Electronic Transactions on Numerical Analysis.
Volume 11, pp. 121-130, 2000.

ISSN 1068-9613.

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APPROXIMATION OF HYPERGEOMETRIC FUNCTIONS WITH MATRICIAL ARGUMENT THROUGH THEIR DEVELOPMENT IN SERIES OF ZONAL POLYNOMIALS*

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Abstract. Hypergeometric functions with matricial argument are being used in several fields of mathematics. This article tries to obtain a good approximation of this family of functions, since there are no general expressions for them, calculating zonal polynomials of high degrees and developing the functions in a truncated series.

Key words. hypergeometric functions with matricial argument, zonal polynomials.

AMS subject classifications. 33C20, 33C99, 12Y05.

1. Introduction. Hypergeometric functions with matricial argument is an interesting problem to study from a purely analytic point of view. However, they appear in the practice of different fields of mathematics, so knowledge of them is necessary for applications of theories associated with these fields.

In harmonic analysis these functions were introduced by Bochner[1] through Bessel functions with matricial argument. Herz[6] defined them through Laplace and inverse Laplace transforms. Siegel[13] and Selberg[11] investigations in number theory allowed Gindikin[3] to introduce some generalizations of Gaussian hypergeometric functions associated with homogeneous cones and analysis on Siegel domains. Applications in number theory have continued until recent works (Terras[15], Shimura[12], and others).

In the field of statistical mathematics, development of the Herz theory carried out by James [8], Constantine[2], and others has contributed to the study of some distributions associated with normal populations. Takemura[14] studies these functions as eigenfunctions of operators, also with normal populations.

In the last few years hypergeometric functions with matricial argument have been used in generating probability distributions, as a generalization of the use of classical hypergeometric functions in this field (see Rodrí guez[10], Gutiérrez[5]).

In this article we will present a development of this family of functions in a truncated series that permits us to know their explicit values with a high degree of precision. So we can solve in part the problem of nonexistence of expressions for these functions. We can also obtain approximated summation results from normalizing distributions of probability.

Calculation of zonal polynomials is the most important aspect in the development of the truncated series; we have programmed an algorithm for the calculation of these polynomials that is based on an algorithm due to James [8]. It allows us to calculate all the polynomials up to degree 20. (There are 2,714 polynomials.) Higher degrees can be calculated easily, but more time would be needed. (We spent about 8 days to obtain the 627 zonal polynomials of degree 20 with a 350 MHz Pentium II processor.)

2. Hypergeometric functions and zonal polynomials. We now introduce hypergeometric functions with matricial argument.

*Received February 16, 2000. Accepted for publication December 4, 2000. Recommended by F. Marcellán.
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DEFINITION 2.1. *We define hypergeometric functions with matricial argument by*

$$ {}_pF_q\left(a_1,...,a_p;b_1,...,b_q;X\right)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(a_1\right)_\kappa...\left(a_p\right)_\kappa}{\left(b_1\right)_\kappa...\left(b_q\right)_\kappa}\frac{C_\kappa\left(X\right)}{k!}, \tag{2.1}$$

where $\sum_\kappa$ denotes summation over all partitions $\kappa=(k_1,...,k_m)$ of the integer k, $C_\kappa(X)$ denotes the zonal polynomial of X corresponding to κ and the coefficients $()_\kappa$, called generalized hypergeometric coefficients, are

$$\left(a\right)_\kappa=\prod_{i=1}^{m}\left(a-\frac{1}{2}\left(i-1\right)\right)_{k_i}, \tag{2.2}$$

where $\left(a\right)_k=a\left(a+1\right)...\left(a+k-1\right),\left(a\right)_0=1.$

X is a complex $m\times m$ symmetric matrix and the parameters a_i, b_j are arbitrary complex numbers. Any parameter b_j in the denominator can be zero or an integer less than or equal to $(m-1)/2$. The series converges if $p\le q$; it converges when $\|X\|<1$ if $p=q+1$ ($\|X\|$ is the highest absolute value of the eigenvalues of X); and, unless the series is finite, it diverges if $p>q+1$ and $X\neq 0$.

We will use the following definition of zonal polynomials by James [7] , because it allows us to obtain the algorithm for the calculation of the polynomials. Other definitions, based on representation group theory, are more difficult to be used for these calculations.

DEFINITION 2.2. *Let X be a symmetric $m\times m$ complex matrix with eigenvalues $x_1,...,x_m$ and let $\kappa=(k_1,...,k_m)$ be a partition of k in no more than m parts. The zonal polynomial of X corresponding to κ, denoted by $C_\kappa(X)$, is a symmetric homogeneous polynomial of degree k in the eigenvalues $x_1,...,x_m$ such that*

1. *The highest weight term in the expansion of $C_\kappa(X)$ is $x_1^{k_1}...x_m^{k_m}$, i.e.,*

$$C_\kappa\left(X\right)=d_\kappa x_1^{k_1}...x_m^{k_m}+\text{terms with lower weight,} \tag{2.3}$$

where d_κ is a constant.

2. *$C_\kappa(X)$ is an eigenfunction of the differential operator Δ_X , where*

$$\Delta_X=\sum_{i=1}^{m}x_i^2\frac{\partial^2}{\partial x_i^2}+\sum_{i=1}^{m}\sum_{1=j\neq i}^{m}\frac{x_i^2}{x_i-x_j}\frac{\partial}{\partial x_i}, \tag{2.4}$$

with eigenvalue

$$\left[\rho_\kappa+k\left(m-1\right)\right], \tag{2.5}$$

where

$$\rho_\kappa=\sum_{i=1}^{m}k_i\left(k_i-i\right). \tag{2.6}$$

3. *And*

$$\left(trX\right)^k=\left(x_1+...+x_m\right)^k=\sum_{\kappa}^{m}C_\kappa\left(X\right). \tag{2.7}$$

There is no general formula for zonal polynomials. In Muirhead [9] there is an algorithm to calculate them from this definition, but, although it is very interesting from a methodological point of view, we think that it is an algorithm not easy to program for high degrees. We use another algorithm by James[7], because it is easier to program and allows us to calculate zonal polynomials of high degrees.

The nonexistence of general formulae for zonal polynomials is the most important difficulty in using hypergeometric functions with matricial argument. On the other hand, there is no other way to express these functions apart from the series (2.1).

If $m = 1$, zonal polynomials are the powers of a single variable, so hypergeometric functions with matricial argument are classical hypergeometric functions of one variable.

Zonal polynomials depend only on the eigenvalues of X, $x_1, ..., x_m$, so we can use as argument diagonal matrices.

There are two special cases of (2.1):

$$\begin{aligned} {}_0F_0(X) &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(X)}{k!} \\ &= \sum_{k=0}^{\infty}\frac{(trX)^k}{k!} \\ &= etr(X), \end{aligned} \tag{2.8}$$

which is a generalization of the exponential series, and

$$\begin{aligned} {}_1F_0(a;X) &= \sum_{k=0}^{\infty}\sum_{\kappa}(a)_\kappa\frac{C_\kappa(X)}{k!} \quad (\|X\|<1) \\ &= \det(I_m - X)^{-a}. \end{aligned} \tag{2.9}$$

Other explicit results are unknown, so there is not any way for evaluating these functions, except for some particular results (for example, Gauss theorem in the multivariate case).

3. Calculation of zonal polynomials. We now describe the algorithm due to James [8] for calculating zonal polynomials.

It is necessary to express these zonal polynomials as a linear combination of symmetric monomials; this is the most important aspect for the develop of the hypergeometric series.

Symmetric monomials are a base of symmetric polynomials and the expression that determines the algorithm is a change of base equation, so zonal polynomials are a base of symmetric polynomials and they may be used to approximate other symmetric functions too.

If $\kappa = (k_1, ..., k_m)$, the symmetric monomial of $X_{m\times m}$, a symmetric matrix with latent roots $x_1, ..., x_m$, is

$$M_\kappa(X) = \sum ... \sum x_{i_1}^{k_1} x_{i_2}^{k_2} ... x_{i_p}^{k_p}, \tag{3.1}$$

where p is the number of no zeros parts of κ, and the summation is over all partitions $(i_1, ..., i_p)$ of p integers between $1, ..., m$. For example,

$$\begin{aligned} M_{(1)}(X) &= x_1 + ... + x_m, \\ M_{(2)}(X) &= x_1^2 + ... + x_m^2, \\ M_{(1,1)}(X) &= \sum_{i<j}^{m} x_i x_j. \end{aligned} \tag{3.2}$$

Let κ be a partition of the integer k. Then

$$C_{\kappa}(X) = \sum_{\lambda \leq \kappa} c_{\kappa,\lambda} M_{\lambda}(X), \tag{3.3}$$

where $c_{\kappa,\lambda}$ are constants and the summation is over all partitions λ of k such that $\lambda \leq \kappa$. The expression for $c_{\kappa,\lambda}$ is

$$c_{\kappa,\lambda} = \sum_{\lambda < \mu \leq \kappa} \frac{[(l_i + t) - (l_j - t)]}{\rho_{\kappa} - \rho_{\lambda}} c_{\kappa,\mu}, \tag{3.4}$$

where $\lambda = (l_1, ..., l_m)$ and $\mu = (l_1, ..., l_i + t, ..., l_j - t, ..., l_m)$ for $t = 1, ..., l_j$ such that, when parts of the partition μ are arranged in descending order, μ is above λ and below or equal to κ in the lexicographical ordering. The summation in (3.4) is over all such μ, including possibly, nondescending ones, and any empty sum is taken to be zero.

This determines all coefficients $c_{\kappa,\lambda}$ in the expansion of $C_{\kappa}(X)$ except for the coefficient $c_{\kappa,\kappa}$; i.e., determines the zonal polynomial except for normalizing. Using condition (2.6) in the definition, it follows that $c_{(k),)k)} = 1$ and then, all coefficients $c_{(k),\lambda}$ in the expansion of $C_{(k)}(X)$ are given by the expression (3.4). For calculating the highest weight coefficient of $C_{(k-1,1)}(X)$ and the next ones, Muirhead [9] suggests that it is possible to calculate them using condition (3) in definition (2.2), developing

$$(x_1 + ... + x_m)^k \tag{3.5}$$

in terms of the symmetric monomials and identifying coefficients in this expansion (for more details see Muirhead [9]). Although this is a way for calculating the coefficients $c_{\kappa,\kappa}$, we think that it is more difficult to program than the James [8] way.

James [8] proves that

$$c_{\kappa,\kappa} = \frac{2^{2k} k!}{2k!} \chi_{[2\kappa]}(1) \prod_{l=1}^{p} \prod_{i=1}^{l} \left(\frac{1}{2} l - \frac{1}{2}(i-1) + k_i - k_l \right)_{k_i - k_{l+1}}, \tag{3.6}$$

where $\chi_{[2\kappa]}(1)$ is the degree of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols, given by

$$\chi_{[2\kappa]}(1) = \frac{2k! \prod_{i<j}^{p} (2k_i - 2k_j - i + j)}{\prod_{i=1}^{p} (2k_i + p - i)!}, \tag{3.7}$$

where p is the number of no zero parts of κ.

Nevertheless, we have introduced this normalization of zonal polynomials in a different way. There is a general expression for them when $X = I_m$, i.e.,

$$C_{\kappa}(I_m) = 2^{2k} k! \left(\frac{1}{2} m \right)_{\kappa} \frac{\prod\limits_{i<j}^{p} (2k_i - 2k_j - i + j)}{\prod\limits_{i<j}^{p} (2k_i + p - i)!}, \tag{3.8}$$

where p is the number of no zero parts of κ and

$$\left(\frac{1}{2} m \right)_{\kappa} = \prod_{i=1}^{p} \left(\frac{1}{2}(m - i + 1) \right)_{k_i},$$

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k	Num. of partitions	Num. of steps	Num. of addends
2	2	3	1
5	7	162	110
10	42	16,992	13,100
15	176	549,407	452,009
20	627	10,736,310	9,159,310

TABLE 3.1

Some computational aspects: The number of partitions is also the number of zonal polynomials. Steps are the number of steps in the algorithm for the calculation of zonal polynomials of degree k. Addends are the number of addends in this calculation.

with $(a)_k = a\,(a+1)\ldots(a+k-1)\,, (a)_0 = 1$. So we started the algorithm with all coefficients $c_{\kappa,\kappa} = 1$ and then we normalized the polynomial using (3.8). Actually, both ways, the original way by James and the way we programmed it, are the same, and they will take a very similar time in their performance. Nevertheless, we think that the expression (3.8) requires less computation than (3.6).

We calculated zonal polynomials up to degree 20 using Matlab. Although the performance for high degrees requires a lot of time, the program will be able to calculate other degrees. The problem is that for $k = 20$ there are 627 partitions, so 627 zonal polynomials, and this number is increased very fast by higher degrees. In table 3.1 appears details of the computation of all zonal polynomials up to degree 20.

4. Development in truncated series. The nonexistence of general expressions for hypergeometric functions with matricial argument is a very serious problem for using them in all their applications.

There is a field where this problem is particularly important: We can use hypergeometric functions with matricial argument as generating probability functions of multivariate distributions of probability (see Rodríguez [10], Gutiérrez[5]). But if we do not have expressions for them we can give only some general properties of generated distributions, never probabilities for them.

In this article we describe a development of the hypergeometric series that gives us their approximate value; in the case we use them as generating probability functions, they also allow us to obtain a great part of the probabilities of the generated distributions.

The well-known Taylor's theorem for evaluating a regular function $g(t)$ is

$$g(t) = g(0) + \frac{g'(0)}{1!} + \frac{g''(0)}{2!} + \ldots .$$

We have tried to obtain a similar expression for hypergeometric functions with matricial argument. These functions are defined by (2.1), i.e.,

$${}_pF_q\left(a_1, ..., a_p; b_1 ..., b_q; X_{m\times m}\right) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa \ldots (a_p)_\kappa}{(b_1)_\kappa \ldots (b_q)_\kappa}\frac{C_\kappa(X)}{k!}.$$

In this expression, for each degree k, the term

$$\sum_{\kappa}\frac{(a_1)_\kappa \ldots (a_p)_\kappa}{(b_1)_\kappa \ldots (b_q)_\kappa}\frac{C_\kappa(X)}{k!}$$

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includes all zonal polynomials of degree k; they are a linear combination of symmetric monomials with degree less than or equal to k, so if we stop the series at degree M, i.e.,

$$\sum_{k=0}^{M}\sum_{\kappa}\frac{(a_1)_\kappa\dots(a_p)_\kappa}{(b_1)_\kappa\dots(b_q)_\kappa}\frac{C_\kappa(X)}{k!},$$

using (3.3), this truncated series is

$$\begin{aligned}&\sum_{k=0}^{M}\sum_{\kappa}\frac{(a_1)_\kappa\dots(a_p)_\kappa}{(b_1)_\kappa\dots(b_q)_\kappa}\frac{\sum_{\lambda\leq\kappa}c_{\kappa,\lambda}M_\lambda(X)}{k!}\\&=\sum_{k=0}^{M}\sum_{\kappa}\sum_{\lambda\leq\kappa}\frac{1}{k!}\frac{(a_1)_\kappa\dots(a_p)_\kappa}{(b_1)_\kappa\dots(b_q)_\kappa}c_{\kappa,\lambda}M_\lambda(X).\end{aligned}$$

Grouping symmetric monomials, we have,

$$\sum_{r=0}^{M}\sum_{\rho}f_\rho M_\rho(X),$$

where the summation is over all partitions ρ of r.

In this expression, if

$$\rho=(r_1,\dots,r_m),$$

as the symmetric monomial is

$$M_\rho(X)=\sum_{\sigma}x_1^{\sigma(1)}\dots x_m^{\sigma(m)},$$

where the summation is over all permutations σ of $(1,\dots,m)$, the coefficient f_ρ is the same for all the terms

$$x_1^{\sigma(1)}\dots x_m^{\sigma(m)}$$

for any permutation σ; moreover, there are no more terms with this expression $x_1^{\sigma(1)}\dots x_m^{\sigma(m)}$ in the queue of the series.

For example, we describe the development of the series ${}_2F_1(a,b;c;x_1,x_2)$:

$${}_2F_1(a,b;c;x_1,x_2)=$$

$$\begin{gathered}1+\\ \frac{(a)_{(1,0)}\cdot(b)_{(1,0)}}{(c)_{(1,0)}}\cdot\frac{C_{(1,0)}(x_1,x_2)}{1!}+\\ \frac{(a)_{(2,0)}\cdot(b)_{(2,0)}}{(c)_{(2,0)}}\cdot\frac{C_{(2,0)}(x_1,x_2)}{2!}+\frac{(a)_{(1,1)}\cdot(b)_{(1,1)}}{(c)_{(1,1)}}\cdot\frac{C_{(1,1)}(x_1,x_2)}{2!}+\\ \frac{(a)_{(3,0)}\cdot(b)_{(3,0)}}{(c)_{(3,0)}}\cdot\frac{C_{(3,0)}(x_1,x_2)}{3!}+\frac{(a)_{(2,1)}\cdot(b)_{(2,1)}}{(c)_{(2,1)}}\cdot\frac{C_{(2,1)}(x_1,x_2)}{3!}+\\ \frac{(a)_{(4,0)}\cdot(b)_{(4,0)}}{(c)_{(4,0)}}\cdot\frac{C_{(4,0)}(x_1,x_2)}{4!}+\frac{(a)_{(3,1)}\cdot(b)_{(3,1)}}{(c)_{(3,1)}}\cdot\frac{C_{(3,1)}(x_1,x_2)}{4!}+\\ +\dots.\end{gathered}$$

In terms of symmetric monomials,

$$
\begin{gathered}
{}_2F_1\left(a,b;c;x_1,x_2\right)= \\
\\
\frac{(a)_{(1,0)}\cdot(b)_{(1,0)}}{(c)_{(1,0)}}\cdot M_{(1,0)}+ \\
\frac{(a)_{(2,0)}\cdot(b)_{(2,0)}}{(c)_{(2,0)}}\cdot\frac{1}{2}M_{(2,0)}+ \\
\left\{\frac{(a)_{(2,0)}\cdot(b)_{(2,0)}}{(c)_{(2,0)}}\cdot\frac{1}{3}+\frac{(a)_{(1,1)}\cdot(b)_{(1,1)}}{(c)_{(1,1)}}\cdot\frac{2}{3}\right\}\cdot M_{(1,1)}+ \\
\frac{(a)_{(3,0)}\cdot(b)_{(3,0)}}{(c)_{(3,0)}}\cdot\frac{1}{6}M_{(3,0)}+ \\
\left\{\frac{(a)_{(3,0)}\cdot(b)_{(3,0)}}{(c)_{(3,0)}}\cdot\frac{1}{10}+\frac{(a)_{(2,1)}\cdot(b)_{(2,1)}}{(c)_{(2,1)}}\cdot\frac{4}{10}\right\}\cdot M_{(2,1)}+ \\
\frac{(a)_{(4,0)}\cdot(b)_{(4,0)}}{(c)_{(4,0)}}\cdot\frac{1}{24}M_{(4,0)}+ \\
\left\{\frac{(a)_{(4,0)}\cdot(b)_{(4,0)}}{(c)_{(4,0)}}\cdot\frac{1}{42}+\frac{(a)_{(3,1)}\cdot(b)_{(3,1)}}{(c)_{(3,1)}}\cdot\frac{1}{7}\right\}\cdot M_{(3,1)}+ \\
\left\{\frac{(a)_{(4,0)}\cdot(b)_{(4,0)}}{(c)_{(4,0)}}\cdot\frac{3}{4\cdot 35}+\frac{(a)_{(3,1)}\cdot(b)_{(3,1)}}{(c)_{(3,1)}}\cdot\frac{2}{21}+\right. \\
\left.\frac{(a)_{(2,2)}\cdot(b)_{(2,2)}}{(c)_{(2,2)}}\cdot\frac{2}{15}\right\}\cdot M_{(2,2)}+\ldots\,.
\end{gathered}
$$

If hypergeometric functions with matricial argument are generating probability functions, this development of the series permits us to identify probabilities of the generated distributions, because the coefficient f_ρ is

$$P\left[(X_1,...,X_m)=\left(r_{\sigma(1)},...,r_{\sigma(m)}\right)\right],$$

except for normalizing, for each permutation σ. So, if we want to know the probability

$$P\left[(X_1,...,X_m)=(i_1,...,i_m)\right]$$

for a distribution generated by ${}_pF_q\left(a_1,...,a_p;b_1...,b_q;x_1,...,x_m\right)$, we have to develop the series up to degree

$$M=i_1+...+i_m,$$

and search in the $M-$term of the symmetric monomial corresponding to the partition $(i_1,...,i_m)$.

Of course, if any numerator parameter in the series is a negative integer then the series is a finite sum and we can obtain the complete distribution using this method.

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a	b	c	${}_2F_1\left(a,b;c;I_2\right)$	$\sum_{k=0}^{20}\sum_{\kappa}\frac{(a)_\kappa(b)_\kappa}{(c)_\kappa}\frac{C_\kappa(I_2)}{k!}$	Difference
2	2	10	3.0569	3.0547	0.0022
3	3	10	20.0909	19.0750	1.0159
5	5	25	13.8359	13.8299	0.006
4	5	20	15.7342	15.7046	0.0296
8	10	50	55.1096	55.0856	0.024
10	10	50	172.7901	172.2222	0.5679
12	12	65	260.1934	259.5012	0.6922
15	15	85	699.2147	695.7047	3.51

TABLE 5.1
Development of the function ${}_2F_1\left(a,b;c;I_2\right)$.

5. Computational results. We now give some practical results where the truncated development of the series approximates the exact value of the function. In all the cases we have developed series up to degree 20.

We would like to highlight that it is possible to have a better approximation by calculating zonal polynomials of degrees higher than 20, if you spend enough time.

The results refer to matrices $\lambda \cdot I_m$, where λ is such that it determines the convergence of the series. These results are used in normalizing distributions generated by functions ${}_pF_q\left(a_1,...,a_p;b_1,...,b_q;X_{m\times m}\right)$. We think that it is not very difficult to obtain other results for matrices with any eigenvalues, $\lambda_1,...,\lambda_m$; the unique problem to be solved is to obtain expressions for symmetric monomials in these eigenvalues.

Of course, precision of the truncated series to approximate hypergeometric functions is higher when the parameters =

$b_1,...,b_q$ in the denominators are very much bigger than the parameters $a_1,...,a_p$ in the numerators or when λ is smaller than unity; however, we describe examples where both the parameters in the numerators and denominators are similar, and λ is smaller than, equal to or bigger than unity.

1. The multivariate extension of the Gauss summation theorem is known, i.e.,

$$ {}_2F_1\left(a;b;c;I_m\right)=\frac{\Gamma_m\left(c\right)\Gamma_m\left(c-a-b\right)}{\Gamma_m\left(c-a\right)\Gamma_m\left(c-b\right)}, \tag{5.1}$$

   where $\Gamma_m\left(.\right)$ is the multivariate gamma function, defined by

$$ \Gamma_m\left(a\right)=\int_{A>0} etr\left(-A\right)\det A^{a-\frac{m+1}{2}}dA. \tag{5.2}$$

   If $Re\left(a\right)>\frac{1}{2}\left(m-1\right)$,

$$ \Gamma_m\left(a\right)=\pi^{\frac{m(m-1)}{4}}\prod_{i=1}^{m}\Gamma\left(a-\frac{1}{2}\left(m-1\right)\right). \tag{5.3}$$

   Then we may compare the value of the development of ${}_2F_1$ in the identity matrix of dimension two with the exact value, using (5.1). Some examples are shown in table 5.1.
2. Although there are no more expressions for other hypergeometric functions apart from ${}_2F_1$, we can see the convergence of the hypergeometric series. In tables 5.2, 5.3, and 5.4 appear the values of truncated series developed for different degrees for the functions ${}_1F_1$, ${}_2F_1$, and ${}_3F_2$.

$\sum_{k=0}^{M}\sum_{\kappa}\frac{(a)_\kappa}{(c)_\kappa}\frac{C_\kappa(\lambda I_m)}{k!}$	$a=2$ $c=3$ $\lambda=3.5$ $m=2$	$a=5$ $c=7$ $\lambda=2.5$ $m=3$	$a=8$ $c=12$ $\lambda=1$ $m=5$
$M=1$	5.6667	6.3571	4.3333
$M=2$	17.1000	20.9090	9.9242
$M=5$	89.9875	126.6973	25.2013
$M=10$	167.5575	247.4603	29.0010
$M=15$	173.6505	256.7171	29.0283
$M=18$	173.7336	256.8337	29.0284
$M=19$	173.7360	256.8369	29.0284
$M=20$	173.7368	256.8379	29.0284

TABLE 5.2
Development of the function $_1F_1\left(a;c;\lambda\cdot I_m\right)$.

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$\sum_{k=0}^{M}\sum_{\kappa}\frac{(a)_\kappa(b)_\kappa}{(c)_\kappa}\frac{C_\kappa(\lambda I_m)}{k!}$	$a=2$ $b=2$ $c=3$ $\lambda=0.5$ $m=2$	$a=5$ $b=5$ $c=20$ $\lambda=0.65$ $m=3$	$a=5$ $b=5$ $c=25$ $\lambda=0.5$ $m=7$
$M=1$	2.3333	3.4375	4.500
$M=2$	3.5333	6.7978	11.0275
$M=5$	5.4763	15.5009	36.0281
$M=10$	6.0506	19.3368	53.4960
$M=15$	6.0856	19.5987	54.6481
$M=18$	6.0871	19.6091	54.6777
$M=19$	6.0872	19.6100	54.6794
$M=20$	6.0873	19.6104	54.6801

TABLE 5.3
Development of the function $_2F_1\left(a,b;c;\lambda\cdot I_m\right)$.

6. Acknowledgements. The authors would like to thank the referee for helpful comments.

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$\sum_{k=0}^{M}\sum_{\kappa}\frac{(a_1)_\kappa(a_2)_\kappa(a_3)\kappa}{(b_1)_\kappa(b_2)_\kappa}\frac{C_\kappa(\lambda I_m)}{k!}$	$a_1=2$, $a_2=3$, $a_3=4$, $b_1=7$, $b_2=8$, $\lambda=1$, $m=2$	$a_1=7$, $a_2=8$, $a_3=9$, $b_1=10$, $b_2=11$, $\lambda=0.3$, $m=2$	$a_1=6.5$, $a_2=6.5$, $a_3=6.5$, $b_1=15$, $b_2=15$, $\lambda=0.75$, $m=3$
$M=1$	1.8571	3.7491	3.7462
$M=2$	2.4103	7.8901	7.9855
$M=5$	3.0529	19.4990	21.4205
$M=10$	3.2261	24.2864	29.6901
$M=15$	3.2493	24.4855	30.5997
$M=18$	3.2529	24.4888	30.6577
$M=19$	3.2536	24.4890	30.6637
$M=20$	3.2541	24.4891	30.6672

TABLE 5.4
Development of the function ${}_3F_2\left(a_1,a_2,a_3;b_1,b_2;\lambda\cdot I_m\right)$.

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