B)\right)\\ &\le \|A-\lambda_j I\|_2\frac{\epsilon(3+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}.\end{aligned}$$

Therefore,

$$\|\delta s\|_2^2 \le \epsilon^2\sum_{j=1}^n\left[\|A-\lambda_j I\|_2^2\left(\frac{(3+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}{1-\epsilon(2+2\sqrt{2})\kappa_2(B)c_{F2}(M_j)}\right)^2\right], \tag{2.25}$$

and the result (2.21) follows. □

THEOREM 2.6. *Assume that*

$$\epsilon(2+2\sqrt{2})\kappa_2(B)\|\Phi^{-1}\|_2\|V\|_2\|B\|_2\sqrt{\sum_{j=1}^n(c_{F2}(M_j))^2} < 1,$$

so that inequality (2.4) is satisfied and

$$\|\Phi^{-1}\|_2\|\Delta\Phi\|_2 < 1.$$

Then there exists a solution $\widetilde{K}$ to the perturbed problem (2.2) which satisfies

$$\begin{aligned}\|K-\widetilde{K}\|_F \le \epsilon\|\Phi^{-1}\|_2\kappa_2(B)\Bigg((3+2\sqrt{2})\sqrt{\sum_{j=1}^n[\|A-\lambda_j I\|_2 c_{F2}(M_j)]^2}\\ +(2+2\sqrt{2})\|V\|_2\|B\|_2\|K\|_F\sqrt{\sum_{j=1}^n(c_{F2}(M_j))^2}\Bigg) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{2.26}$$

Proof. Combining (2.19) and (2.22) results in the equation

$$\Phi(\mathrm{vec}(K)-\mathrm{vec}(\widetilde{K})) = -\delta s + \Delta\Phi\mathrm{vec}(K) + \Delta\Phi(\mathrm{vec}(\widetilde{K})-\mathrm{vec}(K)).$$

It follows that

$$\|K-\widetilde{K}\|_F \le \|\Phi^{-1}\|_2\|\delta s\|_2 + \|\Phi^{-1}\|_2\|\Delta\Phi\|_2\|K\|_F + \|\Phi^{-1}\|_2\|\Delta\Phi\|_2\|K-\widetilde{K}\|_F,$$

so that

$$\|K-\widetilde{K}\|_F \le \frac{1}{1-\|\Phi^{-1}\|_2\|\Delta\Phi\|_2}(\|\Phi^{-1}\|_2\|\delta s\|_2 + \|\Phi^{-1}\|_2\|\Delta\Phi\|_2\|K\|_F).$$

Therefore, from Lemma 2.5,

$$\|K-\widetilde{K}\|_F \le \Bigg(\epsilon(3+2\sqrt{2})\|\Phi^{-1}\|_2\kappa_2(B)\sqrt{\sum_{j=1}^n[\|A-\lambda_j I\|_2 c_{F2}(M_j)]^2}$$

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$$+\epsilon(2+2\sqrt{2})\kappa_2(B)\|\Phi^{-1}\|_2\|V\|_2\|B\|_2\|K\|_F\sqrt{\sum_{j=1}^{n}(c_{F2}(M_j))^2}\Bigg) \div$$
$$\left(1-\epsilon(2+2\sqrt{2})\kappa_2(B)\|\Phi^{-1}\|_2\|V\|_2\|B\|_2\sqrt{\sum_{j=1}^{n}(c_{F2}(M_j))^2}\right)+\mathcal{O}(\epsilon^2),$$

and the result follows. □

A difference between (2.26) and the analogous result in [12] is that $\|K\|_F$ appears on the right hand side of (2.26) whereas the right hand side of the result in [12] contains $\|\widetilde{K}\|_2$. Further comparisons between these two results appear in the numerical results found in §4.

3. Computational Effort. The matrix V in Theorems 2.4 and 2.6 is assumed to be nonsingular and to satisfy (1.7). The bulk of the effort in computing V is the construction of V_j for each distinct λ_j. Using real arithmetic and Householder reflections, $O(n^3)$ operations are needed to calculate each V_j. Assuming n distinct eigenvalues, therefore, the computation of $V_1,\ldots,V_n$ requires $O(n^4)$ operations, and this effort is irrespective of the method used to construct V once the eigenspaces have been identified. Given $V_1,\ldots,V_n$, the algorithm in [10], for example, constructs V using $O(n^3)$ + $O(n^2m)$.

The bounds (2.17) and (2.26) involve the calculation of two-norms, and therefore singular values, of n different matrices. The operation count appearing subsequently assumes that the Golub-Reinsch SVD algorithm is used to compute the singular values of a given matrix [6]. For the bound (2.17), calculating $\kappa_2(B)$ and the norms $\|M_j^+\|_2$, $j=1,2,\ldots,n$, requires approximately $(4nm^2-4m^3/3)$ and $(4n^2m^2-4m^4/3)$ flops, respectively. The calculation of $\|A-\lambda_j I\|_F$, $j=1,\ldots,n$ requires $5n^2-4n+2$ flops. Therefore, neglecting lower-order terms, approximately $(4n^2m^2-4m^4/3)$ flops are required to compute the bound (2.17).

For the bound (2.26), since $\|\Phi^{-1}\|_2=\|B^+\|_2\|V^{-1}\|_2$, $\|\Phi^{-1}\|_2$ may be computed more efficiently using the product $\|B^+\|_2\|V^{-1}\|_2$. Calculating $\|B^+\|_2$ and $\kappa_2(B)$ requires $(4nm^2-4m^3)$ flops and calculating $\|V^{-1}\|_2$ requires $8n^3/3$ flops. To calculate $\|A-\lambda_j I\|_2$, $8/3n^3$ flops are required for each λ_j. Therefore, $8n^4/3$ flops are required to calculate $\|A-\lambda_j I\|_2$, $j=1,\ldots,n$. Additionally, $(4n^2m^2-4m^4/3)$ flops are required to compute $\|M_j^+\|_2$, $j=1,2,\ldots,n$. Moreover, $(2nm-1)$ flops are required to calculate $\|K\|_F$. Therefore, neglecting lower order terms, approximately $8n^4/3+4n^2m^2-4m^4/3$ are required to calculate the bound (2.26). By way of comparison, since the bound in [12] requires the calculation of $\sigma_n([A-\lambda_j I\, B])$ for $j=1,2,\ldots,n$, approximately $8n^4/3+4n^3m$ floating point operations are necessary. Therefore, the effort required to compute the bound (2.26) is comparable to that in [12].

4. Numerical Results. In order to compare analytical results with computational results, for each example an ensemble of randomly-generated perturbations, satisfying (2.1), was calculated. Out of each ensemble, the data which resulted in the largest perturbation in the result (denoted with a superscript '1') and the data which resulted in the smallest perturbation in the result (denoted with a superscript '2') are presented. In Examples 4.1 and 4.3, computed results are compared with (2.17). For these two examples, given a random perturbation, each matrix V^1 was computed so that $\|\widetilde{V}^1-V^1\|_F$ was maximized, where $\widetilde{V}^1$ was nearest to V^1 in a least norm sense [15]. That is, for $j=1,\ldots,n$, v_j and $\widetilde{v}_j$ were computed so that

$$\|\widetilde{v}_j-v_j\|_2=\max_{\substack{v_j\in\mathcal{V}_j\\ \|v_j\|_2=1}}\ \min_{\widetilde{v}_j\in\widetilde{\mathcal{V}}_j}\|\widetilde{v}_j-v_j\|_2.$$

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This calculation is consistent with the definition of the distance between subspaces in [15] and was performed using the singular value decomposition [15]. In a similar manner, V^2 and $\widetilde{V}^2$ were calculated so that $\|\widetilde{V}^2 - V^2\|_F$ was minimized.

In Examples 4.2 and 4.4, computed results are compared with (2.26). For these two examples, the algorithm in [10] was used to compute V in order to minimize $\kappa_2(V)$. Other methods for computing V can be found in [16]. Given V and perturbations $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{\Lambda}$, of A, B, and Λ, respectively, $\widetilde{V}$ was calculated using the singular value decomposition so that $\|\widetilde{V} - V\|_F$ was minimized. Once $\widetilde{V}$ was computed, $\widetilde{K}$ was calculated as the solution to the linear system (2.19) so that $(\widetilde{A} - \widetilde{B}\widetilde{K})\widetilde{V} = \widetilde{V}\widetilde{\Lambda}$. The norm of the difference between K and $\widetilde{K}$, that is, $\|\widetilde{K} - K\|_F$, is compared to both the upper bound given by (2.26) and the upper bound derived for the multi-input pole placement problem in [12]. Since the bound in [12] is a generalization of the bound derived in [14], these numerical results also serve to compare the upper bound (2.26) with the result in [14]. It should be noted that although the upper bound in [12] is for $\|\widetilde{K} - K\|_2$, the inequality $\|\cdot\|_F \leq \sqrt{\text{rank}(K)}\, \|\cdot\|_2$ allows this upper bound to be used for $\|\widetilde{K} - K\|_F$ [7]. All computations were performed on an Intel Celeron-based PC running Windows 98 using MATLAB version 5.3.

Example 4.1. Consider the system (A, B) where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

along with the eigenvalue matrix $\Lambda = \text{diag}(-1, -1, -1.5)$. For the maximum perturbation, the computed matrix of eigenvectors is

$$V^1 = \begin{bmatrix} 0.49463199537210 & -0.59937029244746 & -0.70725907677428 \\ 0.71461764483424 & -0.53057563557225 & -0.52675102469606 \\ -0.49463199537210 & 0.59937029244746 & 0.47150605118285 \end{bmatrix}.$$

Define perturbations satisfying (2.1) as

$$\Delta A_1 = \epsilon \begin{bmatrix} -0.21564477662453 & 0.28295205139480 & 0.49270485964721 \\ -0.05357721271427 & -0.06769241288139 & -0.20677488775496 \\ -0.50459970263128 & -0.53800232918814 & 0.19079147624726 \end{bmatrix},$$

$$\Delta B_1 = \epsilon \begin{bmatrix} -0.36959236540992 & -0.01929590375419 \\ -0.24961697602371 & 0.34258048096381 \\ 0.72676019179571 & 0.39392734671949 \end{bmatrix},$$

$$\Delta \Lambda_1 = \epsilon\, \text{diag}(0.13009539513388, -0.68669398496239, -0.69870789437494).$$

For the minimum perturbation, the computed matrix of right eigenvectors is

$$V^2 = \begin{bmatrix} 0.70634080108141 & -0.60492174830267 & -0.81949401635109 \\ 0.04653327256211 & 0.51782174236013 & 0.17307167530692 \\ -0.70634080108141 & 0.60492174830267 & 0.54632934423406 \end{bmatrix}.$$

Define perturbations satisfying (2.1) as

$$\Delta A_2 = \epsilon \begin{bmatrix} 0.55315682618267 & 0.02154667012989 & 0.13275833578443 \\ -0.22424698870527 & 0.20277685783903 & 0.61139214347771 \\ -0.04972271936733 & -0.12336744276203 & -0.43935304375176 \end{bmatrix},$$

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$$\Delta B_2 = \epsilon \begin{bmatrix} 0.26919917686954 & 0.03609568137167 \\ -0.24717566122336 & 0.52105598100158 \\ 0.06601086832521 & -0.76764335949926 \end{bmatrix},$$

$$\Delta\Lambda_2 = \epsilon\ \mathrm{diag}(-0.36126442754700, 0.22876621756179, 0.11072784164490)$$

In order to compare resulting perturbations of V with the upper bound in (2.17), define Δ_V as

$$\Delta_V = \epsilon(2+\sqrt{2})\kappa_2(B)\sqrt{\sum_{j=1}^{n}(c_{F2}(M_j))^2}.$$

Table 4.1 displays results for a range of ϵ-values. Note that the estimate given by (2.17) is of the same order of magnitude as the computed value of $\|\widetilde{V}^1 - V^1\|_F$.

TABLE 4.1
Results for Δ_V, Example 4.1

ϵ	1.0e-02	1.0e-04	1.0e-06	1.0e-08	1.0e-10	1.0e-12
Δ_V	8.6e-02	8.6e-04	8.6e-06	8.6e-08	8.6e-10	8.6e-12
$\|\widetilde{V}^1 - V^1\|_F$	2.0e-02	2.0e-04	2.0e-06	2.0e-08	2.0e-10	1.9e-12
$\|\widetilde{V}^2 - V^2\|_F$	6.8e-15	2.4e-07	2.5e-09	2.5e-11	2.5e-13	3.5e-15

Example 4.2. With A and B as above, let

$$K = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Lambda = \mathrm{diag}(-1,-1,1)$$

Define perturbations satisfying (2.1) as

$$\Delta A_1 = \epsilon \begin{bmatrix} -0.02646755126766 & -0.37133013122124 & -0.32329657116399 \\ -0.28959827450764 & -0.18364471536169 & -0.12370435133218 \\ 0.03853719691009 & 0.32394645383195 & 0.39695213859877 \end{bmatrix},$$

$$\Delta B_1 = \epsilon \begin{bmatrix} -0.11023089696840 & 0.40216789149869 \\ -0.34024244779514 & -0.23571317194120 \\ 0.58689170212728 & -0.55708405440022 \end{bmatrix},$$

$$\Delta\Lambda_1 = \epsilon\ \mathrm{diag}(-0.56095601039052, 0.01233336655997, -0.71310822652332),$$

$$\Delta A_2 = \epsilon \begin{bmatrix} -0.36549298982614 & 0.32574118686489 & -0.14209090090193 \\ 0.12273704460413 & -0.34080917254253 & -0.00605430177760 \\ -0.28847482180794 & 0.40285602889843 & 0.05792323150546 \end{bmatrix},$$

$$\Delta B_2 = \epsilon \begin{bmatrix} 0.30944065088079 & -0.30483942789852 \\ 0.11549631486344 & 0.61838896225908 \\ 0.64435570671299 & 0.01951467987861 \end{bmatrix},$$

and

$$\Delta\Lambda_2 = \epsilon \text{ diag}(0.12907761526293, -0.59260614193590, 0.06863445367600).$$

In order to compare resulting perturbations of K with the upper bound in (2.26), define $(\Delta_K)_{CC}$ as

$$(\Delta_K)_{CC} = \epsilon \|\Phi^{-1}\|_2 \kappa_2(B) \left((3+2\sqrt{2}) \sqrt{\sum_{j=1}^{n} [\|A - \lambda_j I\|_2 c_{F2}(M_j)]^2} \right. \\ \left. + (2+2\sqrt{2}) \|V\|_2 \|B\|_2 \|K\|_F \sqrt{\sum_{j=1}^{n} (c_{F2}(M_j))^2} \right). \tag{4.1}$$

Note that this bound serves for both perturbations. Furthermore, define $(\Delta_{K^1})_{MX}$ and $(\Delta_{K^2})_{MX}$ as upper bounds for $\|\widetilde{K}^1 - K\|_F$ and $\|\widetilde{K}^2 - K\|_F$, respectively, in [12]. Table 4.2 displays results for a range of ϵ-values.

TABLE 4.2
Results for $(\Delta_K)_{CC}$, Example 4.2

ϵ	1.0e-02	1.0e-04	1.0e-06	1.0e-08	1.0e-10	1.0e-12
$(\Delta_K)_{CC}$	2.82e-01	2.82e-03	2.82e-05	2.82e-07	2.82e-09	2.82e-11
$(\Delta_{K^1})_{MX}$	1.14e-01	1.13e-03	1.13e-05	1.13e-07	1.13e-09	1.13e-11
$\|\widetilde{K}^1 - K\|_F$	2.84e-02	2.84e-04	2.84e-06	2.84e-08	2.84e-10	2.84e-12
$(\Delta_{K^2})_{MX}$	1.01e-01	1.01e-03	1.01e-05	1.01e-07	1.01e-09	1.01e-11
$\|\widetilde{K}^2 - K\|_F$	4.94e-03	4.96e-05	4.96e-07	4.96e-09	4.96e-11	4.96e-13

In this example, note that both the upper bound given by (2.26) and the upper bound given in [12] are within one order of magnitude of $\|\widetilde{K}^1 - K\|_F$.

Example 4.3. [1] Consider the system (A, B) where

$$A = \begin{bmatrix} -20.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -25.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ -0.7440 & -0.0320 & 0 & -0.1540 & -0.0042 & 1.5400 & 0 \\ 0.3370 & -1.1200 & 0 & 0.2490 & -1.0000 & -5.2000 & 0 \\ 0.0200 & 0 & 0.0386 & -0.9960 & -0.0003 & -0.1170 & 0 \\ 0 & 0 & 0 & 0.5000 & 0 & 0 & -0.5000 \end{bmatrix},$$

$$B = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

along with the eigenvalues

$$\{-2.0000 \pm 1.5000i, -1.5000 \pm 1.5000i, -20.0000, -25.0000, -0.5000\}.$$

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For the maximum perturbation result, the computed eigenvectors are

$$\begin{bmatrix} 0.6007 \pm 0.2800i \\ 0.6249 \pm 0.0651i \\ 0.0831 \mp 0.0961i \\ 0.1295 \pm 0.1542i \\ -0.0220 \pm 0.3167i \\ -0.0023 \pm 0.0788i \\ 0.0041 \mp 0.0473i \end{bmatrix} \begin{bmatrix} 0.5112 \pm 0.4112i \\ 0.6098 \pm 0.0750i \\ 0.1434 \mp 0.0041i \\ 0.1018 \pm 0.2256i \\ -0.2089 \pm 0.2211i \\ -0.0496 \pm 0.1028i \\ 0.0364 \mp 0.0582i \end{bmatrix} \begin{bmatrix} -0.1351 \\ 0.9890 \\ -0.0030 \\ -0.0035 \\ 0.0607 \\ -0.0000 \\ 0.0001 \end{bmatrix} \begin{bmatrix} -0.9669 \\ -0.2536 \\ -0.0001 \\ -0.0293 \\ 0.0020 \\ -0.0004 \\ 0.0006 \end{bmatrix} \begin{bmatrix} -0.1566 \\ 0.4343 \\ 0.7495 \\ -0.0000 \\ -0.3748 \\ -0.0677 \\ -0.2829 \end{bmatrix}.$$

Define perturbations satisfying (2.1) as
$\Delta A_1 =$

$$\epsilon \begin{bmatrix} -3.8192 & -3.3826 & 0.0820 & -0.9856 & 1.8974 & 2.5714 & 3.2129 \\ 3.4555 & 2.1405 & 1.6639 & -2.4112 & 0.4222 & -1.3083 & -2.3635 \\ 2.2656 & -2.0906 & 2.8754 & 3.5109 & 1.4215 & 0.0840 & -1.9934 \\ 2.3509 & 2.7341 & -2.1501 & 2.8360 & 2.5907 & -1.5920 & 0.3138 \\ -1.2737 & -1.6610 & 1.8700 & -2.8914 & 1.6040 & -1.6882 & 1.8227 \\ -2.3913 & -1.5716 & 0.7964 & 2.8471 & -3.5891 & 2.8926 & -2.3599 \\ -0.7342 & 2.7644 & 3.8267 & -1.9896 & -1.9955 & 2.4706 & -2.8546 \end{bmatrix},$$

$$\Delta B_1 = \epsilon \begin{bmatrix} 8.1053 & -1.8532 \\ -1.7706 & -1.4891 \\ -9.5757 & 4.1846 \\ 6.8511 & 2.6593 \\ 3.4265 & -0.5449 \\ -7.0891 & 11.5788 \\ -7.9824 & 11.2792 \end{bmatrix},$$

and

$$\Delta\Lambda_1 = \epsilon\, \mathrm{diag}(0.3560 \pm 0.4559i, -0.5490 \pm 0.1466i, 2.4850, 6.1808, 0.0178).$$

For the minimum perturbation, the computed eigenvectors are

$$\begin{bmatrix} -0.4152 \mp 0.2351i \\ 0.6122 \pm 0.3133i \\ 0.0569 \mp 0.1879i \\ -0.0720 \mp 0.1070i \\ 0.1682 \pm 0.4612i \\ 0.0032 \mp 0.0477i \\ -0.0058 \pm 0.0298i \end{bmatrix} \begin{bmatrix} -0.3715 \mp 0.2613i \\ 0.5418 \pm 0.3085i \\ 0.1517 \mp 0.2128i \\ -0.0662 \mp 0.1362i \\ 0.0916 \pm 0.5468i \\ 0.0226 \mp 0.0638i \\ -0.0213 \pm 0.0362i \end{bmatrix} \begin{bmatrix} -0.2547 \\ 0.9650 \\ -0.0031 \\ -0.0080 \\ 0.0615 \\ -0.0001 \\ 0.0002 \end{bmatrix} \begin{bmatrix} 0.8654 \\ 0.5002 \\ -0.0004 \\ 0.0265 \\ 0.0110 \\ 0.0004 \\ -0.0005 \end{bmatrix} \begin{bmatrix} 0.1166 \\ -0.3233 \\ -0.5580 \\ 0.0000 \\ 0.2790 \\ 0.0504 \\ -0.7000 \end{bmatrix}.$$

Define perturbations satisfying (2.1) as
$\Delta A_2 =$

$$\epsilon \begin{bmatrix} 3.0760 & 2.6045 & -1.4300 & 3.1049 & 2.6324 & 2.2352 & -3.2888 \\ 2.3436 & 0.4874 & 3.7785 & 2.3391 & -0.3691 & 3.7375 & -0.2633 \\ -0.0472 & -1.4392 & 3.0293 & -2.7895 & 0.2311 & -2.9460 & 1.4481 \\ -2.8082 & -1.7018 & -1.2077 & 1.1342 & 1.5224 & 2.9446 & 3.7304 \\ -0.4955 & 2.2313 & -2.9869 & 0.4704 & 2.6611 & -1.9103 & 0.6989 \\ -1.9873 & 3.8598 & -1.6331 & 0.6989 & -2.6065 & 2.6475 & -0.7115 \\ -0.0875 & 3.9339 & -3.5926 & -0.9245 & -0.2940 & -1.2704 & 3.6772 \end{bmatrix},$$

$$\Delta B_2 = \epsilon \begin{bmatrix} 9.3574 & 7.8692 \\ 4.5651 & 5.0172 \\ -7.9163 & -7.7619 \\ -6.7618 & -7.5011 \\ 5.7692 & -5.8986 \\ -6.6062 & -5.7049 \\ 7.2259 & -2.8500 \end{bmatrix},$$

and

$$\Delta\Lambda_2 = \epsilon \text{ diag}(0.1841 \mp 0.5068i, -0.2936 \pm 0.2815i, -0.6861, 5.8224, -0.0671).$$

Table 4.3 displays results for a range of ϵ-values. Note that the estimate given by (2.17) is one order of magnitude greater then the computed value of $\|\widetilde{V}^1 - V^1\|_F$.

TABLE 4.3
Results for Δ_V, Example 4.3

ϵ	1.0e-07	1.0e-09	1.0e-11	1.0e-13
Δ_V	2.91e-04	2.91e-06	2.91e-08	2.91e-10
$\|\widetilde{V}^1 - V^1\|_F$	2.97e-05	2.97e-07	2.97e-09	2.97e-011
$\|\widetilde{V}^2 - V^2\|_F$	4.19e-07	4.19e-09	4.19e-11	4.19e-13

Example 4.4. With A, B and eigenvalues as in Example 4.3, computed eigenvectors

$$\begin{bmatrix} 0.5525 \mp 0.2293i \\ -0.2073 \pm 0.5166i \\ 0.1929 \pm 0.0626i \\ 0.1683 \pm 0.0164i \\ -0.4797 \pm 0.1641i \\ 0.0436 \pm 0.0446i \\ -0.0253 \mp 0.0308i \end{bmatrix} \begin{bmatrix} 0.0463 \mp 0.3533i \\ -0.8124 \mp 0.0415i \\ -0.0761 \pm 0.1694i \\ 0.0613 \mp 0.1126i \\ -0.1399 \mp 0.3683i \\ 0.0612 \mp 0.0144i \\ -0.0354 \pm 0.0032i \end{bmatrix} \begin{bmatrix} -0.9980 \\ -0.0482 \\ -0.0008 \\ -0.0374 \\ 0.0151 \\ -0.0009 \\ 0.0010 \end{bmatrix} \begin{bmatrix} -0.3466 \\ 0.9367 \\ -0.0019 \\ -0.0092 \\ 0.0487 \\ -0.0001 \\ 0.0002 \end{bmatrix} \begin{bmatrix} 0.1182 \\ -0.3279 \\ -0.5660 \\ 0.0000 \\ 0.2830 \\ 0.0511 \\ 0.6895 \end{bmatrix},$$

and

$$K = \begin{bmatrix} 0.1693 & -0.0972 & 0.8889 & -4.2885 & 0.5320 & 4.0489 & -0.0311 \\ -0.0301 & 0.0937 & -4.6356 & -0.5097 & -2.2873 & 1.7809 & -2.4823 \end{bmatrix},$$

define perturbations satisfying (2.1) as
$\Delta A_1 =$

$$\epsilon \begin{bmatrix} 1.0818 & -0.5301 & -3.2246 & -0.9528 & -3.6388 & 1.2752 & -2.6901 \\ -3.7376 & 2.4787 & 0.6683 & -0.6055 & 2.6487 & 1.5129 & 2.3012 \\ -1.0068 & -0.0985 & 1.8877 & -2.2400 & -2.1470 & 3.0329 & -4.1027 \\ 1.4494 & 1.1704 & 0.0023 & 3.2571 & 2.6906 & -2.7734 & 1.1614 \\ 1.2856 & 0.0208 & 4.0648 & -2.6443 & -2.2752 & 2.2656 & 3.8843 \\ -3.8387 & -2.0093 & -2.5948 & -0.5333 & 3.0975 & -2.2155 & 2.3436 \\ 0.0592 & -0.2996 & -1.8969 & 0.8775 & 2.0849 & 1.8696 & 3.9161 \end{bmatrix},$$

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$$\Delta B_1 = \epsilon \begin{bmatrix} -12.2212 & -10.4005 \\ 1.0649 & 5.3107 \\ -3.1903 & 2.8906 \\ 0.6625 & 1.3633 \\ -1.6827 & 1.0726 \\ 3.7369 & -13.5984 \\ -0.0950 & -10.6973 \end{bmatrix},$$

$$\Delta\Lambda_1 = \epsilon \text{ diag}(-0.0301 \mp 0.2104i, 0.1588 \mp 0.3612i, 1.9563, 0.0275, 0.0466),$$

$$\Delta A_2 =$$

$$\epsilon \begin{bmatrix} 3.5501 & 0.7677 & -0.0470 & 0.6981 & 3.6748 & 2.3717 & -0.1520 \\ 0.5828 & 2.5052 & 0.2798 & 1.1927 & -1.0432 & -1.9981 & -3.9968 \\ -3.9275 & 3.8399 & 4.0487 & 2.6892 & 0.4788 & 3.1016 & 3.4917 \\ -0.6373 & 1.0905 & -3.3759 & 2.8012 & 1.0362 & -1.4872 & -2.7135 \\ 0.6396 & -3.4510 & -1.2366 & 1.9567 & -1.9062 & 0.4253 & -0.1461 \\ -3.3654 & 2.5050 & -1.3715 & 0.2911 & -0.0656 & -1.4468 & -3.9716 \\ 1.4148 & 0.8038 & 2.8124 & -3.6424 & 0.3130 & -3.3626 & -1.2145 \end{bmatrix},$$

$$\Delta B_2 = \epsilon \begin{bmatrix} 3.3611 & 7.4928 \\ 12.2752 & 6.4126 \\ 2.9557 & 0.8021 \\ -6.0221 & -6.6566 \\ -0.0368 & 12.0282 \\ 6.3241 & -7.8475 \\ -1.2250 & -5.2959 \end{bmatrix},$$

and

$$\Delta\Lambda_2 = \epsilon \text{ diag}(-0.3548 \mp 0.0969i, 0.5605 \mp 0.4749i, 0.7368, -0.9905, -0.1539).$$

Table 4.4 displays results for a range of ϵ-values.

TABLE 4.4
Results for $(\Delta_K)_{CC}$, Example 4.4

ϵ	1.0e-05	1.0e-07	1.0e-09	1.0e-11	1.0e-13
$(\Delta_K)_{CC}$	2.43e-00	2.43e-02	2.43e-04	2.43e-06	2.43e-08
$(\Delta_{K^1})_{MX}$	1.56e+02	1.56e-00	1.56e-02	1.56e-04	1.56e-06
$\Vert\widetilde{K}^1 - K\Vert_F$	1.08e-02	1.08e-04	1.08e-06	1.08e-08	1.08e-10
$(\Delta_{K^2})_{MX}$	1.49e+02	1.49e-00	1.49e-02	1.49e-04	1.49e-06
$\Vert\widetilde{K}^2 - K\Vert_F$	7.63e-04	7.64e-06	7.64e-08	7.64e-10	7.63e-12

Note that the upper bound given by (2.26) is two orders of magnitude tighter than the upper bound given in [12]. This example points out one of the main differences between the upper bound given by (2.26) and the upper bound in [12]. The bound in (2.26) includes the condition number of the matrix of right eigenvectors, whereas the bound in [12] includes the condition number of the closed-loop matrix $(A - BK)$. For this example, $\kappa_2(V) = 28.0115$, and $\kappa_2(A - BK) = 620.8399$. It may be possible to combine the best features of these two bounds into a "hybrid" bound which supercedes both results.

Using the method in [17], a distance, d, to the nearest uncontrollable system, was computed for each example. For Example 1, $d = 1$, and the computed results based on (2.26) and the analogous result in [12] were each within one order of magnitude of the true perturbation. For Example 2, where $d = 0.037$, (2.26) yielded a result two orders of magnitude larger than the exact value while the bound in [12] produced a value four orders of magnitude larger than the exact value. These results are consistent with the following observations. First, the denominator in the perturbation bound for K in [12] contains a term closely related to the distance to uncontrollability. A similar term appears in (2.26) in a more subtle manner. In [5], it is shown that $\kappa_2(V)$ is bounded below by a factor which is inversely proportional to the distance to uncontrollability. And $\kappa_2(V)$ is present in (2.26) through the term $\|\Phi^{-1}\|_2\|V\|_2$. A topic for future study is the relationship between distance to uncontrollability, condition number of the eigenvector matrix, and the perturbation bound for the gain matrix K.

5. Conclusion. This paper contains a perturbation analysis of the multi-input eigenstructure assignment problem. Upper bounds for perturbations on the eigenvector matrix are derived even though this matrix is not unique. Upper bounds on perturbations in the gain matrix are developed. All bounds are to first order in the relative perturbations in the data.

Numerical results for a range of perturbations are provided, confirming analytical results.

Although numerical results are reasonably close to predicted bounds, tighter analytical results may be obtainable. It may be possible to redo the analysis using an alternate definition to that in (2.14), which is smaller and thus closer to the true condition number of M_j. Also, it may be worthwhile to derive entry-wise perturbation results, especially for sparse systems such as in Examples 4.3 and 4.4.

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