Appendix to: Some Geometry and Combinatorics for the S-invariant of Ternary Cubics.

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The formula for the coefficient A of M in S, with M as in Example 4.1, is given as $A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$, where the A_i are defined (as functions of s) as follows:

$$A_1 = \sum_{l=0}^{s}\sum_{j=0}^{l}\sum_{i=0}^{s-l}\binom{s}{l}^3\binom{l}{j}^2\left(\binom{l}{j+1}-\binom{l}{j}\right)\binom{s-l}{i}^2\left(\binom{s-l}{i+1}-\binom{s-l}{i}\right).$$

$$A_2 = \sum_{l=0}^{s-1}\sum_{j=0}^{l}\binom{s}{l}^2\binom{s}{l+1}\binom{l}{j}\left(\binom{l}{j+1}\binom{l+1}{j+2}+\binom{l}{j+1}\binom{l+1}{j+1}-2\binom{l}{j}\binom{l+1}{j+1}\right)$$
$$\sum_{i=0}^{s-l}\binom{s-l}{i}\left(\binom{s-l}{i}\binom{s-l-1}{i-1}-\binom{s-l}{i+1}\binom{s-l-1}{i}\right).$$

$$A_3 = \sum_{l=0}^{s-2}\sum_{j=0}^{l}\binom{s}{l}^2\binom{s}{l+2}\binom{l}{j}\left(\binom{l}{j+1}\binom{l+2}{j+2}-\binom{l}{j}\binom{l+2}{j+1}\right)$$
$$\sum_{i=0}^{s-l}\binom{s-l}{i}\binom{s-l-2}{i-1}\left(\binom{s-l}{i+1}-\binom{s-l}{i}\right).$$

$$A_4 = \sum_{l=0}^{s-1}\sum_{j=0}^{l+1}\binom{s}{l+1}^2\binom{s}{l}\left(\binom{l}{j+1}\binom{l+1}{j+1}+\binom{l}{j}\binom{l+1}{j+1}-2\binom{l+1}{j}\binom{l}{j-1}\right)$$
$$\binom{l+1}{j}\sum_{i=0}^{s-l-1}\binom{s-l-1}{i}\binom{s-l}{i+1}\left(\binom{s-l-1}{i}-\binom{s-l-1}{i+1}\right).$$

$$A_5 = \sum_{l=0}^{s-2}\sum_{j=0}^{l}\binom{s}{l}\binom{s}{l+1}\binom{s}{l+2}\binom{l+1}{j}\left(\binom{l}{j+1}\binom{l+2}{j+2}-\binom{l}{j}\binom{l+2}{j+1}\right)$$
$$\sum_{i=0}^{s-l-1}\binom{s-l-1}{i}\binom{s-l}{i+1}\left(\binom{s-l-2}{i}-\binom{s-l-2}{i-1}\right).$$

$$A_6 = \sum_{l=0}^{s-2}\sum_{j=1}^{l+2}\binom{s}{l+2}^2\binom{s}{l}\binom{l+2}{j}\left(\binom{l}{j-2}\binom{l+2}{j-1}-\binom{l}{j-1}\binom{l+2}{j}\right)$$
$$\sum_{i=0}^{s-l-2}\binom{s-l-2}{i}\binom{s-l}{i+1}\left(\binom{s-l-2}{i-1}-\binom{s-l-2}{i}\right).$$

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If we take the formula for S in terms of cofactors, as used in Section 4, but write it as a sum over $p \leq q$, these numbers represent the coefficients of M in the terms with $(p, q) = (3, 3)$, $(2, 3)$, $(2, 2)$, $(1, 3)$, $(1, 2)$ and $(1, 1)$, respectively. If we take as an example $s = 4$ in the given formulae, the above numbers are $A_1 = 5804$, $A_2 = -3048$, $A_3 = 2352$, $A_4 = -4552$, $A_5 = -2256$, $A_6 = 2352$ and $A = 652$. In fact, for the monomial M of this example, we have $A_3 = A_6$ for all s; this latter identity may be seen by writing A_3 in terms of $l' = s - 2 - l$, rearranging the sums over i and j, and then comparing with the formula for A_6.