Zeros of $\{-1, 0, 1\}$ Power Series and Connectedness Loci for Self-Affine Sets

Pablo Shmerkin and Boris Solomyak

CONTENTS

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We consider the set Ω_2 of double zeros in $(0, 1)$ for power series with coefficients in $\{-1, 0, 1\}$. We prove that Ω_2 is disconnected, and estimate $\min \Omega_2$ with high accuracy. We also show that $[2^{-1/2} - \eta, 1) \subset \Omega_2$ for some small, but explicit, $\eta > 0$ (this was known only for $\eta = 0$). These results have applications in the study of infinite Bernoulli convolutions and connectedness properties of self-affine fractals.

1. INTRODUCTION

Let

$$\mathcal{B} = \left\{1 + \sum_{n=1}^{\infty} a_n x^n : \ a_n \in \{-1, 0, 1\}\right\}. \qquad (1–1)$$

We investigate the set

$$\Omega_2 = \{x \in (0, 1) : \ \exists f \in \mathcal{B}, \ f(x) = f'(x) = 0\}. \qquad (1–2)$$

Thus Ω_2 is the set of zeros of power series with coefficients in $\{-1, 0, 1\}$ of order greater than or equal to two (we call them "double zeros" for short). Since $\mathcal{B}$ is a normal family, Ω_2 is relatively closed in $(0, 1)$, so there exists $\alpha_2 = \min \Omega_2$.

We prove that the set Ω_2 is disconnected and show that $\alpha_2 \approx 0.6684756$ (see Theorem 2.6). In fact, we found 58 distinct components in Ω_2, and we conjecture that there are infinitely many components. Numerical evidence indicates that the structure of Ω_2 is very complicated.

Let $\widetilde{\alpha}_2 := \sup((0, 1)\backslash\Omega_2)$. It is known that $[2^{-1/2}, 1) \subset \Omega_2$ (see Lemma 2.3); hence $\widetilde{\alpha}_2 \le 2^{-1/2}$. We show that $\widetilde{\alpha}_2 \le 2^{-1/2} - 4 \times 10^{-6}$ (see Theorem 2.11). Numerical evidence suggests that $\widetilde{\alpha}_2 \approx 0.67$, but this is harder to prove rigorously.

The study of Ω_2 is motivated by work on infinite Bernoulli convolutions [Solomyak 95, Peres and Solomyak 96, Peres and Solomyak 98, Peres and Schlag 00] and on some fractal sets [Jordan and Pollicott 06], where a key

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step is checking a certain *transversality condition*. This condition holds precisely on $(0,1)\setminus\Omega_2$. The cited papers used the estimate $\alpha_2 \geq 0.649\dots$ obtained in [Solomyak 95] by computing the smallest double zero of a larger class of power series $\widetilde{\mathcal{B}} := \{1+\sum_{n=1}^{\infty} a_n x^n \, : \, a_n \in [-1,1]\}$. Theorem 2.6 extends considerably (by more than 12%) the set of parameters for which the results of [Peres and Solomyak 98, Peres and Schlag 00, Jordan and Pollicott 06] apply.

Another motivation comes from the study of connectedness loci for certain families of self-similar and self-affine fractals in the plane. One of them is the "Mandelbrot set for pairs of linear maps" $\mathcal{M}$, studied in [Barnsley and Harrington 85, Barnsley 93, Bousch 98, Indlekofer et al. 95, Solomyak 98, Bandt 02, Solomyak and Xu 03]. We introduce two other connectedness loci, $\mathcal{N}$ and $\mathcal{O}$; the latter one, associated with a linear map having a 2×2 Jordan block, coincides with $\Omega_2 \cup (-\Omega_2)$. Theorem 2.6 yields new information about "spikes," or "antennas"—peculiar features of $\mathcal{M}$ and $\mathcal{N}$. Our second main result is Theorem 2.11, in which we obtain explicit neighborhoods of (previously unknown) interior points of all three connectedness loci.

Let us make a few comments about the proofs. We use a C++ program, based on a modification of Bandt's algorithm from [Bandt 02], with rigorous estimates, to rule out double zeros in specific intervals. This program also indicates whether there is a possible root in the interval, and provides a polynomial that is the initial part of a power series in $\mathcal{B}$ with a double zero in the interval. Once such a polynomial is found, we use a simple argument (see Section 3) to prove the existence of the function. It is completely rigorous, and its application uses only *Mathematica* (or any similar package) to plot polynomials of degree up to ≈ 50. Thus, the lower estimate for α_2 is computer-assisted in a more substantial way than the upper estimate (which is just "*Mathematica*-assisted").

In order to show that $\widetilde{\alpha}_2 < 2^{-1/2} - 4\times 10^{-6}$ and obtain neighborhoods of interior points in the connectedness loci, we use a covering argument inspired by [Indlekofer et al. 95] and [Solomyak and Xu 03]. At one point we need to check that a certain set is covered by the union of 3^5 parallelograms, which we do using a computer.

The paper is organized as follows. In Section 2 we provide the background on iterated function systems and discuss the relation between zeros of functions in $\mathcal{B}$ and connectedness of self-affine fractals. We then state our results. In Section 3 we show how to find double roots close to a local minimum of a polynomial with certain properties. In Section 4 we establish the covering results and estimate $\widetilde{\alpha}_2$. In Section 5 we prove the existence of gaps in Ω_2 and estimate α_2. Section 6 is devoted to some variants and generalizations. Section 7 contains proofs of several auxiliary results.

2. PRELIMINARIES ON IFS AND STATEMENT OF RESULTS

An *iterated function system* (IFS) is a finite collection of (strict) contractions $\{f_1,\dots,f_m\}$ on a complete metric space. Given such a system, there is a unique nonempty compact set E satisfying $E=\bigcup_{i\leq m} f_i(E)$, called the *attractor* of the IFS; see [Hutchinson 81]. We consider only IFS on $\mathbb{R}^d$ of the form $\{T_i\mathbf{x}+\mathbf{b}_i\}_{i\leq m}$, where T_i are linear maps and $\mathbf{b}_i \in \mathbb{R}^d$. Their attractors are called *self-affine*. For the maps to be contractive (in some norm) it is necessary and sufficient that all the eigenvalues of T_i be less than 1 in absolute value.

We investigate when attractors are connected in the simplest case $m=2$. The following result is well known.

Proposition 2.1. [Hata 85] *The attractor E of an IFS $\{f_1,f_2\}$ is connected if and only if $f_1(E)\cap f_2(E)\neq\varnothing$.*

Remark 2.2. Of course, the "only if" direction is obvious.

For IFS of two affine maps there is a simple sufficient condition for connectedness. We can assume $\mathbf{b}_1=\mathbf{0}$ without loss of generality, by making a change of variable.

Lemma 2.3. (Folklore.) *Let $\{T_1\mathbf{x}, T_2\mathbf{x}+\mathbf{b}\}$ be an IFS of contracting affine maps, such that $\max\{\|T_1\|,\|T_2\|\}<1$ in some operator norm, and $|\det(T_1)|+|\det(T_2)|\geq 1$. Then the attractor is connected.*

We include a proof for completeness (see Section 7).

Next we specialize even more, assuming that $T=T_1=T_2$, and state a criterion for connectedness in terms of zeros of power series.

Let $E=E(T,\mathbf{b})$ be the attractor of the IFS $\{T\mathbf{x},T\mathbf{x}+\mathbf{b}\}$, i.e., the unique nonempty compact set in $\mathbb{R}^d$ satisfying

$$E = TE \cup (TE+\mathbf{b}). \tag{2–1}$$

Observe that

$$E(T,\mathbf{b}) = \Big\{\sum_{n=0}^{\infty} a_n T^n \mathbf{b} : \ a_n \in \{0,1\}\Big\}, \tag{2–2}$$

since the right-hand side is well defined and satisfies (2–1).

We can assume, without loss of generality, that $\mathbf{b}$ is a cyclic vector for T, that is, $H := \text{Span}\{T^k\mathbf{b} : k \geq 0\} = \mathbb{R}^d$. Indeed, otherwise we can replace T by the restriction of T to H and consider the corresponding IFS on H.

Combining Proposition 2.1 and (2–2) easily implies the following criterion, which is known, at least in special cases. Recall that $\mathcal{B}$ is defined in (1–1).

Proposition 2.4. *Let T be a linear contraction with (possibly complex) eigenvalues λ_j, for $j = 1, \dots, m$, having algebraic multiplicities $k_j \geq 1$ and geometric multiplicities equal to one. Let $\mathbf{b}$ be a cyclic vector for T. Then $E(T, \mathbf{b})$ is connected if and only if there exists $f \in \mathcal{B}$ such that*

$$f(\lambda_j) = \cdots = f^{(k_j-1)}(\lambda_j) = 0, \quad j = 1, \dots, m. \quad (2\text{–}3)$$

In particular, connectedness does not depend on $\mathbf{b}$.

Since this is a key statement relating connectedness of self-affine sets to zeros of power series, we include a short proof in Section 7. Combining Proposition 2.4 and Lemma 2.3 yields the following result. We denote by $\mathbb{D}$ the open unit disk in the complex plane.

Corollary 2.5. *Let $\Lambda = \{\lambda_1, \dots, \lambda_m\} \subset \mathbb{D}$ and let $k_j = k(\lambda_j) \geq 1$ be such that for any nonreal $\lambda \in \Lambda$, we have $\overline{\lambda} \in \Lambda$ and $k(\overline{\lambda}) = k(\lambda)$. If $\prod_{j=1}^m |\lambda_j|^{k_j} \geq \frac{1}{2}$, then there exists $f \in \mathcal{B}$ having zeros at λ_j of multiplicity $\geq k(\lambda_j)$ for $j = 1, \dots, m$.*

In particular, we obtain that for any $k \geq 1$, every $\lambda \in [2^{-1/k}, 1)$ is a zero of multiplicity $\geq k$ for some power series in $\mathcal{B}$. In [Beaucoup et al. 98, Section 3] it is asked whether there exist power series (or polynomials) with coefficients in $\{-1, 0, 1\}$ having a kth-order root strictly inside the unit circle for arbitrary k. Corollary 2.5 answers the question for power series in a strong quantitative way, but the question for polynomials is much harder and remains open.

From now on, we restrict ourselves to the case $d = 2$. Applying an invertible linear transformation as a conjugacy, we can assume without loss of generality that T is one of the following:

$$\text{(i)}\ \ T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \qquad \text{(ii)}\ \ T = \begin{pmatrix} \gamma & 0 \\ 0 & \lambda \end{pmatrix},$$

$$\text{(iii)}\ \ T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

FIGURE 1. Part of the connectedness locus $\mathcal{N}$.

where a, b, λ, γ are real, $a^2 + b^2 < 1$, $b \neq 0$, $|\lambda|, |\gamma| < 1$, and $\gamma \neq \lambda$. Each of the cases leads to a set that we call the *connectedness locus* for the corresponding family of self-affine sets. Namely, we consider the sets

$$\begin{aligned}
\mathcal{M} &:= \{z = a + ib \in \mathbb{D} : \ \exists f \in \mathcal{B}, \ f(z) = 0\}, \\
\mathcal{N} &:= \{(\gamma, \lambda) \in (-1,1)^2 : \ \exists f \in \mathcal{B}, \ f(\gamma) = f(\lambda) = 0\}, \\
\mathcal{O} &:= \{\lambda \in (-1,1) : \ \exists f \in \mathcal{B}, \ f(\lambda) = f'(\lambda) = 0\}.
\end{aligned}$$

Thus, $\mathcal{M}, \mathcal{N}, \mathcal{O}$ are essentially the sets of parameters for which the attractors in cases (i), (ii), (iii) respectively are connected. (It is natural to allow $\gamma = \lambda$ in $\mathcal{N}$ and $b = 0$ in $\mathcal{M}$ to ensure that the sets are relatively closed in $\mathbb{D}$.)

By Lemma 2.3,

$$\begin{aligned}
\mathcal{M} \supset \mathcal{M}_t &:= \{\lambda \in \mathbb{D} : \ |\lambda| \geq 2^{-1/2}\}, \\
\mathcal{N} \supset \mathcal{N}_t &:= \{(\gamma, \lambda) \in (-1,1)^2 : \ |\gamma\lambda| \geq 1/2\}, \\
\mathcal{O} \supset \mathcal{O}_t &:= \{\lambda \in (-1,1) : \ |\lambda| \geq 2^{-1/2}\}.
\end{aligned}$$

We refer to $\mathcal{M}_t, \mathcal{N}_t, \mathcal{O}_t$ as "trivial parts" of the corresponding sets.

The set $\mathcal{M}$ has been studied by several authors; see [Barnsley and Harrington 85, Barnsley 93, Bousch 98, Solomyak 98, Bandt 02, Solomyak and Xu 03]. In particular, Bousch [Bousch 98] proved that $\mathcal{M}$ is connected and locally connected. The set $\mathcal{N}$ has not been studied as much, although partial results are obtained in [Solomyak 05], where it is shown that a large "chunk" of $\mathcal{N}$ is connected (all of $\mathcal{N}$ is conjectured to be connected). An approximation to $\mathcal{N} \cap (0,1)^2$ is depicted in Figure 1, with the nontrivial part shown in black. The picture is created with a program of C. Bandt; note that the visible disconnected pieces are a computing artefact.

By symmetry, we have $\mathcal{O} = \Omega_2 \cap (-\Omega_2)$. The set Ω_2, defined in (1–2), is our main object of study. Since $\mathcal{B}$ is a normal family, Ω_2 is relatively closed in $(0,1)$, so there exists $\alpha_2 = \min \Omega_2$.

Theorem 2.6. *We have the following:*

(i) $\alpha_2 \in (0.6684755, 0.6684757)$;

(ii) Ω_2 *is disconnected. In fact, the intervals* I_j, *for* $j \leq 5$, *lie in distinct components of* $(0,1) \setminus \Omega_2$, *where* $I_j = .668 + 10^{-3} I'_j$, *and* $I'_1 = (.478, .489)$, $I'_2 = (.632, .653)$, $I'_3 = (1.282, 1.306)$, $I'_4 = (1.327, 1.333)$, $I'_5 = (1.343, 1.352)$.

This theorem is proved in Section 5, using results of Section 3 for part of the proof.

Remark 2.7. Numerical evidence suggests that there are infinitely many components of $(0,1)\setminus\Omega_2$. We do not have a proof of that. The topological structure of Ω_2 appears to be very complicated. It can be proved rigorously that the five "gaps" above are the largest.

Remark 2.8. Let $\widetilde{\alpha}_2 := \sup((0,1) \setminus \Omega_2)$. We have $[2^{-1/2}, 1) \subset \Omega_2$ by Lemma 2.3, and hence $\widetilde{\alpha}_2 \leq 2^{-1/2}$. We are able to show in fact that $\widetilde{\alpha}_2 \leq 2^{-1/2} - 4 \times 10^{-6}$ (see Theorem 2.11 below). Our rigorous numerical results also yield $\widetilde{\alpha}_2 \geq 0.669355$, and it seems that $\widetilde{\alpha}_2 \leq .67$, but we do not have a proof.

Returning to the sets $\mathcal{M}$, $\mathcal{N}$, we note that they are related to $\mathcal{O}$. In fact, we have the following, where $F \subset \mathbb{R}$ is denoted by $\mathrm{Diag}(F) := \{(\lambda, \lambda) : \lambda \in F\}$.

Lemma 2.9.

(i) $\mathrm{clos}(\mathcal{M} \setminus \mathbb{R}) \cap \mathbb{R} \subset \mathcal{O}$;

(ii) $\mathrm{clos}(\mathcal{N} \setminus \mathrm{Diag}(\mathbb{R})) \cap \mathrm{Diag}(\mathbb{R}) \subset \mathrm{Diag}(\mathcal{O})$.

This follows by an easy compactness argument. See [Solomyak 98, Lemma 2.5] for the proof of (i); (ii) is proved in Section 7.

As a consequence of Lemma 2.9, the pictures of $\mathcal{M}$ near the real axis reveal something about the structure of $\mathcal{O}$. In fact, Figure 7 in [Bandt 02] served as an inspiration for our work. Lemma 2.9 also explains an interesting feature of $\mathcal{M}$ and $\mathcal{N}$, namely the "antennas." The antenna for $\mathcal{M}$ (let us denote it by $\Gamma(\mathcal{M})$) is defined as the connected component of $\left[\frac{1}{2}, 1\right) \setminus \mathrm{clos}(\mathcal{M} \setminus \mathbb{R})$ containing $\frac{1}{2}$ (there is obviously a symmetric antenna on the negative real axis); it was first discovered in [Barnsley and Harrington 85] and studied in [Solomyak 98, Bandt 02]. Similarly, the antenna for $\mathcal{N}$, denoted by $\Gamma(\mathcal{N})$, may be defined as the connected component of $\mathrm{Diag}([\frac{1}{2}, 1)) \setminus \mathrm{clos}(\mathcal{N} \setminus \mathrm{Diag}(\mathbb{R}))$ containing $(\frac{1}{2}, \frac{1}{2})$. Bandt [Bandt 02] noted that the "tip of the antenna" $\sup \Gamma(\mathcal{M})$ is the infimum of the set of double zeros of power series in $\mathcal{B}$ with infinitely many coefficients not equal to $+1$. It is not hard to show that $\sup \Gamma(\mathcal{N})$ is the infimum of the set of double zeros of power series in $\mathcal{B}$ with infinitely many coefficients not equal to -1. It seems very likely that $\sup \Gamma(\mathcal{M}) = \sup \Gamma(\mathcal{N}) = \alpha_2$, but we do not know how to prove this. However, as a byproduct of our investigation, we obtain the following corollary, proved in Section 5.

Corollary 2.10. *We have* $\sup \Gamma(\mathcal{M})$, $\sup \Gamma(\mathcal{N}) \in (0.6684755, 0.6684757)$.

Our second main result concerns nontrivial interior points of the connectedness loci. In [Indlekofer et al. 95] and [Solomyak and Xu 03] some chunks of interior points of $\mathcal{M}\setminus\mathcal{M}_t$ were found. Although numerical experimentation indicates that $\mathcal{N}\setminus\mathcal{N}_t$ and $\mathcal{O}\setminus\mathcal{O}_t$ also have nonempty interior, this had not been proved rigorously before.

Theorem 2.11. *Let* $\eta = 4 \times 10^{-6}$. *Then*

$$\mathcal{N} \supset \Delta_1 = \Big\{(\gamma, \lambda) \in (0,1)^2 : \ |\gamma - 2^{-1/2}|, \ |\lambda - 2^{-1/2}| < \eta\Big\}, \tag{2–4}$$

$$\mathcal{M} \supset \Delta_2 = \Big\{\lambda \in \mathbb{D} : \ |\lambda - 2^{-1/2}| < 3\eta/4\Big\}, \tag{2–5}$$

$$\mathcal{O} \supset \Delta_3 = \Big\{\lambda \in (0,1) : \ |\lambda - 2^{-1/2}| < \eta\Big\}. \tag{2–6}$$

This theorem is proved in Section 5. The value of η is very small and clearly not optimal, but nevertheless it is an explicit constant.

3. EXISTENCE OF DOUBLE ROOTS I

We denote by $\mathcal{B}_n$ the subset of $\mathcal{B}$ consisting of polynomials having degree less than or equal to n.

Let us say that (P, n, a, b) is *good* if $P \in \mathcal{B}_n$, $0.5 < a < b < 1$ (in reality, we will consider only $0.66 < a < b < 0.68$),

$$P(a) > \frac{a^{n+1}}{(1-a)}, \quad P(b) > \frac{b^{n+1}}{(1-b)}, \tag{3–1}$$

$P(x) > 0$ for all $x \in [a, b]$, and

$$\exists\, x \in (a,b) : \ P(x) < \frac{x^{n+1}}{(1-x)}. \tag{3–2}$$

Lemma 3.1. *Suppose that (P, n, a, b) is good. Let $Q(x) = P(x) - x^m$, where m is the minimal integer greater than or equal to $n + 1$ such that $Q(x) > 0$ on $[a, b]$. Then (Q, m, a, b) is good.*

Proof: It is clear that $Q \in \mathcal{B}_m$. We easily check that

$$Q(a) = P(a) - a^m > \frac{a^n}{(1-a)} - a^m > \frac{a^{m+1}}{(1-a)},$$

since $a \in (0, 1)$, and similarly, $Q(b) > b^{m+1}/(1 - b)$. It remains to check the last condition. Either $m = n + 1$, in which case we note that

$$Q(x) = P(x) - x^{n+1} < \frac{x^{n+1}}{(1-x)} - x^{n+1} = \frac{x^{n+2}}{(1-x)},$$

or $m > n + 1$, in which case there exists $t \in (a, b)$ such that $P(t) - t^{m-1} \leq 0$. Then

$$\begin{aligned} Q(t) = P(t) - t^m &= (P(t) - t^{m-1}) + (t^{m-1} - t^m) \\ &\leq t^{m-1} - t^m < \frac{t^{m+1}}{(1-t)}, \end{aligned}$$

since t is greater than $\frac{1}{2}$. Clearly $t \neq a, b$, and the proof is complete. □

Corollary 3.2. *Suppose that (P, n, a, b) is good. Then there exists $f \in \mathcal{B}$ such that P is the initial part of f, and f has a double zero in (a, b).*

Proof: Iterating Lemma 3.1, we obtain a sequence of polynomials $Q_j \in \mathcal{B}_{m_j}$ such that $m_j \to \infty$ and Q_j is the initial part of Q_{j+1} for all j. Then $Q_j \to f \in \mathcal{B}$ uniformly on compact subsets of the unit disk in the complex plane. Since Q_j are all positive on $[a, b]$, we have that $f(x) \geq 0$ on $[a, b]$. Since $Q_j(x_j) < x_j^{m_j}$ for some $x_j \in (a, b)$, we have that $\min_{[a,b]} f = 0$. On the other hand, (3–1) implies that any function in $\mathcal{B}$ with initial part P is strictly positive at a and b. It follows that f has a zero in (a, b) of order at least two. □

Remark 3.3. The function $f(x) = 1 + \sum_{j=1}^{\infty} a_j x^j$ obtained in Corollary 3.2 has the property that $a_j \in \{0, -1\}$ for all $j \geq n + 1$ and there are infinitely many 0's and -1's. The only thing to check is that there are infinitely many 0's; the rest is obvious by the construction in Lemma 3.1. Suppose that $a_j = -1$ for all $j \geq N$ for some $N \in \mathbb{N}$. Then at some step in our construction we have good (Q, N, a, b), which implies $Q(x) < x^{N+1}/(1 - x)$ for some $x \in (a, b)$. Then $f(x) = Q(x) - x^{N+1}/(1 - x) < 0$, which is a contradiction.

Corollary 3.2 is not very efficient numerically, since it allows us to find a double zero with an error of order $b^{n/2}$. The next statement shows that this can be improved considerably.

Corollary 3.4. *Assume that (P, n, a, b) is good, $b \leq 0.68$, and $n > 10$. Further, suppose that P' has a zero at $y \in (a, b)$, and*

$$P''(x) > C'' \quad \text{for all } x \in [a, b]. \tag{3–3}$$

Then there exists $f \in \mathcal{B}$ such that P is the initial part of f, and f has a double root in the interval $(y - \eta, y + \eta)$, where

$$\eta = \frac{1 + n(1-b)}{C''(1-b)^2}\, b^n < \frac{4}{C''}(n+1)b^n.$$

Proof: From Corollary 3.2 we know that there exists $f \in \mathcal{B}$ such that P is its initial part, and f has a double root in (a, b); let r be this root. Then $P'(y) = f'(r) = 0$, whence, using the intermediate value theorem,

$$|P'(r) - f'(r)| = |P'(r) - P'(y)| > C''|r - y|. \tag{3–4}$$

Note, however, that

$$\begin{aligned} |P'(r) - f'(r)| \leq \sum_{i=n+1}^{\infty} i r^{i-1} &= \frac{(1 + n(1-r))r^n}{(1-r)^2} \\ &< \frac{1 + n(1-b)}{(1-b)^2}\, b^n. \end{aligned} \tag{3–5}$$

Combining (3–4) and (3–5) yields the corollary. □

Example 3.5. Consider

$$\begin{aligned} P(x) = 1 &- x^1 - x^2 - x^3 + x^4 + x^6 + x^7 + x^9 + x^{10} \\ &+ x^{12} + x^{13} + x^{14} + x^{16} + x^{17} + x^{18} + x^{20} \\ &+ x^{21} + x^{23} + x^{24} + x^{25} + x^{26} + x^{27} + x^{28} \\ &+ x^{29} + x^{31} + x^{32} + x^{33} + x^{34} + x^{35} + x^{36} \\ &+ x^{37} - x^{38} + x^{39} + x^{40} + x^{41} + x^{42} + x^{43} \\ &- x^{44} + x^{45} - x^{46} - x^{47} + x^{48} - x^{49} - x^{50} \\ \in \mathcal{B}_{50}. \end{aligned}$$

Let $a = 0.668470$, $b = 0.668482$. We claim that $(P, 50, a, b)$ is good. We checked that $P(x) > 0$ on $[a, b]$ by plotting the graph of P on $[a, b]$ in *Mathematica*; see Figure 2.

We have

$$P(a)(1-a)a^{-51} \approx 1.03199, \quad P(b)(1-b)b^{-51} \approx 1.06665,$$

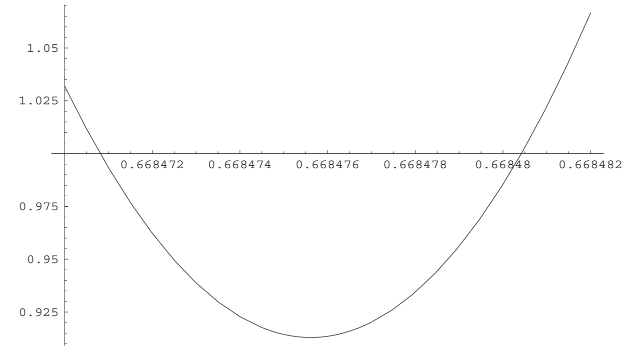

FIGURE 2. Checking the existence of double roots: the polynomial P was obtained with the help of a C++ program. We plotted $P(x)(1-x)/x^{51}$ on the interval $(0.668470, 0.668482)$; it is clear from the picture that $(P, 51, 0.668470, 0.668482)$ is good.

which satisfies (3–1). On the other hand,

$$P(r)(1-r)r^{-51} \approx 0.912958, \quad \text{for } r = 0.6684756,$$

which satisfies (3–2). Thus, Corollary 3.2 applies, and we have a double zero in $[0.668470, 0.668482]$.

Using Corollary 3.4, we obtain a more precise estimate. We have

$$P(r + 2 \times 10^{-8}) - P(r) \approx 8.89847 \times 10^{-15} > 0,$$
$$P(r - 2 \times 10^{-8}) - P(r) \approx 2.04733 \times 10^{-15} > 0,$$

which implies that there exists

$$y \in (0.66847558, 0.66847562)$$

such that $P'(y) = 0$. We checked that $P''(x) > 20$ on $[a, b]$ by plotting the graph of P'' on $[a, b]$ in *Mathematica*. Thus, Corollary 3.4 applies. Since $51b^{50} \cdot (4/20) < 2 \times 10^{-8}$, we obtain that there exists $f \in \mathcal{B}$ with a double zero in $(0.66847556, 0.66847564)$.

4. EXISTENCE OF DOUBLE ROOTS II AND CONNECTEDNESS LOCI

Here we prove Theorem 2.11. The proof is based on several lemmas.

Lemma 4.1. *Let T be a contracting linear map on $\mathbb{R}^2$, and let $\mathbf{b} \in \mathbb{R}^2$ be a vector such that $T^k\mathbf{b}$, $k \geq 0$, span $\mathbb{R}^2$. Denote the attractor of $\{T\mathbf{x}, T\mathbf{x} + \mathbf{b}\}$ by E. Let $\mathbf{v}$ be a point of the form*

$$\mathbf{v} = \sum_{i=1}^{k} a_i T^{-i}\mathbf{b},$$

where $a_i \in \{-1, 0, 1\}$ and $a_k = 1$. Assume that there exists a set U containing $\mathbf{v}$ and $j \in \mathbb{N}$ such that

$$U \subset \bigcup_{u \in \{-1,0,1\}^j} (T + u_1\mathbf{b}) \circ \cdots \circ (T + u_j\mathbf{b})(U). \quad (4\text{–}1)$$

Then K is connected.

Proof: There are similar results in [Indlekofer et al. 95], for example, but we sketch a proof for completeness. Let $\widetilde{E}$ denote the attractor of the IFS $\{T\mathbf{x} - \mathbf{b}, T\mathbf{x}, T\mathbf{x} + \mathbf{b}\}$. Condition (4–1) implies that U is contained in $\widetilde{E}$. In particular, $\mathbf{v} \in \widetilde{E}$. Recalling the form of $\mathbf{v}$, we get that

$$\sum_{i=1}^{k} a_i T^{-i}\mathbf{b} = \sum_{i=0}^{\infty} c_i T^i \mathbf{b}, \quad \text{for some } c_i \in \{-1, 0, 1\}.$$

Applying T^k from the left on both sides, we obtain a power series $f \in \mathcal{B}$ such that $f(T)\mathbf{b} = 0$. Write $f = f_1 - f_2$, where f_i has coefficients $0, 1$ only. In particular, since $a_k \neq 0$, the power series f_1 and f_2 have different constant terms, whence

$$f_1(T)\mathbf{b} = f_2(T)\mathbf{b} \in TE \cap (TE + \mathbf{b}).$$

By Proposition 2.1, the set E is connected. □

Lemma 4.2. *Let*

$$T = \begin{pmatrix} 2^{-1/2} & 0.7 \\ 0 & 2^{-1/2} \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let also $\mathbf{p} = (-2.10, 0.20)$ and $\mathbf{q} = (4.90, 2.45)$. Denote by U the open parallelogram with the vertices $\pm\mathbf{p}, \pm\mathbf{q}$, and denote by V the open parallelogram with the vertices $\pm 0.95\mathbf{p}, \pm 0.95\mathbf{q}$. Then

$$\mathbf{v} = T^{-1}\mathbf{b} - T^{-2}\mathbf{b} - T^{-3}\mathbf{b} - T^{-4}\mathbf{b} + T^{-5}\mathbf{b}$$
$$\approx (2.95837, 1.75736) \in V,$$

and

$$U \subset \bigcup_{u \in \{-1,0,1\}^5} (T + u_1\mathbf{b}) \circ \cdots \circ (T + u_5\mathbf{b})(V). \quad (4\text{–}2)$$

Proof: This is the part of the proof that is computer-assisted. The parallelograms U and V and the vector $\mathbf{v}$ were obtained through experimentation with *Mathematica.*

The coordinates of $\mathbf{v}$ in the base $\{\mathbf{p}, \mathbf{q}\}$ are $(\alpha, \beta) \approx (0.223, 0.699)$. Since $|\alpha| + |\beta| < 0.95$, we get that $\mathbf{v} \in V$. We checked (4–2) rigorously as follows: we changed

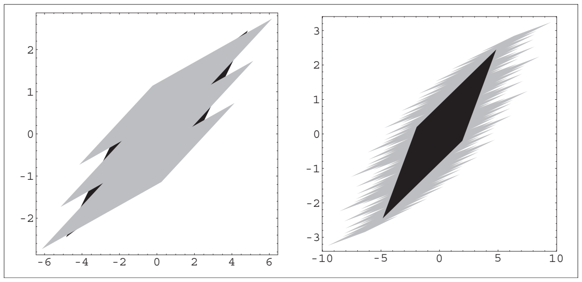

FIGURE 3. The parallelogram U is pictured in black. On the left, the gray figure is $(TV-\mathbf{b})\cup(TV)\cup(TV+\mathbf{b})$; notice that U comes close to being covered by the iterates of V already in the first step. However, five iterations steps are needed in order to get a complete covering. This is illustrated by the picture on the right; there, the gray figure corresponds to the right-hand side in (4–2).

coordinates so that the parallelograms on the right-hand side of (4–2) become squares with sides parallel to the axes of length 2; let $\widetilde{\mathcal{V}}$ be the collection of such squares. In the new coordinate system, U becomes a parallelogram $\widetilde{U}$. We covered $\widetilde{U}$ by a grid of squares of size η for some small η ($\eta = \frac{1}{25}$ worked, and close to 10^5 small squares were needed to cover), and verified that each of those small squares is contained in an element of $\widetilde{\mathcal{V}}$.[1]

Figure 3 depicts the situation graphically. □

Proof of Theorem 2.11: Let T, $\mathbf{b}$, $\mathbf{p}$, $\mathbf{q}$, U, V be as in Lemma 4.2. Using Lemma 4.2 we see that for sufficiently small η,

$$\|T'-T\| < \eta \quad \Longrightarrow \quad V \subset (T'+u_1\mathbf{b})\circ\cdots\circ(T'+u_5\mathbf{b})(V). \tag{4–3}$$

The bulk of the proof will consist in obtaining an explicit value of η such that (4–3) holds.

Lemma 4.3. *The condition (4–3) holds for* $\eta = 4\times 10^{-6}$.

The proof of the lemma is straightforward, but technical, so we postpone it till Section 7. As mentioned above, the existence of such positive η is obvious.

An immediate consequence of (4–3) and Lemmas 4.1 and 4.2 is

$$\|T'-T\| < \eta \tag{4–4}$$
$$\Longrightarrow \text{ the attractor of } \{T'\mathbf{x}, T'\mathbf{x}+\mathbf{b}\} \text{ is connected.}$$

It is convenient to use the ℓ^∞ norm in $\mathbb{R}^2$ and the corresponding operator matrix norm; recall that the latter is computed as the maximum of ℓ^1 norms of the rows. Let

$$M_1(\gamma,\lambda) = \begin{pmatrix} \gamma & 0.7 \\ 0 & \lambda \end{pmatrix}, \quad \gamma \neq \lambda,$$
$$M_2(r,\varepsilon) = \begin{pmatrix} \rho & 0.7 \\ -\varepsilon & \rho \end{pmatrix}, \quad \rho,\varepsilon > 0,$$
$$M_3(\lambda) = \begin{pmatrix} \lambda & 0.7 \\ 0 & \lambda \end{pmatrix}.$$

Note that $M_1(\gamma,\lambda)$ is conjugate to the diagonal matrix with eigenvalues γ and λ; likewise, $M_3(\lambda)$ is conjugate to the standard Jordan block with eigenvalue λ. Note also that

$$\|M_1(\gamma,\lambda)-T\| = \max\left(|2^{-1/2}-\gamma|, |2^{-1/2}-\lambda|\right);$$
$$\|M_3(\gamma,\lambda)-T\| = |2^{-1/2}-\lambda|.$$

From this, Lemma 4.3, and (4–4) we obtain (2–4) and (2–6).

It remains to consider the complex-eigenvalue case. The matrix $M_2(\rho,\varepsilon)$ has eigenvalues $\rho \pm i\sqrt{0.7\varepsilon}$. Let λ be a nonreal complex number such that $|\lambda - 2^{-1/2}| < \frac{3}{4}\eta$; without loss of generality assume that $\mathrm{Im}(\lambda) > 0$, and write $\lambda = \rho + i\sqrt{0.7\varepsilon}$. We have that $|\rho - 2^{-1/2}| < \frac{3}{4}\eta$ and

$$\sqrt{0.7\varepsilon} < \frac{3}{4}\eta \;\Rightarrow\; \varepsilon < \frac{9}{16\times 0.7}\eta^2 < \frac{1}{4}\eta.$$

Therefore $|\rho - 2^{-1/2}| + \varepsilon < \eta$, and from this we conclude that $\|M_2(\rho,\varepsilon)-T\| < \eta$. Invoking (4–4) again, this shows that (2–5) is satisfied, which completes the proof. □

Remark 4.4. Although the proof of Theorem 2.11 is inspired by analogous results for self-similar sets that appeared in [Indlekofer et al. 95] and [Solomyak and Xu 03], the more complicated geometry of self-affine sets introduces some additional difficulties. For example, the use of a computer first to find the sets U, V and the vector $\mathbf{v}$ and then to check that the covering property holds becomes essential (in the self-similar case, some attractors are actually rectangles, which allows one to do a purely algebraic analysis in certain regions; see [Solomyak and Xu 03]). The chunks of interior points of $\mathcal{M}\setminus\mathcal{M}_t$ that were found in those papers are located away from the real line.

Remark 4.5. Let $K_{\gamma,\lambda}$ be the attractor of $\{T_{\gamma,\lambda}, T_{\gamma,\lambda}+(1,1)\}$, where $T_{\gamma,\lambda}$ is a diagonal map with eigenvalues γ,λ. In [Shmerkin 06] the almost sure Hausdorff dimension of $K_{\gamma,\lambda}$ was found in the region $(0,1)^2\setminus\mathcal{N}_t$. The

[1] Full C++ code is available as supplemental material online (http://www.expmath.org/expmath/volumes/15/15.4/shmerkin/covering.cpp).

result was new only in the region $\mathcal{N}\backslash\mathcal{N}_t$; hence Theorem 2.11 makes that result effective, by showing that $\mathcal{L}_2(\mathcal{N}\backslash\mathcal{N}_t) > 0$.

Remark 4.6. The value of η found in Lemma 4.3 is extremely small, but graphical experimentation suggests that the same covering argument, even with the same covering, works for a large range of parameters. We believe that it should be possible to extend the result to show that actually $(0.7, 2^{-1/2}) \subset \Omega_2$ (we recall that computer results suggest that indeed $(0.67, 2^{-1/2}) \subset \Omega_2$, but this appears harder to prove rigorously).

5. EXISTENCE OF GAPS

We describe an algorithm that we use to rigorously prove the existence of gaps in the set Ω_2. As a byproduct, the results of Section 3 yield bounds on how large the gaps may be. In practice, we are able to obtain an accurate description of the set Ω_2 up to an error of 10^{-8}.

Our algorithm is based on Bandt's algorithm to study the Mandelbrot set for pairs of linear maps [Bandt 02]. The idea is the following: assume that $f \in \mathcal{B}$ has a double zero in $(a,b) \subset \left(\frac{1}{2}, 1\right)$, and let $P_n \in \mathcal{B}_n$ be the initial part of f up to exponent n. Then, letting r be the double root,

$$|P_n(r)| = |P_n(r) - f(r)| \leq \sum_{i=n+1}^{\infty} r^i < \frac{b^{n+1}}{1-b};$$

$$|P_n'(r)| = |P_n'(r) - f'(r)| \leq \sum_{i=n+1}^{\infty} i r^{i-1} < \frac{(1+n(1-b))b^n}{(1-b)^2}.$$

Hence, using that $a < r < b$ and the intermediate value theorem,

$$\begin{aligned} |P_n(b)| &\leq |P_n(r)| + \|P_n'\|_{L^\infty(r,b)}(b-r) \\ &< \frac{b^{n+1}}{1-b} + \frac{b-a}{(1-b)^2}; \\ |P_n'(b)| &\leq |P_n'(r)| + \|P_n''\|_{L^\infty(r,b)}(b-r) \\ &< \frac{(1+n(1-b))b^n}{(1-b)^2} + \frac{b-a}{2(1-b)^3}. \end{aligned} \tag{5–1}$$

Conditions (5–1) are easily checkable. If for some n and (a,b), at least one of them fails for all $P \in \mathcal{B}_n$, then $(a,b) \cap \Omega_2 = \varnothing$.

We discuss some of the details of our implementation.[2] We used floating-point arithmetic in order to get practical performance (in principle, one could use exact rational arithmetic instead). To keep the algorithm rigorous we computed the theoretical floating-point error of our calculations and added a corresponding error term to the inequalities (5–1). First, we note that if the second inequality in (5–1) holds, then the right-hand side provides an upper bound for P_n' on all of (a,b). Hence one can use this much sharper estimate for $\|P_n'\|_{L^\infty(r,b)}$ in the first equation. Second, in order to reduce the number of arithmetic operations and, mainly, avoid dealing with very small numbers and the consequent loss of precision, we multiplied both sides of (5–1) by b^{-n}. Thus the numerical tests that we used are

$$\begin{aligned} b^{-n}|P_n'(b)| &< \frac{1+n(1-b)}{(1-b)^2} + \frac{(b-a)b^{-n}}{2(1-b)^3} + \eta \\ &:= K(n,a,b); \\ b^{-n}|P_n(b)| &< \frac{b}{1-b} + (b-a)K(n,a,b) + \eta. \end{aligned} \tag{5–2}$$

Here η is the error term. In practice, taking $\eta = 10^{-14}$ suffices. This was calculated based on the IEEE floating-point standard. We used the MinGW compiler on a Windows XP platform.

In order to make the algorithm efficient, we exploited the tree structure of $\mathcal{B}_n$. The basic routine takes as arguments an interval (a,b), a depth d, and a polynomial $P \in \mathcal{B}_n$. The routine returns a Boolean value, which we denote by $C((a,b),d,P)$. This value indicates whether the inequalities (5–2) are satisfied for at least one polynomial of degree at most $n+d$ with initial part P. Hence if $C((a,b),d,1)$ returns *false* for some d, we must have $(a,b) \cap \Omega_2 = \varnothing$. The structure of the routine is as follows:

(i) Check (5–2) for P and the interval (a,b). If any of the inequalities fails to hold, return *false* and exit.

(ii) If $d = 0$, then return *true* and exit.

(iii) For $i = -1$, 0, and 1, run $C((a,b), d-1, P(x) + ix^{n+1})$. If any of these returns *true*, then return *true* and exit.

(iv) Return *false*.

Assuming that $C((a,b),d,1)$ returns *true*, this routine easily produces a polynomial of degree d for which (5–2)

[2] Full C++ code can be found online (http://www.expmath.org/expmath/volumes/15/15.4/shmerkin/doubleroots.cpp).

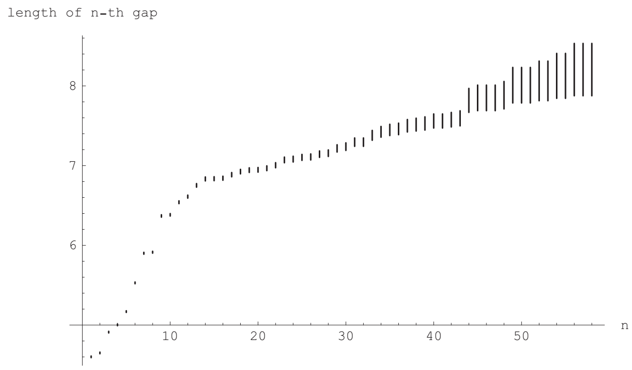

FIGURE 4. This figure shows the distribution of the largest gaps in the set Ω_2. The negatives of the base-10 logarithms of the length of the gaps are contained in the vertical segments (it is possible to check this rigorously based on the information provided by the algorithm, but we have not verified all the details, so the graph is essentially heuristic).

holds (this is done by keeping track of which i produces a *true* in Step 5 of the routine). This polynomial can in turn be used to show that there is a double root near b using the results of Section 3.

Note that in order to use a large depth one needs to run the algorithm on very small intervals, due to the presence of the term $(b-a)b^{-n}$ on the right-hand side of (5–2). Even on a standard desktop PC, it took less than three hours to scan the interval $(0.66847, 0.66936)$ for gaps using a grid of 10^7 subintervals and running the main procedure on each. Figure 4 summarizes our findings. We plotted the logarithms of the lengths of the gaps we found, as well as the lengths of the complementary intervals. Although we do not know how to prove it, we believe that at least the few largest of those pieces that the algorithm did not rule out contain intervals of the set Ω_2 of approximately the same length.

We finish this section by explaining how we verified the assertions of Theorem 2.6.

Proof of Theorem 2.6: We used the algorithm to show that $(0.5, 0.6684755)$ and $I_j\,(1 \le j \le 5)$ are contained in $(0,1)\backslash\Omega_2$. We did not need to go deeper than 50 iterations for this. For example, we subdivided the interval I_1 into 110 intervals (a_i, b_i) of length 10^{-6} and ran $C((a_i, b_i), 40, 1)$ for each i. The routine returned a *false* value for all, which implies that $I_1 \in (0,1)\backslash\Omega_2$.

It follows from Example 3.5 that $\alpha_2 < 0.6684757$. To check that the intervals I_j lie in different connected components of $(0,1)\backslash\Omega_2$ we used the algorithm to produce suitable polynomials, and then followed the scheme of Example 3.5 to show the existence of points $\theta_j (1 \le j \le 4)$, $\theta_j \in (\max I_j, \min I_{j+1}) \cap \Omega_2$.

j	θ'_j	P_j
1	0.668550	11̄1̄1̄101101101110111111̄0111111̄1̄11̄1̄01111111111111̄1̄11̄1̄
2	0.668900	11̄1̄1̄10110110111111̄1111̄1111̄11̄1̄111̄1̄111̄11111̄101̄1̄01̄01̄1̄1̄01̄01̄1̄1̄
3	0.669310	11̄1̄1̄1011011100111010111101111111̄0001̄1̄1̄1̄1̄1̄1̄1̄1̄1̄100011
4	0.669336	11̄1̄1̄10110111001110110011111111̄0110101011111̄111111111̄1̄0

TABLE 1. Numbers that lie within 10^{-7} of Ω_2.

Table 1 summarizes our findings. The symbols on the right represent the coefficients of the polynomials P_j; $\bar{1}$ corresponds to the coefficient -1. The numbers θ'_j are within 10^{-7} of a double root; more precisely, for each j there exist $f_j \in \mathcal{B}$ with initial part P_j, and θ_j a double root of f_j such that $|\theta_j - \theta'_j| < 10^{-7}$. □

Proof of Corollary 2.10: The double zero y obtained in Example 3.5 is a root of $f \in \mathcal{B}$ that has infinitely many 0's among the coefficients, by Remark 3.3. Replacing $a_j = 0$ by $+1$ for j large yields a function in $\mathcal{B}$ with a complex zero close to y. On the other hand, replacing $a_j = 0$ by -1 for j large yields a function in $\mathcal{B}$ with two real zeros close to y. Thus, $y \in \text{clos}(\mathcal{M} \setminus \mathbb{R})$ and $(y, y) \in \text{clos}(\mathcal{N} \setminus \text{Diag}(\mathbb{R}))$. In view of Lemma 2.9, this implies that both "tips of antennas" belong to (α_2, y), and the claim follows from Theorem 2.6. □

6. VARIANTS AND GENERALIZATIONS

The algorithm described in the previous section adapts without difficulty to more general settings; we consider some examples below.

6.1 The Set of Triple Roots

Here we restrict ourselves to the family $\mathcal{B}$, but consider higher-order roots. Geometrically, the set of roots of multiplicity n corresponds to the connectedness locus of self-affine sets associated with Jordan blocks of order n; recall Proposition 2.4. We denote this set by Ω_n.

The computer algorithm extends in a straightforward way to higher-multiplicity roots. Indeed, it is enough to replace (5–1) by the set of tests

$$|P^{(i)}(b)| < H_n^{(i)}(b) + \frac{b-a}{(i+1)(1-b)^{i+2}}, \quad 0 \le i < n,$$

where $H_n(x) = x^n/(1-x)$. The algorithm yields intervals in $(0,1)\backslash\Omega_n$; in particular, it gives lower bounds on $\alpha_n = \min \Omega_n$. We remark, however, that in practice the program becomes very slow for $n \ge 4$. For $n = 3$, we have the following result.

Proposition 6.1.

(i) $\alpha_3 = \min \Omega_3 > 0.743$,

(ii) $(0.746, 0.7465) \subset (0,1) \backslash \Omega_3$,

(iii) $(2^{-1/3}, 1) \subset \Omega_3$.

Sketch of a proof: Parts (i) and (ii) are direct applications of the algorithm, while (iii) follows from Lemma 2.3. (Note that $2^{-1/3} \approx 0.7937$.) □

Remark 6.2. Numerical experimentation suggests that $\alpha_3 \in (0.743, 0.744)$ and, in particular, that Ω_3 is disconnected. However, the techniques of Section 3 do not seem to apply, so this remains a conjecture.

It is interesting to compare our results with those of [Beaucoup et al. 98], where multiple roots of the following family were considered:

$$\widetilde{\mathcal{B}} = \left\{ 1 + \sum_{i=1}^{\infty} a_i x^i : a_i \in [-1, 1] \right\}.$$

Let β_n be the smallest root of multiplicity (at least) n of some $f \in \widetilde{\mathcal{B}}$. The values of β_n for $n \le 27$ were computed in [Beaucoup et al. 98]; in particular, $\beta_2 \approx 0.64914$ and $\beta_3 \approx 0.72788$. Observe that $\alpha_2 - \beta_2 > 0.01934$ and $\alpha_3 - \beta_3 > 0.0151$. Thus going from a continuous to a discrete set of coefficients does have a substantial impact on the set of multiple roots.

6.2 The Set of Double Zeros with Coefficients $0, \pm 1, \pm 2$

We can generalize the set Ω_2 in another direction by enlarging the set of allowed coefficients. For concreteness, we will work with the coefficient set $\{0, \pm 1, \pm 2\}$; specifically, let

$$\mathcal{B}' = \left\{ 1 + \sum_{i=1}^{\infty} a_i x^i : a_i \in \{0, -1, 1, -2, 2\} \right\}.$$

Denote by Ω_2' the set of double zeros in $(0,1)$ of elements of $\mathcal{B}'$. It turns out that a specific power series plays a very special role in the study of this set. Let

$$Q(x) = 1 - 2x - 2x^2 + \sum_{i=3}^{\infty} 2x^i = 1 - 2x - 2x^2 + \frac{2x^3}{1-x}.$$

Note that $\frac{1}{2}$ is a double root of $Q(x)$. In fact, more is true: $Q(x)$ is a so-called $(*)$-function for the class $\mathcal{B}'$ on the interval $(0, x_0)$ for all $x_0 < \frac{1}{2}$; see [Solomyak 04] for the relevant definitions and proofs. A consequence of this is that $\min \Omega_2'$ is precisely $\frac{1}{2}$.

Here we prove that $\frac{1}{2}$ is actually an isolated point of Ω_2'; this is due to the special form of the function $Q(x)$, and in particular the fact that all but finitely many coefficients are $+2$. More precisely, we have the following result:

Proposition 6.3. $\min \left(\Omega_2' \backslash \left\{ \frac{1}{2} \right\} \right) \in (0.5436, 0.5438)$.

Before proving the proposition we remark that the setup of Section 3 works here with minor modifications. In this context, we say that (P, n, a, b) is *good* if $P \in \mathcal{B}_n'$ (the family of polynomials in $\mathcal{B}'$ of degree at most n), $0.5 < a < b < 1$,

$$P(a) > 2\frac{a^{n+1}}{(1-a)}, \quad P(b) > 2\frac{b^{n+1}}{(1-b)},$$

$P(x) > 0$ for all $x \in [a, b]$, and

$$\exists\, x \in (a, b) : \; P(x) < 2\frac{x^{n+1}}{(1-x)}.$$

The proofs of Section 3 apply almost verbatim. In particular, if (P, n, a, b) is good, then there exists a sequence $i_1 < i_2 < \cdots$ such that $f(x) = P(x) - \sum_{j=1}^{\infty} 2x^{i_j}$ has a double root in (a, b).

Proof of Proposition 6.3: We will use the following result, which follows from a modification of the proof of Theorem 2 in [Beaucoup et al. 98] (see [Shmerkin 06] for a complete proof): if f is a power series with coefficients in $[-1, 1]$, and $\alpha_1, \ldots, \alpha_k$ are complex roots of f in the unit disk, counted with multiplicity, then

$$|\alpha_1 \cdots \alpha_k| \ge \left(1 + \frac{1}{k}\right)^{-k/2} (k+1)^{-1/2}. \tag{6–1}$$

We will use this result with $k = 3$. A slight modification of the algorithm shows that $f \in \mathcal{B}'$ may have roots in $(0.5, 0.51)$ only if it starts with $1 - 2x - 2x^2$ (this can also be checked directly by considering all possible cases). Then it follows from the definition of Q that $f(x) < Q(x)$ for all $x > 0$. Since $f(0) = 1$ and $f\left(\frac{1}{2}\right) < Q\left(\frac{1}{2}\right) = 0$, f has a root in the interval $\left(0, \frac{1}{2}\right)$. Suppose that α is a double root of f in $\left(\frac{1}{2}, 1\right)$. We obtain from (6–1) that

$$\frac{1}{2}\alpha^2 > \left(1 + \frac{1}{3}\right)^{-3/2} (3+1)^{-1/2} = \frac{3\sqrt{3}}{16},$$

whence $\alpha > (6\sqrt{3}/16)^{1/2} > 0.8$. We conclude that f cannot have double roots in the interval $(0.5, 0.51)$, whence $(0.5, 0.51) \subset (0,1) \backslash \Omega_2'$.

Another straightforward application of the algorithm shows also that $(0.51, 0.5436) \subset (0,1)\backslash\Omega_2'$.

Finally, let P be the polynomial of degree 26 with coefficients

$$(1,-2,-1,1,1,1,2,1,1,2,1,1,2,1,1,2,1,2,1,1,2, \\ -2,-2,-1,-2,2,-1).$$

We checked with *Mathematica* that $(P, 27, 0.5436, 0.5438)$ is good. This implies that $\min\Omega_2' \in (0.5436, 0.5438)$, completing the proof. □

Remark 6.4. The set $\Omega_2'\backslash\left\{\frac{1}{2}\right\}$ seems to be connected (i.e., no "gaps" appear when the program is run), but we do not have a proof of this. Still, if what the numerical experimentation suggests holds true, then the sets Ω_2 and Ω_2' have strikingly different topological structures.

7. REMAINING PROOFS

Proof of Lemma 2.3: Let E be the attractor of the IFS, that is, $E = T_1E \cup (T_2E + \mathbf{b})$. Let $\|\cdot\|$ be the norm in $\mathbb{R}^d$ such that $\|T_ix\| \le r\|x\|$, $i = 1, 2$, for some $r \in (0,1)$ and all $x \in \mathbb{R}^d$. Denote by F_ε the ε-neighborhood of a set $F \subset \mathbb{R}^d$ in this norm. Then $(T_iF)_\varepsilon \supset T_iF_{\varepsilon/r}$.

Suppose that E is disconnected. Then $T_1E \cap (T_2E + \mathbf{b}) = \varnothing$ by Proposition 2.1, and since these are compact sets, we can find $\varepsilon > 0$ such that $(T_1E)_\varepsilon \cap (T_2E+\mathbf{b})_\varepsilon = \varnothing$. Then $E_\varepsilon = (T_1E)_\varepsilon \cup (T_2E + \mathbf{b})_\varepsilon$ is a disjoint union, so

$$\begin{aligned}\mathcal{L}^d(E_\varepsilon) &= \mathcal{L}^d((T_1E)_\varepsilon) + \mathcal{L}^d((T_2E)_\varepsilon)\\ &\ge \mathcal{L}^d(T_1E_{\varepsilon/r}) + \mathcal{L}^d(T_2E_{\varepsilon/r})\\ &= (|\det(T_1)| + |\det(T_2)|)\cdot\mathcal{L}^d(E_{\varepsilon/r})\\ &\ge \mathcal{L}^d(E_{\varepsilon/r}).\end{aligned}$$

This is a contradiction, since $E_{\varepsilon/r} \setminus E_\varepsilon$ has positive Lebesgue measure. □

Proof of Proposition 2.4: If $E = E(T, \mathbf{b})$ is connected, then $TE \cap (TE+\mathbf{b}) \ne \varnothing$. In view of (2–2), we obtain that there exist $\{0,1\}$-sequences $\{a_n\}_0^\infty$ and $\{a_n'\}_0^\infty$ such that $a_0 = 1$, $a_0' = 0$, and $\sum_{n=0}^\infty (a_n - a_n')T^n\mathbf{b} = \mathbf{0}$. Setting $f(x) = \sum_{n=0}^\infty (a_n - a_n')x^n$, we get $f \in \mathcal{B}$ and $f(T)\mathbf{b} = \mathbf{0}$.

Now let us write $\mathbf{b} = \sum_{j=1}^m c_j\mathbf{e}_j$, where $\mathbf{e}_j \in \mathrm{Ker}(T - \lambda_jI)^{k_j}$. Since $\mathbf{b}$ is a cyclic vector for T, we have $c_j \ne 0$ and $\mathbf{e}_j \notin \mathrm{Ker}(T - \lambda_jI)^{k_j-1}$ for $j \le m$. Then $f(T)\mathbf{b} = \mathbf{0}$ implies that $f(T)\mathbf{e}_j = \mathbf{0}$ for all $j \le m$. We have

$$\begin{aligned}f(T)\mathbf{e}_j = f(\lambda_j)\mathbf{e}_j &+ f'(\lambda_j)(T - \lambda_jI)\mathbf{e}_j\\ &+ \cdots + f^{(k_j-1)}(\lambda_j)(T - \lambda_jI)^{k_j-1}\mathbf{e}_j.\end{aligned}$$

Since the vectors $\{(T - \lambda_jI)^\ell\mathbf{e}_j : \ell = 1, \ldots, k_j - 1\}$ are linearly independent, (2–3) follows.

Conversely, if $f(x) = 1 + \sum_{n=1}^\infty b_nx^n \in \mathcal{B}$ satisfies (2–3), then $f(T)\mathbf{b} = \mathbf{0}$ for all $\mathbf{b}$. Writing $b_n = a_n - a_n'$ for some $a_n, a_n' \in \{0,1\}$, we obtain that $TE \cap (TE+\mathbf{b}) \ne \varnothing$, and hence E is connected by Proposition 2.1. □

Proof of Lemma 2.9: (ii) Suppose that $\lambda \in (-1,1)$ and that (λ,λ) is such that there exists a sequence $(\gamma_n, \lambda_n) \in \mathcal{N}$, with $\gamma_n < \lambda_n$ converging to (λ,λ). Then there are power series $f_n \in \mathcal{N}$ such that $f_n(\lambda_n) = f(\gamma_n) = 0$. By compactness, passing to a subsequence, we can assume that $f_n \to f \in \mathcal{B}$ coefficientwise. Then $f_n^{(k)}(\lambda) \to f^{(k)}(\lambda)$ for all $k \ge 0$. By Lemma 2.3, $|\lambda| < 2^{-1/2}$. Let $C_k = \max\{|f^{(k)}(x)| : f \in \mathcal{B},\ |x| \le 2^{-1/2}\}$, which is finite (and easy to compute explicitly). Then we have for n sufficiently large,

$$|f_n(\lambda)| = |f_n(\lambda) - f_n(\lambda_n)| \le C_1|\lambda - \lambda_n| \to 0, \quad n \to \infty.$$

Next, there exists $t_n \in (\gamma_n, \lambda_n)$ such that $f_n'(t_n) = 0$, and we have for n sufficiently large,

$$|f_n'(\lambda)| = |f_n'(\lambda) - f_n'(t_n)| \le C_2|\lambda - t_n| \to 0, \quad n \to \infty.$$

It follows that $f(\lambda) = f'(\lambda) = 0$, as desired. □

Proof of Lemma 4.3: In the following calculations, η will be a fixed positive number, to be determined later. We use the ℓ^∞ norm on $\mathbb{R}^2$.

Given a word $u \in \{-1,0,1\}^5$ let

$$\begin{aligned}T_u &= (T + u_1\mathbf{b}) \circ \cdots \circ (T + u_5\mathbf{b}),\\ T_u' &= (T' + u_1\mathbf{b}) \circ \cdots \circ (T' + u_5\mathbf{b}).\end{aligned}$$

Observe that

$$T_u\mathbf{x} = T^5\mathbf{x} + \sum_{i=1}^5 u_iT^{i-1}\mathbf{b},$$

$$(T_u')^{-1}\mathbf{x} = (T')^{-5}\mathbf{x} - (T')^{-5}\left(\sum_{i=1}^5 u_i(T')^{i-1}\mathbf{b}\right).$$

Hence

$$\begin{aligned}(T_u')^{-1}T_u\mathbf{x} &= \left((T')^{-5}T^5\right)\mathbf{x} + (T')^{-5}\sum_{i=1}^5 u_iT^{i-1}\mathbf{b}\\ &\quad - (T')^{-5}\sum_{i=1}^5 u_i(T')^{i-1}\mathbf{b}\\ &=: S\mathbf{x} + \mathbf{d},\end{aligned} \tag{7–1}$$

where

$$S = (T')^{-5}T^5, \qquad \mathbf{d} = (T')^{-5}\sum_{i=1}^5 (T^{i-1} - (T')^{i-1})\mathbf{b}.$$

Observe that (for $\eta < 10^{-2}$),

$$\begin{aligned}\|T^k - (T')^k\| &\le \sum_{j=1}^{k} \|T^j (T')^{k-j} - T^{j-1}(T')^{k-j+1}\| \\ &= \sum_{j=1}^{k} \|T^{j-1}(T - T')(T')^{k-j-1}\| \qquad (7\text{–}2) \\ &\le k\eta \max(\|T\|, \|T'\|)^{k-1} < k\,(1.5)^{k-1}\,\eta.\end{aligned}$$

Hence if $R = (T')^5 - T^5$, then $\|R\| \le 5\,(1.5)^4\,\eta < 26\eta$, and we can estimate

$$\begin{aligned}\|(T')^{-5}\| &= \|\left(T^5(I - T^{-5}R)\right)^{-1}\| \\ &\le \|(I - T^{-5}R)^{-1}\|\|T^{-5}\| \\ &\le \|T^{-5}\| \sum_{j=0}^{\infty} (\|T^{-5}\|\|R\|)^j \\ &= \frac{\|T^{-5}\|}{1 - \|T^{-5}\|\|R\|} \\ &< \frac{33.6}{1 - 33.6 \times 26\,\eta} < 34, \qquad (7\text{–}3)\end{aligned}$$

as long as $\eta < 10^{-5}$. Therefore,

$$\|S - I\| = \|(T')^{-5}R\| \le \|(T')^{-5}\|\|R\| < 34 \times 26\,\eta = 884\,\eta.$$

We have by (7–2),

$$\|\mathbf{d}\| \le \|(T')^{-5}\| \sum_{j=0}^{4} j\,(1.5)^{j-1}\eta\|\mathbf{b}\| < 34 \times 24.25\,\eta < 825\,\eta. \qquad (7\text{–}4)$$

Note that

$$\begin{aligned}\max\{\|\mathbf{x}\| : \mathbf{x} \in V\} &= 0.95 \max(\|\mathbf{p}\|, \|\mathbf{q}\|) \\ &= 0.95 \times 4.9 < 4.7.\end{aligned}$$

It follows from the previous estimates that for $x \in V$,

$$\|S\mathbf{x} - \mathbf{x}\| \le \|S - I\|\|\mathbf{x}\| < 884 \times 4.7\,\eta < 4155\,\eta.$$

In particular, this implies that $SV \subset V_\delta$, where $\delta = 4155\,\eta$, and V_δ denotes the δ neighborhood of V. Recalling (7–1), we get that

$$(T'_u)^{-1} T_u V \subset (V_\delta)_{\|\mathbf{d}\|} = V_{\delta + \|\mathbf{d}\|} \subset V_{\delta'}, \qquad (7\text{–}5)$$

where $\delta' = 5 \times 10^3\,\eta$, since $\|\mathbf{d}\| < 825\,\eta$ by (7–4).

Let $W(r) = \{(x, y) : |x| + |y| < r\}$ and let M be the matrix with the columns $\mathbf{p}, \mathbf{q}$. Note that $U = MW(1)$ and $V = MW(0.95)$. Note also that

$$\operatorname{dist}(W(0.95), \mathbb{R}^2 \backslash W(1)) = \frac{0.05}{2} = 2.5 \times 10^{-2},$$

where $\operatorname{dist}(\cdot, \cdot)$ denotes the distance induced by the ℓ^∞ norm. An easy calculation yields $\|M^{-1}\| = 1.2$. Therefore,

$$\operatorname{dist}(V, \mathbb{R}^2 \backslash U) \ge \frac{\operatorname{dist}(W(0.95), \mathbb{R}^2 \backslash W(1))}{\|M^{-1}\|} > 2 \times 10^{-2}. \qquad (7\text{–}6)$$

It follows from (7–5) and (7–6) that if η is so small that $\delta' \le 2 \times 10^{-2}$, then

$$(T'_u)^{-1} T_u V \subset U. \qquad (7\text{–}7)$$

Since $\delta' = 5 \times 10^3 \eta$, this will be the case for $\eta = 4 \times 10^{-6}$. Now we will fix this value of η (since $\eta < 10^{-5}$, the previous calculations apply).

Let A be the "multiplication by 0.95" map, so that $V = AU$. By applying A to both sides of (4–2) we get

$$V \subset A\Bigg(\bigcup_{u \in \{-1,0,1\}^5} T_u(V) \Bigg) = \bigcup_{u \in \{-1,0,1\}^5} T_u AV.$$

However, we deduce from (7–7) that $T_u V \subset T'_u A^{-1} V$, or $T'_u V \supset T_u AV$. Combining this with the last displayed formula, we conclude that (4–3) holds. □

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Pablo Shmerkin, Department of Mathematics and Statistics, P.O. Box 35 (MaD), University of Jyväskylä, 40014, Finland (shmerkin@math.jyu.fi)

Boris Solomyak, Box 354350, Department of Mathematics, University of Washington, Seattle, WA 98195 (solomyak@math.washington.edu)

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