The Superpolynomial for Knot Homologies

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CONTENTS

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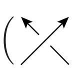
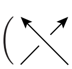
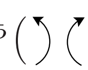
We propose a framework for unifying the sl(N) Khovanov–Rozansky homology (for all N) with the knot Floer homology. We argue that this unification should be accomplished by a triply graded homology theory that categorifies the HOMFLY polynomial. Moreover, this theory should have an additional formal structure of a family of differentials. Roughly speaking, the triply graded theory by itself captures the large-N behavior of the sl(N) homology, and differentials capture nonstable behavior for small N, including knot Floer homology. The differentials themselves should come from another variant of sl(N) homology, namely the deformations of it studied by Gornik, building on work of Lee.

While we do not give a mathematical definition of the triply graded theory, the rich formal structure we propose is powerful enough to make many nontrivial predictions about the existing knot homologies that can then be checked directly. We include many examples in which we can exhibit a likely candidate for the triply graded theory, and these demonstrate the internal consistency of our axioms. We conclude with a detailed study of torus knots, developing a picture that gives new predictions even for the original sl(2) Khovanov homology.

1. INTRODUCTION

1.1 Knot Homologies

Here, we are interested in homology theories of knots in $\mathbf{S}^3$ associated with the HOMFLY polynomial. For a knot K, its HOMFLY polynomial $\bar{P}(K)$ is determined by the skein relation

$a\bar{P}$ () $- a^{-1}\bar{P}$ () $= (q - q^{-1})\bar{P}$ (),

together with the requirement $\bar{P}(\text{unknot}) = (a - a^{-1})/(q - q^{-1})$. The HOMFLY polynomial unifies the quantum sl(N) polynomial invariants of K, which are denoted by $\bar{P}_N(K)(q)$ and are equal to $\bar{P}(K)(a = q^N, q)$. Here, the original Jones polynomial $J(K)$ is just $\bar{P}_2(K)$. The HOMFLY polynomial encodes the classical Alexander polynomial as well.

A number of different knot homology theories have been discovered related to these polynomial invariants.

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Although the details of these theories differ, the basic idea is that for a knot K, one can construct a doubly graded homology theory $H_{i,j}(K)$ whose graded Euler characteristic with respect to one of the gradings gives a particular knot polynomial. Such a theory is referred to as a *categorification* of the knot polynomial.

For example, the Jones polynomial J is the graded Euler characteristic of the doubly graded *Khovanov Homology* $H^{\mathrm{Kh}}_{i,j}(K)$; that is,

$$J(q) = \sum_{i,j} (-1)^j q^i \dim H^{\mathrm{Kh}}_{i,j}(K).$$

Here, the grading i is called the *Jones grading*, and j is called the *homological grading*. Khovanov originally constructed $H^{\mathrm{Kh}}_{i,j}$ combinatorially in terms of skein theory [Khovanov 99], but it is conjectured to be essentially the same as Seidel and Smith's *symplectic Khovanov homology*, which is defined by considering the Floer homology of a certain pair of Lagrangians [Seidel and Smith 04].

Khovanov's theory was generalized by Khovanov and Rozansky [Khovanov and Rozansky 05] to categorify the quantum sl(N) polynomial invariant $\bar{P}_N(q)$. Their homology $\overline{\mathrm{HKR}}^N_{i,j}(K)$ satisfies

$$\bar{P}_N(q) = \sum_{i,j} (-1)^j q^i \dim \overline{\mathrm{HKR}}^N_{i,j}(K).$$

For $N = 2$, this theory is expected to be equivalent to the original Khovanov homology. There are also important deformations of the original Khovanov homology [Lee 02b, Bar-Natan 05a, Khovanov 04b], as well as of the sl(N) Khovanov–Rozansky homology [Gornik 04]. In a sense, the deformed theory of Lee [Lee 02b] also can be regarded as a categorification of the sl(1) polynomial invariant.

Another knot homology theory that will play an important role here is knot Floer homology, $\widehat{\mathrm{HFK}}_j(K; i)$, introduced in [Ozsváth and Szabó 04a, Rasmussen 03]. It provides a categorification of the Alexander polynomial:

$$\Delta(q) = \sum_{i,j} (-1)^j q^i \dim \widehat{\mathrm{HFK}}_j(K; i).$$

Unlike Khovanov–Rozansky homology, knot Floer homology is not known to admit a combinatorial definition; in the end, computing $\widehat{\mathrm{HFK}}$ involves counting pseudo-holomorphic curves.

The polynomials above are closely related; indeed, they can all be derived from a single invariant, namely the HOMFLY polynomial. While the above homology theories categorify polynomial knot invariants in the same class, their constructions are very different! Despite this, our objective here is summarized in the following goal.

Goal 1.1. Unify the Khovanov–Rozansky sl(N) homology (for all N), knot Floer homology, and various deformations thereof into a single theory.

We do not succeed here in defining such a unified theory. Instead, we postulate a very detailed picture of what such a theory should look like: it is a triply graded homology theory categorifying the HOMFLY polynomial together with a certain additional formal structure. Although we don't know a definition of this triply graded theory, our description of its properties is powerful enough to give us many nontrivial predictions about knot homologies that can be verified directly.

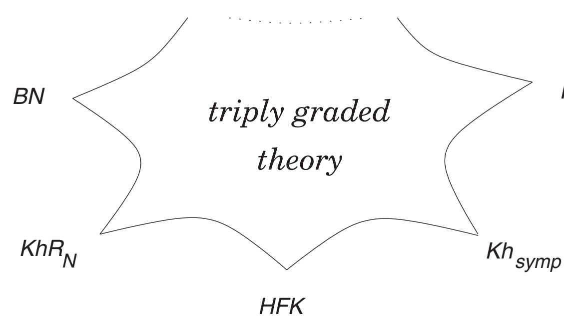


FIGURE 1. Triply graded theory as a unification of knot homologies.

There are several reasons to hope for the type of unified theory asked for in Goal 1.1. In the recent work [Gukov et al. 05], a physical interpretation of the Khovanov–Rozansky homology naturally led to the unification of the sl(N) homologies when N is sufficiently large. At the small-N end, the sl(2) Khovanov homology and $\widehat{\mathrm{HFK}}$ seem to be very closely related. For instance, their total ranks are very often (but not always) equal (see [Rasmussen 05a] for more). One hope for our proposed theory is that it will explain the mysterious fact that while the connections between $\overline{\mathrm{HKR}}_2$ and HFK hold very frequently, they are not universal.

1.2 The Superpolynomial

We now work toward a more precise statement of our proposed unification, starting with a review of the work [Gukov et al. 05]. To concisely describe the homology groups $\overline{\mathrm{HKR}}^N_{i,j}(K)$, it will be useful to introduce the graded Poincaré polynomial $\overline{\mathrm{KhR}}_N(q,t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$, which encodes the dimensions of these groups via

$$\overline{\mathrm{KhR}}_N(q,t) := \sum_{i,j} q^i t^j \dim \overline{\mathrm{HKR}}^N_{i,j}(K). \tag{1–1}$$

The Khovanov–Rozansky homology has finite total dimension, so $\overline{\mathrm{KhR}}_N$ is a *finite polynomial*, that is, one with only finitely many nonzero terms. The Euler characteristic condition on $\overline{\mathrm{HKR}}^N_{i,j}(K)$ is concisely expressed by $\bar{P}_N(q) = \overline{\mathrm{KhR}}_N(q, t = -1)$.

The basic conjecture of [Gukov et al. 05] is as follows.

Conjecture 1.2. *There exists a finite polynomial* $\bar{\mathcal{P}}(K) \in \mathbb{Z}[a^{\pm1}, q^{\pm1}, t^{\pm1}]$ *such that*

$$\overline{\mathrm{KhR}}_N(q,t) = \frac{1}{q - q^{-1}} \bar{\mathcal{P}}(a = q^N, q, t) \tag{1–2}$$

for all sufficiently large N.

We will refer to $\bar{\mathcal{P}}(K)$ as the *superpolynomial* for K. This conjecture essentially says that, for sufficiently large N, the dimension of the sl(N) knot homology grows linearly in N, and the precise form of this growth can be encoded in a finite set of the integer coefficients. Therefore, if one knows the sl(N) knot homology for two different values of N, both of which are in the "stable range" $N \geq N_0$, one can use (1–2) to determine the sl(N) knot homology for all other values of $N \geq N_0$.

In some examples, it seems that (1–2) holds true for all values of N, not just large N. In [Gukov et al. 05], this was used to compute $\bar{\mathcal{P}}(K)$ for certain knots. However, this is not always true. The simplest knot for which (1–2) holds for all $N \geq 3$ but not for $N = 2$ is the 8-crossing knot 8_{19}. Notice that Conjecture 1.2 implies that for all knots, the HOMFLY polynomial is a specialization of the superpolynomial

$$\bar{P}(K)(a,q) = \frac{1}{q - q^{-1}} \bar{\mathcal{P}}(a, q, t = -1). \tag{1–3}$$

The motivation for Conjecture 1.2 in [Gukov et al. 05] was based on the geometric interpretation of the sl(N) knot homology and the 3-variable polynomial $\bar{\mathcal{P}}(a, q, t)$. In fact, we can offer two (related) geometric interpretations of $\bar{\mathcal{P}}(a, q, t)$:

- as an index (cf. elliptic genus):

  $$\bar{\mathcal{P}}(a,q,t) = \mathrm{Str}_{\mathcal{H}}[a^Q q^s t^r] = \mathrm{Tr}_{\mathcal{H}}[(-1)^F a^Q q^s t^r].$$

  Here $\mathcal{H} = \mathcal{H}_{\mathrm{BPS}}$ is a $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$-graded Hilbert space of the so-called BPS states. Specifically, F is the $\mathbb{Z}_2$ grading, and Q, s, and r are the three $\mathbb{Z}$ gradings. Following the notation in [Gukov et al. 05], we also introduce the graded dimension of this Hilbert space:

  $$D_{Q,s,r} := (-1)^F \dim \mathcal{H}^{F,Q,s,r}_{\mathrm{BPS}}. \tag{1–4}$$

  Notice that the integer coefficients of the polynomial $\bar{\mathcal{P}}(a, q, t)$ are precisely the graded dimensions (1–4).

- as an enumerative invariant: The triply graded integers $D_{Q,s,r}$ are related to the dimensions of the cohomology groups

  $$H^k(\mathcal{M}_{g,Q}), \tag{1–5}$$

  where $\mathcal{M}_{g,Q}$ is the moduli space of holomorphic Riemann surfaces with boundary in a certain Calabi–Yau 3-fold. We will return to this relationship in Section 4.

1.3 Reduced Superpolynomial

The setup of the last section needs to be modified in order to bring knot Floer homology into the picture. Let $P(K)(a, q)$ be the *reduced* or *normalized* HOMFLY polynomial of the knot K, determined by the convention that $P(\text{unknot}) = 1$. This switch brings the Alexander polynomial naturally into the picture since it arises by a specialization $\Delta(q) = P(K)(a = 1, q)$. There is a categorification of $P(K)(a = q^N, q)$ called the *reduced* Khovanov–Rozansky homology (see [Khovanov 03, Section 3] and [Khovanov and Rozansky 05, Section 7]). We will use $\mathrm{KhR}_N(K)(q, t)$ to denote the Poincaré polynomial of this theory.

For this reduced theory, there is also a version of Conjecture 1.2. Essentially, it says that, for sufficiently large N, the total dimension of the reduced sl(N) knot homology is independent of N, and the graded dimensions of the homology groups change linearly with N:

Conjecture 1.3. *There exists a finite polynomial* $\mathcal{P}(K) \in \mathbb{Z}_{\geq 0}[a^{\pm1}, q^{\pm1}, t^{\pm1}]$ *such that*

$$\mathrm{KhR}_N(q,t) = \mathcal{P}(a = q^N, q, t) \tag{1–6}$$

for all sufficiently large N.

In contrast with the previous case, in the reduced case the superpolynomial is required to have nonnegative coefficients. This is forced merely by the form of (1–6), since for large N distinct terms in $\mathcal{P}(a, q, t)$ cannot coalesce when we specialize to $a = q^N$. Moreover, one also has

$$P(K)(a,q) = \mathcal{P}(a,q,t=-1). \tag{1–7}$$

Thus we will view $\mathcal{P}(a,q,t)$ as the Poincaré polynomial of some new triply graded homology theory $\mathcal{H}_{i,j,k}(K)$ categorifying the normalized HOMFLY polynomial.

As with the unreduced theory, for some simple cases (1–6) holds for all $N \geq 2$. However, in general there will be exceptional values of N for which this is not the case. To account for this, we introduce an additional structure on $\mathcal{H}_*(K)$: a family of differentials $\{d_N\}$ for $N > 0$. The complete details of this structure we postpone until Section 3, but the basic idea is this: The $\mathrm{sl}(N)$ homology is the homology of $\mathcal{H}_*(K)$ with respect to the differential d_N. For large N, the differential d_N is trivial, giving the stabilization phenomena of Conjecture 1.3. The main reason for expecting the presence of the differentials d_N comes from Gornik's work on the $\mathrm{sl}(N)$ homology. In particular, in [Gornik 04] Gornik describes a deformation of Khovanov and Rozansky's construction that gives rise to a differential on HKR_N.

We also postulate additional differentials for $N \leq 0$. After a somewhat mysterious regrading, the knot Floer homology arises from the $N = 0$ differential. Consider the Poincaré polynomial

$$\mathrm{HFK}(q,t) := \sum_{i,j} q^i t^j \dim \widehat{\mathrm{HFK}}_j(K;i). \tag{1–8}$$

In the simplest cases, we have the following relationship between the knot Floer homology and the superpolynomial:

$$\mathrm{HFK}(q,t) = \mathcal{P}(a=t^{-1},q,t).$$

For the more general situation, see Section 3.

1.4 Some Predictions

Our conjectures imply that the HOMFLY polynomial, the knot Floer homology, and Khovanov–Rozansky homology should all be related. Unfortunately, this relation is mediated by the triply graded homology group $\mathcal{H}_{i,j,k}(K)$, which is often considerably larger than $\widehat{\mathrm{HFK}}(K)$, $\mathrm{HKR}_2(K)$, or the minimum size dictated by $P(K)$. Thus it seems unlikely that there will be a general relation between the dimensions of either of these groups and the HOMFLY polynomial. On the other hand, our hypotheses about the structure of the triply graded theory enable us to make testable predictions about the $\mathrm{sl}(2)$ Khovanov homology and HOMFLY polynomial for some specific families of knots. We list some of the more important ones here:

1. HKR_N for small knots: Using conjectured properties of the triply graded theory, we make exact predictions for the group $\mathcal{H}(K)$ for many knots with 10 crossings or fewer. These are given in Sections 5 and 8. From them, it is easy to predict the form of $\mathrm{KhR}_N(K)$ for $N > 2$. These predictions have been verified in simple cases [Rasmussen 05b]; to check them in others requires better methods for calculating the Khovanov–Rozansky homology.

2. HOMFLY polynomials of thin knots. In Section 5.1, we describe a class of $\mathcal{H}$-*thin* knots whose triply graded homology has an especially simple form. Let K be such a knot, and let T be the $(2,n)$ torus knot with the same signature as K. Then our conjectures imply that the quotient

$$\frac{P(K) - P(T)}{(1-a^2q^2)(1-a^2q^{-2})}$$

should be an alternating polynomial. Two-bridge knots are expected to be $\mathcal{H}$-thin; we have verified that the relation above holds for all such knots with determinant less than 200.

3. A new pairing on Khovanov homology. Our conjectures suggest that for many knots, the Khovanov polynomial should have the following form:

$$\mathrm{KhR}_2(K) = q^m t^n + (1 + q^6 t^3) Q_-(q,t),$$

where Q_- is a polynomial with positive coefficients. (See Section 5.6 for a complete discussion.) This pattern is easily verified in examples, but so far as we are aware, it had previously gone unnoticed.

4. Khovanov homology of torus knots: In Sections 6 and 7, we use our conjectures to make predictions about the $N = 2$ Khovanov homology of torus knots that can be checked against the computations made by Bar-Natan [Bar-Natan 05b]. These predictions provide some of the best evidence in favor of the conjectures, since the Khovanov homology of torus knots had previously seemed quite mysterious.

1.5 Candidate Theories for the Superpolynomial

The most immediate question raised by Conjecture 1.3 is how to define the underlying knot homology whose Poincaré polynomial is the superpolynomial. In formulating our conjectures, the approach we had in mind was simply to take the inverse limit of KhR_N as $N \to \infty$. This method rests on two basic principles. First, we should have some sort of map from the sl(N) homology to the sl(M) homology for $M < N$, and second, for a fixed knot K the dimension of $\mathrm{HKR}_N(K)$ should be bounded independent of N. We expect that the maps required by the first principle should be defined using the work of Gornik [Gornik 04], although at the moment, technical difficulties prevent us from giving a complete proof of their existence. The proof of the second principle should be more elementary—it should be essentially skein-theoretic in nature.

Very recently, Khovanov and Rozansky have introduced a triply graded theory categorifying the HOMFLY polynomial [Khovanov and Rozansky 06], which gives another candidate for our proposed theory. This theory has some obvious advantages over the approach described above; it is already known to be well-defined, and its definition is in many respects simpler than that of the sl(N) theory. At the same time, there are some gaps between what the theory provides and what our conjectures suggest that it should have. The most important of these is the family of differentials d_N alluded to above. One of our aims in writing this paper is to encourage people to look for these differentials, and, with luck, to find them!

Another approach to constructing a knot homology associated with the superpolynomial might be based on an algebraic structure that unifies sl(N) (or gl(N)) Lie algebras (for all N). A natural candidate for such structure is the infinite-dimensional Lie algebra gl(λ), introduced by Feigin [Feigin 88] as a generalization of gl(N) to non-integer, complex values of the rank N. It is defined as a Lie algebra of the following quotient of the universal enveloping algebra of sl(2):

$$\mathrm{gl}(\lambda) = \mathrm{Lie}\left(U(\mathrm{sl}(2)) \Big/ C - \frac{\lambda(\lambda-1)}{2} \right),$$

where C is the Casimir operator in $U(\mathrm{sl}(2))$. One can also define gl(λ) as a Lie algebra of differential operators on $\mathbb{C}\mathbf{P}(1)$ of "degree of homogeneity" λ:

$$\mathrm{gl}(\lambda) = \mathrm{Lie}(\mathrm{Diff}_\lambda).$$

Representation theory of gl(λ) is very simple and has all the properties that we need: for generic $\lambda \in \mathbb{C}$, gl(λ) has infinite-dimensional representations. Characters of these representations appear in the superpolynomial of torus knots! On the other hand, for $\lambda = N$, we get the usual finite-dimensional representations of gl(N).

1.6 Generalizations

We expect many generalizations of this story. Thus, from the physics point of view, it is clear that a categorification of the quantum sl(N) invariant should exist for arbitrary representation of $U_q(\mathrm{sl}(N))$, not just the fundamental representation.

1.7 Contents of the Paper

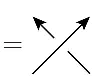
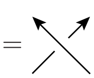
In the next section we summarize our conventions and notation. In Section 3, we introduce families of graded differentials, which play a key role in the reduction to different knot homologies, and give a precise statement of our main conjecture. In Section 4, we explain the geometric interpretation of the triply graded theory. Various examples and patterns are discussed in Section 5; these serve to illustrate the internal consistency of our proposed axioms. Section 6 begins our study of torus knots, and there we give a complete conjecture for the superpolynomials of $(2, n)$ and $(3, n)$ torus knots. While we don't have a complete picture for general (n, m) torus knots, in Section 7 we suggest a limiting "stable" picture as $m \to \infty$. Finally, Section 8 gives information about the superpolynomial for certain 10 crossing knots discussed in Section 5.

2. NOTATION AND CONVENTIONS

In this section, we give our conventions for knot polynomials and the various homology theories. Some of these differ from standard sources; in particular, we view the sl(N) theory as homology rather than cohomology. Also, our convention for the knot Floer homology is the mirror of the standard one. The notation used throughout the paper is collected in Table 1.

2.1 Crossings

Our conventions for crossings are given below:

positive =  negative = 

This convention agrees with [Gukov et al. 05], but differs from [Khovanov 04a, Figure 8] and [Khovanov and Rozansky 05, Figure 45].

2.2 Torus Knots

The torus knot $T_{a,b}$ is the knot lying on a standard solid torus that wraps a times around in the longitudinal direction and b times in the meridional direction. For us, the standard $T_{a,b}$ has *negative* crossings. In particular, the trefoil knot 3_1 in the standard tables [Rolfsen 76, Bar-Natan 05c] is exactly $T_{2,3}$ with our conventions. However, it is important to note that some other torus knots in these tables are positive rather than negative (e.g., 8_{19}

$P(K)(a,q)$	The normalized HOMFLY polynomial of the knot K, where $P(\text{unknot}) = 1$.
$\bar{P}(K)(a,q)$	The unnormalized HOMFLY polynomial of the knot K, where $\bar{P}(\text{unknot}) = (a - a^{-1})/(q - q^{-1})$.
$\text{HKR}^N_{i,j}(K)$	The *reduced* $\text{sl}(N)$ Khovanov–Rozansky homology of the knot K categorifying $P(K)$. Here i is the q-grading and j the homological grading.
$\overline{\text{HKR}}^N_{i,j}(K)$	The *unreduced* $\text{sl}(N)$ Khovanov–Rozansky homology of the knot K categorifying $P(K)$. Here i is the q-grading and j the homological grading.
$\text{KhR}_N(K)(q,t)$	The Poincaré polynomial of the *reduced* $\text{sl}(N)$ Khovanov–Rozansky homology of the knot K. In particular, $\text{KhR}_N(q, t = -1) = P(a = q^N, q)$.
$\overline{\text{KhR}}_N(K)(q,t)$	The Poincaré polynomial of the *unreduced* $\text{sl}(N)$ Khovanov–Rozansky homology of the knot K. In particular, $\overline{\text{KhR}}_N(q, t = -1) = \bar{P}(a = q^N, q)$.
$\mathcal{H}_{i,j,k}(K)$	A triply graded homology theory that categorifies $P(K)$. The indices i and j correspond to the variables a and q of $P(K)$ respectively, and k is the homological grading.
$\mathcal{P}(K)(a,q,t)$	The Poincaré polynomial of $\mathcal{H}_*(K)$, called the reduced superpolynomial of K. In particular, $\mathcal{P}(K)(a, q, t = -1) = P(a, q)$.
$\bar{\mathcal{P}}(K)(a,q,t)$	The *unreduced* superpolynomial of the knot K. This is the Poincaré polynomial of a triply graded theory categorifying $\bar{P}(K)$.
$\mathcal{P}_N(q,t)$	The Poincaré polynomial of the homology of $\mathcal{H}_*(K)$ with respect to the differential d_N.
$\Delta(K)(q)$	The Alexander polynomial of the knot K. With our conventions, it is a polynomial in q^2 and is equal to $P(a = 1, q)$.
$\widehat{\text{HFK}}(K)$	The knot Floer homology of the knot K.
$\text{HFK}(K)(q,t)$	The Poincaré polynomial of $\widehat{\text{HFK}}(K)$, with q corresponding to the Alexander grading, and t the homological grading.

TABLE 1. Notation.

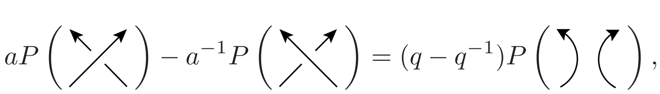
and 10_{124}), and this is why the superpolynomial for 10_{124} given in Section 8 differs from that in Section 6.

2.3 Signature

Our choice of sign for the signature $\sigma(K)$ of a knot K is such that $\sigma(T_{2,3}) = 2$. That is, negative knots have positive signatures.

2.4 Knot Polynomials

For us, the normalized HOMFLY polynomial P of an oriented link L is determined by the skein relation



together with the requirement that $P(\text{unknot}) = 1$. The unnormalized HOMFLY polynomial $\bar{P}(L)$ is determined by the alternative requirement that $\bar{P}(\text{unknot}) = (a - a^{-1})/(q - q^{-1})$.

Several different conventions for the HOMFLY polynomial can be found in the literature; another common one involves the change $a \to a^{1/2}$, $q \to q^{1/2}$. Also, sources sometimes simultaneously switch $a \to a^{-1}$ and $q \to q^{-1}$. For the negative torus knot $T_{2,3}$, the polynomial $P(T_{2,3})$ has all positive exponents of a.

For knots, our conventions are consistent with [Gukov et al. 05] (for links, the skein relation here differs by a sign). The papers of Khovanov and Rozansky [Khovanov 99, Khovanov 03, Khovanov and Rozansky 05, Khovanov and Rozansky 06] use the convention that a and q are replaced with their inverses. For the Knot atlas [Bar-Natan 05c], the conventions for HOMFLY agree with ours if you substitute $z = q - q^{-1}$; however, the Knot atlas's conventions for the Jones polynomial differ from ours by $q \to q^{-1}$.

2.5 Coefficients for Homology

All of our homology groups here, in whatever theory, are with coefficients in $\mathbb{Q}$. We expect that things would work out similarly if we used a different field of coefficients; it is less clear what would happen if we tried to use coefficients in $\mathbb{Z}$.

2.6 Khovanov—Rozansky Homology

For the Khovanov–Rozansky homology, there are at least two separate choices needed to fix a normalization. The first is the normalization of the HOMFLY polynomial, and the second is whether you want to view the theory as homology or cohomology. Most sources view it as cohomology (e.g., [Khovanov 99, Bar-Natan 02]), but here we choose to view it as a homology theory. To make it a homology theory, we take the standard cohomological chain complex and flip the homological grading by $i \mapsto -i$, so that the differentials are now grading decreasing. (One could also make it a homology theory by taking the dual complex with dual differentials, but that is not what we do.)

For instance, to put a Poincaré polynomial $\mathrm{KhR}_2(q,t)$ computed by the Knot Atlas [Bar-Natan 05c] or KhoHo [Shumakovitch 04] into our conventions, one needs to substitute $q \to q^{-1}$ and $t \to t^{-1}$. (The first substitution is due to the differing conventions for the Jones polynomial.) Notice that this change has the same effect as keeping the conventions fixed and replacing a knot by its mirror image.

2.7 Knot Floer Homology

Our conventions for knot Floer homology $\widehat{\mathrm{HFK}}$ are the opposite of the usual ones in [Ozsváth and Szabó 04a, Rasmussen 03]; in particular, our knot Floer homology is the standard knot Floer homology of the mirror. This has the effect of simultaneously flipping both the homological and Alexander gradings (see, e.g., [Ozsváth and Szabó 04a, equation 13]). In addition, we use different conventions for writing Poincaré polynomials HFK from those in [Rasmussen 05a]. For consistency with viewing the Alexander polynomial $\Delta(K)$ as a specialization of the HOMFLY polynomial, we view $\Delta(K)$ as the polynomial in q^2 given by $\Delta(K) = P(K)(a = 1, q)$. The variable t in HFK gives the homological grading. In [Rasmussen 05a], t is the variable for $\Delta(K)$ and u is used for the homological grading; one can translate information there into our conventions via the substitution $t \mapsto q^{-2}$, $u \mapsto t^{-1}$.

3. FAMILIES OF DIFFERENTIALS AND RELATION TO KNOT HOMOLOGIES

As discussed in Section 1.3, we can expect uniform behavior for the sl(N) homology only for large N. In this section, we detail the additional structure that should encode the sl(N) homology for all N, and knot Floer homology as well. We start by assuming homology groups $\mathcal{H}_{i,j,k}(K)$ categorifying the reduced HOMFLY polynomial $P(K)(a,q)$. The Poincaré polynomial of this homology is the superpolynomial given by

$$\mathcal{P}(K)(a,q,t) = \sum a^i q^j t^k \dim \mathcal{H}_{i,j,k}(K).$$

In addition, $\mathcal{H}_*(K)$ should be equipped with a family of differentials $\{d_N\}$ for $N \in \mathbb{Z}$, which will give the different homologies. The differentials should satisfy the following axioms:

Grading: For $N > 0$, d_N is triply graded of degree $(-2, 2N, -1)$, i.e.,

$$d_N : \mathcal{H}_{i,j,k}(K) \to \mathcal{H}_{i-2,j+2N,k-1}(K);$$

d_0 is graded of degree $(-2, 0, -3)$, and for $N < 0$, d_N has degree $(-2, 2N, -1 + 2N)$.

Anticommutativity: $d_N d_M = -d_M d_N$ for all $N, M \in \mathbb{Z}$. In particular, $d_N^2 = 0$ for each $N \in \mathbb{Z}$.

Symmetry: There is an involution $\phi : \mathcal{H}_{i,j,*} \to \mathcal{H}_{i,-j,*}$ with the property that

$$\phi d_N = d_{-N} \phi \quad \text{for all } N \in \mathbb{Z}.$$

To build the connection to the other homology theories, first notice that we get a categorification of $P_N(K)$ by amalgamating groups to define

$$\mathcal{H}^N_{p,k}(K) = \bigoplus_{iN+j=p} \mathcal{H}_{i,j,k}(K).$$

The Poincaré polynomial of these new groups is just $\mathcal{P}(K)(a = q^N, q, t)$. For $N > 0$, the first two axioms above imply that $(\mathcal{H}^N_{l,k}(K), d_N)$ is a bigraded chain complex. We can now state our main conjecture:

Conjecture 3.1. *There is a homology theory $\mathcal{H}_*$ categorifying the HOMFLY polynomial, equipped with differentials $\{d_N\}$ satisfying the three axioms. For all $N > 0$, the homology of $(\mathcal{H}^N_*(K), d_N)$ is isomorphic to the* sl(N) *Khovanov–Rozansky homology. For $N = 0$, $(\mathcal{H}^0_*(K), d_0)$ is isomorphic to the knot Floer homology.*

For the last part of this conjecture, one must do additional regrading of $\mathcal{H}^0_*(K)$ to make it precise; see Section 3.2. Let us denote the Poincaré polynomial of the bigraded homology of $(\mathcal{H}^N_*(K), d_N)$ by $\mathcal{P}_N(K)$; the Khovanov–Rozansky part of the conjecture is thus summarized as $\mathcal{P}_N(K) = \mathrm{KhR}_N(K)$.

A few general comments are in order. First, for any given knot K, the superpolynomial has finite support, so the grading condition forces d_N to vanish for N sufficiently large. Thus the earlier Conjecture 1.3 is a special case of Conjecture 3.1.

Second, we remark that the symmetry property generalizes the well-known symmetry of the HOMFLY polynomial:

$$P(K)(a, q) = P(K)(a, q^{-1}).$$

Finally, the homological grading of d_N for $N < 0$ may strike the reader as somewhat peculiar. As we will explain in Section 3.3, it is a natural consequence of the symmetry ϕ.

3.1 Examples

To illustrate the properties above, we consider three examples, starting with the easy case of the unknot.

Example 3.2. (The unknot.) For the unknot U, all the $\mathrm{sl}(N)$ homology is known and $\mathrm{KhR}_N(U) = 1$ for all $N > 0$. Thus the superpolynomial is clearly given by $\mathcal{P}(U) = 1$, where all the differentials d_N are identically zero.

Example 3.3. (The trefoil.) The HOMFLY polynomial of the negative trefoil knot $T_{2,3}$ is given by $P(T_{2,3}) = a^2q^{-2} + a^2q^2 - a^4$. The corresponding superpolynomial also has three terms:

$$\mathcal{P}(T_{2,3}) = a^2q^{-2}t^0 + a^2q^2t^2 + a^4q^0t^3.$$

To illustrate the differentials, it is convenient to represent $\mathcal{H}(K)$ by a *dot diagram* as shown in Figure 2.

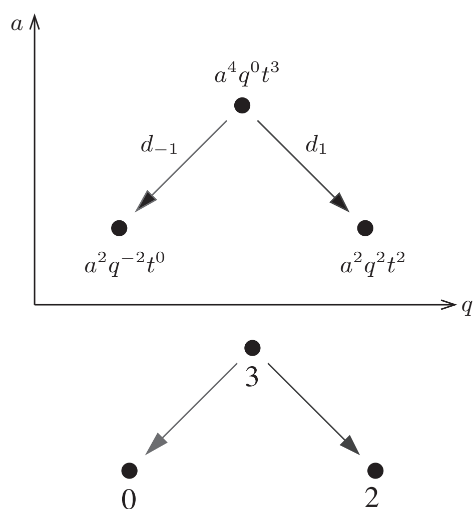


FIGURE 2. Nonzero differentials for the trefoil knot. Above is a fully labeled diagram, and below is the more condensed form that we will use from now on. The minimum a-grading is 2.

We draw one dot for each term in the superpolynomial, so that the total number of dots is equal to the dimension of $\mathcal{H}(K)$. The dots' position on horizontal axis records the power of q, and on the vertical, the power of a. The top image in Figure 2 shows such a diagram for the trefoil, with each dot labeled by its corresponding monomial.

Since the relative a and q gradings are determined by the position of the dots, we omit them from the diagram and just label each dot by its t-grading. To fix the absolute a-grading, we record the a-grading of the bottom row. Determining the absolute q-grading from such a picture is easy, since the line $q = 0$ corresponds to the vertical axis of symmetry. The nonzero components of d_i are shown by arrows of slope $-1/i$. As indicated by the figure, the trefoil has two nontrivial differentials: d_1 and d_{-1}.

Now let's substitute $a = q^N$ and take homology with respect to d_N. For $N > 1$, there are no differentials, and so we just get $\mathcal{P}_N(T_{2,3}) = q^{2N-2}t^0 + q^{2N+2}t^2 + q^{2N}t^3$. For $N = 1$, the differential d_1 kills the two right-hand generators, and we are left with $\mathcal{P}_1(T_{2,3}) = 1$. In this case, it is possible to check directly that $\mathcal{P}_N = \mathrm{KhR}_N$ for all $N > 0$. Note that KhR_1 of any knot is always $1 = q^0t^0$, which is why $d_{\pm 1}$ must be nonzero even in such a simple example as this.

Example 3.4. ($T_{3,4}$.) A more complicated example is provided by the negative $(3, 4)$ torus knot, which is the mirror of the knot 8_{19}. In this case, both the HOMFLY polynomial and the superpolynomial have 11 nontrivial terms:

$$\begin{aligned} P(T_{3,4}) &= a^{10} - a^8(q^{-4} + q^{-2} + 1 + q^2 + q^4) \\ &\quad + a^6(q^{-6} + q^{-2} + 1 + q^2 + q^6), \\ \mathcal{P}(T_{3,4}) &= a^{10}t^8 + a^8(q^{-4}t^3 + q^{-2}t^5 + t^5 + q^2t^7 + q^4t^7) \\ &\quad + a^6(q^{-6}t^0 + q^{-2}t^2 + t^4 + q^2t^4 + q^6t^6). \end{aligned}$$

The superpolynomial is illustrated by the dot diagram in Figure 3.

Here there are five nontrivial differentials: d_{-2}, d_{-1}, d_0, d_1, and d_2. To understand the differentials completely, think of the dots as representing specific basis vectors for $\mathcal{H}_{i,j,k}$; then an arrow means that the corresponding d_N takes the basis element at its tail to $\pm$ the basis element at its tip. In this case, the sign can be inferred from the diagram; those that switch the sign have a small circle at their tails. (To avoid clutter, hereinafter we will leave it to the reader to choose appropriate signs for the differentials.) It is now easy to check that all the d_N anticommute.

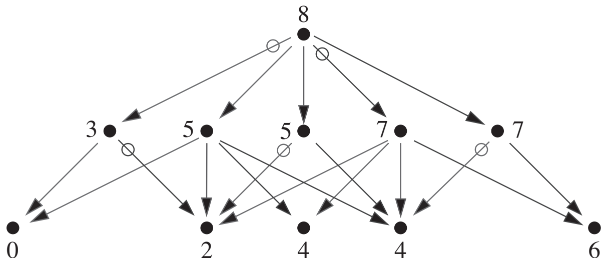


FIGURE 3. Differentials for $T_{3,4}$.The bottom row of dots has a-grading 6. The leftmost dot on that row has q-grading -6, which you can determine by noting that the vertical axis of symmetry corresponds to the line $q = 0$.

The symmetry involution ϕ corresponds to flipping the diagram about its vertical axis of symmetry. For the $\mathcal{H}_*$ off the line itself, ϕ permutes our preferred basis vectors; on $\mathcal{H}_{10,0,8}$ and $\mathcal{H}_{8,0,5}$ it acts by $-\mathrm{Id}$, but is the identity on $\mathcal{H}_{6,0,4}$. You can now easily check the symmetry axiom.

Substituting $a = q^2$ and taking homology with respect to d_2 kills six generators, leaving

$$\mathcal{P}_2(T_{3,4}) = q^6t^0 + q^{10}t^2 + q^{12}t^3 + q^{12}t^4 + q^{16}t^5,$$

which is the ordinary ($N = 2$) Khovanov homology of $T_{3,4}$. As before, $\mathcal{P}_1(T_{3,4}) = 1$; only the bottom leftmost term survives.

3.2 Relation to Knot Floer Homology

In order to recover the knot Floer homology, we must introduce a new homological grading on $\mathcal{H}(K)$, which is given by $t'(x) = t(x)-a(x)$. In other words, the Poincaré polynomial of $\mathcal{H}$ with respect to the new grading is

$$\mathcal{P}'(a, q, t) = \mathcal{P}(a = at^{-1}, q, t).$$

The differential d_0 lowers the new grading t' by 1. Now forget the a-grading (i.e., substitute $a = 1$), and take the homology with respect to d_0. We denote the Poincaré polynomial of this homology by $\mathcal{P}_0(K)(q, t)$, and this homology categorifies the Alexander polynomial $\Delta(K)(q^2) = P(K)(a = 1, q)$. A precise statement of the last part of Conjecture 3.1 is that $\mathcal{P}_0(K) = \mathrm{HFK}(K)$, where HFK is the Poincaré polynomial of knot Floer homology defined in (1–8).

As a first example of this process, consider the trefoil knot. Figure 4 shows the generators for $\mathcal{H}(T_{2,3})$ with respect to the new homological grading t'. The differential d_0 is trivial, so we obtain

$$\mathcal{P}_0(T_{2,3}) = \mathcal{P}(T_{2,3})(a = t^{-1}, q, t) = q^{-2}t^{-2}+q^0t^{-1}+q^2t^0,$$

which is indeed equal to $\mathrm{HFK}(T_{2,3})$.

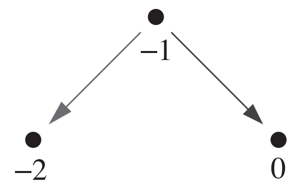


FIGURE 4. Trefoil with new homological gradings.

Next we consider $T_{3,4}$, for which d_0 kills 6 of the 11 generators. We leave it to the reader to check that after regrading and taking homology with respect to d_0, we are left with

$$\mathcal{P}_0(K) = q^{-6}t^{-6} + q^{-4}t^{-5} + q^0t^{-2} + q^4t^{-1} + q^6t^0,$$

which agrees with $\mathrm{HFK}(T_{3,4})$.

3.3 The δ-Grading and Symmetry

It is natural to consider a fourth grading on $\mathcal{H}(K)$, obtained as a linear combination of the a, q, and t gradings. It is defined by

$$\delta(x) = t(x) - a(x) - q(x)/2.$$

When we specialize to $\widehat{\mathrm{HFK}}$ or HKR_2, the δ-grading reduces to the δ-gradings on these two theories defined in [Rasmussen 03]. Indeed, if q_2 is the q-grading on HKR_2 defined by setting $a = q^2$, then

$$t(x) - a(x) - \frac{q(x)}{2} = t(x) - \frac{2a(x) + q(x)}{2} = t(x) - \frac{q_2(x)}{2},$$

where q_2 denotes the q-grading on HKR_2 and the rightmost expression is the *definition* of the δ-grading on HKR_2. Similarly, if t' is the homological grading on $\widehat{\mathrm{HFK}}$, defined by setting $a = 1/t$, then

$$t(x) - a(x) - q(x)/2 = t'(x) - q(x)/2,$$

where the right-hand side is the definition of the δ-grading on $\widehat{\mathrm{HFK}}$.

We can use the δ-grading to justify the somewhat peculiar behavior of d_i for $i < 0$ with respect to the homological grading. In analogy with knot Floer homology, where the δ-grading is preserved by the conjugation symmetry, we expect that the δ-grading will be preserved by the symmetry ϕ of Conjecture 3.1. For $i > 0$, the differential d_i lowers the δ-grading by $1-i$. Since ϕ exchanges d_i and d_{-i}, the differential d_{-i} should lower the δ grading by $1-i$ as well. It is then easy to see that d_{-i} lowers the homological grading by $-1-2i$.

3.4 Canceling Differentials

Let (C, d) be a chain complex. We say that d is a *canceling differential* on C if the homology of C with respect to d is one-dimensional. The presence of a canceling differential is an important feature of all the reduced knot homologies. For $\widehat{\mathrm{HFK}}$, this was known from the start—essentially, it is the fact that $\widehat{\mathrm{HF}}(\mathbf{S}^3) \cong \mathbb{Z}$. For the sl(2) Khovanov homology, it follows from work of Turner [Turner 04], which itself builds on work of Lee [Lee 02b] and Bar-Natan [Bar-Natan 05a]. Finally, the existence of such a differential for HKR_N can be derived by combining Turner's results with the work of Gornik [Gornik 04] in the unreduced case.

Conjecture 3.1 provides a unified explanation for the presence of these canceling differentials. Indeed, for any knot K, $\mathcal{P}_1(K) = 1$, which implies that d_1 should be a canceling differential on $\mathcal{H}(K)$. We expect that the known differentials on the various specializations of $\mathcal{H}$ are all induced by the action of d_1.

To state this more precisely, let us suppose that Conjecture 3.1 is true. Since d_1 anticommutes with d_N, the pair $(\mathcal{H}(K), d_1 + d_N)$ is also a chain complex. Consider the grading on $\mathcal{H}(K)$ obtained by setting $a = q^N$. This grading is preserved by d_N, but is strictly lowered by d_1. In other words, it makes $(\mathcal{H}(K), d_1 + d_N)$ into a filtered complex whose associated graded complex is $(\mathcal{H}(K), d_N)$. Since we are using rational coefficients, we can reduce this complex to a chain-homotopy-equivalent complex of the form $(H_*(\mathcal{H}(K), d_N), d_1')$. (See Lemma 4.5 of [Rasmussen 03] for a proof.)

Proposition 3.5. *If we assume that Conjecture 3.1 holds, then d_1' is a canceling differential on $H_*(\mathcal{H}(K), d_N)$ whenever $N \neq 1$.*

Proof: We again consider the complex $(\mathcal{H}(K), d_1 + d_N)$, but with a different grading, namely, the one defined by setting $a = q$. It is easy to see that d_1 preserves the new grading, while d_N strictly raises it, so this grading also makes $(\mathcal{H}(K), d_1 + d_N)$ into a filtered complex. Reducing as before, we obtain a chain-homotopy-equivalent complex $(H_*(\mathcal{H}(K), d_1), d_N')$. Assuming that the conjecture is true, $H_*(\mathcal{H}(K), d_1) \cong \mathrm{HKR}_1(K)$ is one-dimensional, so

$$\begin{aligned} H_*(H_*(\mathcal{H}(K), d_N), d_1') &\cong H_*(\mathcal{H}(K), d_1 + d_N) \\ &\cong H_*(H_*(\mathcal{H}(K), d_1), d_N') \\ &\cong H_*(\mathcal{H}(K), d_1) \end{aligned}$$

is one-dimensional as well. □

An interesting consequence of Conjecture 3.1 is that it predicts the existence of a *second* canceling differential on HKR_N. Indeed, the symmetry property implies that d_{-1} is also a canceling differential on $\mathcal{H}$, and the same argument used for d_1 implies that it should descend to a differential on any specialization of $\mathcal{H}$.

In the case of $\widehat{\mathrm{HFK}}$, it is well known that two such differentials exist, and that they are exchanged by the conjugation symmetry (see, e.g., [Rasmussen 03, Proposition 4.2]). To illustrate this fact, we consider the knot Floer homology of the trefoil. There, $\widehat{\mathrm{HFK}}(T_{2,3})$ has three generators, corresponding to monomials $q^{-2}t^{-2}$, q^0t^{-1}, and q^2t^0 in the Poincaré polynomial. Looking at Figure 4, we see that the differential induced by d_{-1} takes the second generator to the first, while the differential induced by d_1 takes the second generator to the third. This is indeed the differential structure on $\widehat{\mathrm{HFK}}(T_{2,3})$.

In general, the differential induced by d_{-1} should correspond to the usual differential on $\widehat{\mathrm{HFK}}$ (that is, the one that lowers the Alexander grading), while the differential induced by d_1 corresponds to its conjugate symmetric partner. As a check, let us consider how the two induced differentials behave with respect to the homological grading t'. Since both d_0 and d_{-1} lower the homological grading by 1, the induced map d_{-1*} will lower t' by 1 as well. This is in accordance with the behavior of the usual differential on $\widehat{\mathrm{HFK}}$. In contrast, d_1 raises t' by 1, so the behavior of d_{1*} with respect to t' is somewhat more complicated. In fact, it is not hard to see that if some component of d_{1*} raises the q-grading by $2k$, it will raise t' by $2k-1$. This is precisely the behavior exhibited by the "conjugate" differentials in knot Floer homology.

In contrast, the differential d_N that gets us from $\mathcal{H}(K)$ to $\mathrm{HKR}_N(K)$ lowers the usual homological grading on $\mathcal{H}(K)$ by 1, as does d_1. Thus the differential induced by d_1 on $\mathrm{HKR}_N(K)$ will respect the homological grading on that group. We expect that d_{1*} corresponds to the differential of Lee, Turner, and Gornik. As an example consider the sl(2) homology of the trefoil. Here, we have $\mathcal{P}_2(T_{2,3}) = q^2t^0 + q^6t^2 + q^8t^3$, and the differential induced by d_1 takes the third term to the second. This agrees with the standard canceling differential on the reduced Khovanov homology.

As far as we are aware, the presence of a second canceling differential on the Khovanov homology has not been considered before. Although we do not know how to construct such a differential directly, in Section 5.6 we describe some evidence that supports the idea that HKR_2 admits an additional canceling differential induced by d_{-1}.

3.5 Analogue of s and τ

Given a canceling differential on a filtered chain complex, one can define a simple invariant by considering the filtration grading of the (unique) generator on homology. Applying this fact to knot Floer homology, Ozsváth and Szabó [Ozsváth and Szabó 03b] defined a knot invariant $\tau(K)$, which carries information about the four-ball genus of K. Subsequently, an analogous invariant s was defined using the Khovanov homology [Rasmussen 04].

On the triply graded homology theory $\mathcal{H}(K)$, the canceling differential d_1 can be used to define a similar invariant. Since there are two polynomial gradings on $\mathcal{H}(K)$, it initially looks as though we should get two invariants. In reality, however, the generator of the homology with respect to d_1 always lies on the line where $q(x) = -a(x)$. This is because when we specialize to the sl(1) theory by substituting $a = q$, the generator corresponds to the unique term in $\mathcal{P}_1(K) = 1$. After taking homology with respect to d_1, the surviving term will have the form $a^S q^{-S} t^0$. The number S will be an invariant of K analogous to s and τ.

For example, if K is the $(3, 4)$ torus knot, a glance at Figure 3 shows that $S(K) = 6$. This example illustrates an interesting feature of S, namely, that it is in some sense easier to compute than either s or τ. Indeed, to compute S, we need only consider those generators of $\mathcal{H}(K)$ that lie along the line $a(x) = -q(x)$. In many cases (like the one above) the number of generators we need to consider is quite small.

In analogy with the known properties of S and τ, we expect that S will be a lower bound for the four-ball genus of K (see Section 5.4). It is not clear, however, whether it contains any new information, since in all the examples we have considered, it appears that $S(K) = s(K) = 2\tau(K)$. We hope that further consideration of the construction of S will shed new light on the relationship between s and τ, either by proving that all three quantities are equal, or by suggesting where to look for a counterexample.

3.6 Motivation for the Conjecture

We conclude this section by briefly sketching the background to Conjecture 3.1, and indicating how strongly we believe its various parts. Our main reason for expecting the presence of the differentials d_N for $N > 0$ comes from Gornik's work on the sl(M) homology. In [Gornik 04], Gornik describes a deformation of Khovanov and Rozansky's construction that gives rise to a canceling differential on HKR_M. In fact, this construction may be easily modified to obtain a whole family of deformations, one for each monic polynomial of degree M. It follows that any monic polynomial of degree M gives rise to a differential on HKR_M. If we let $d_N^{(M)}$ be the differential corresponding to the polynomial $X^M - X^N$, we expect that the differential d_N of the conjecture can be obtained as the limit of $d_N^{(M)}$ as $M \to \infty$. In analogy with Gornik's work, we expect that taking the homology of $\mathrm{HKR}_M(K)$ with respect to this differential $d_N^{(M)}$ will give the group $\mathrm{HKR}_N(K)$, thus matching the behavior predicted by Conjecture 3.1. (Indeed, this observation was the genesis of the conjecture.) For $N > 0$, the behavior expressed by the grading axiom was chosen to agree with the known behavior of $d_N^{(M)}$. Finally, the fact that d_{N_1} and d_{N_2} $(N_1, N_2 > 0)$ anticommute should follow from the linearity of the space of deformations. More precisely, if we let $d_{N_1,N_2}^{(M)}$ be the differential corresponding to the polynomial $X^M - X^{N_1} - X^{N_2}$, then $d_{N_1,N_2}^{(M)} = d_{N_1}^{(M)} + d_{N_2}^{(M)}$, so the fact that $\left(d_{N_1,N_2}^{(M)}\right)^2 = 0$ implies that $d_{N_1}^{(M)}$ and $d_{N_2}^{(M)}$ anticommute.

The rest of the conjecture is more speculative. Our original reason for expecting the presence of the differentials d_N for $N \leq 0$ was based on analogy with the knot Floer homology. We believe that the strong internal consistency of the theory, as seen in the examples of Section 5, together with the apparently correct predictions it makes (such as the computations of the stable sl(2) Khovanov homology of the torus knots in Section 7.3), indicate that there must be *something* meaningful going on. It is possible, however, that we have erred in stating the exact details. Below, we outline some potential weak points of Conjecture 3.1.

- We are not currently aware of any construction that might give rise to the d_N's for $N \leq 0$. Our reasons for expecting their existence are based on analogy with the case $N > 1$, which suggests that there should be a differential d_0 giving rise to knot Floer homology, and with knot Floer homology itself, whose symmetries suggest the presence of d_N for $N < 0$.

- The statement in the conjecture about the gradings of differentials is somewhat stronger than would be expected from Gornik's work. A priori, the differentials coming from Gornik's theory should shift the (a, q) bigrading by some multiple of $(-2, 2N)$. The requirement that this multiple be always one is imposed to ensure that d_N shifts both t and t' by a constant amount. (Some further support for this idea is provided by the fact that there are a number of ten-crossing knots that at first glance look as if d_1 might lower the (a, q) bigrading by $(-4, 4)$. In all these examples, however, further examination suggests that this is not the case.)

- Finally, there is some chance that taking homology with respect to d_0 does not give the knot Floer homology, but some other categorification of the Alexander polynomial that happens to look a lot like it. An interesting test case for this possibility is provided by the presence of mutant knots with different genera. For example, there are several mutant pairs of 11-crossing knots, one of which has genus one bigger than the other. These knots have the same HOMFLY polynomial and KhR_2, but their knot Floer homologies must differ. It is an interesting question to determine whether these knots have the same superpolynomial and (if they do) the same differentials.

4. GEOMETRIC INTERPRETATION

In this section, we explain in more detail the geometric interpretation of the triply graded knot homology in the language of open Gromov–Witten theory. As discussed in Section 1.2, this relation was part of the original motivation for the triply graded theory, and we hope it can be useful for developing both sides of the correspondence. In this section, we mainly consider the unreduced homology, which has a more direct relation to the geometry of holomorphic curves.

The geometric setup consists of the following data: a noncompact Calabi–Yau 3-fold X and a Lagrangian submanifold $\mathcal{L} \subset X$. Therefore, for every knot $K \subset \mathbf{S}^3$, we need to define X and $\mathcal{L}$. The Calabi–Yau space X is independent of the knot; it is defined as the total space of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{C}\mathbf{P}^1$:

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}\mathbf{P}^1. \tag{4–1}$$

On the other hand, the information about the knot K is encoded in the topology of the Lagrangian submanifold, which we denote by $\mathcal{L}_K$ to emphasize that it is determined by the knot:

$$K \rightsquigarrow \mathcal{L}_K.$$

A systematic construction of the Lagrangian submanifold $\mathcal{L}_K$ from a braid diagram of K was proposed by Taubes [Taubes 01]. It involves two steps. First, one constructs a two-dimensional noncompact Lagrangian submanifold $\mathcal{L}_K^{(2)} \subset \mathbb{C}^2$, which has the property that its intersection with a large-radius 3-sphere $\mathbf{S}^3 \subset \mathbb{C}^2$ is isotopic to the knot K. Then, we identify $\mathbb{C}^2 \otimes \mathcal{O}(-1)$ with a fiber of X and define $\mathcal{L}_K$ to be a particular subbundle $\mathcal{L}_K^{(2)} \to \mathbf{S}^1$ of the bundle (4–1) restricted to the equator $\mathbf{S}^1 \subset \mathbb{C}\mathbf{P}^1$. The construction is such that $\mathcal{L}_K$ is Lagrangian with respect to the standard Kähler form on X. Moreover, for every knot K, the resulting 3-manifold $\mathcal{L}_K$ has the first Betti number $b_1(\mathcal{L}_K) = 1$.

Given a Calabi–Yau space X and a Lagrangian submanifold $\mathcal{L}_K \subset X$, it is natural to study holomorphic Riemann surfaces in X with Lagrangian boundary conditions on $\mathcal{L}_K$:

$$(\Sigma, \partial\Sigma) \hookrightarrow (X, \mathcal{L}_K). \tag{4–2}$$

Specifically, we consider embedded surfaces Σ that satisfy the following conditions:

1. Σ is a holomorphic Riemann surface with a fixed genus g and one boundary component, $\partial\Sigma \cong \mathbf{S}^1$.
2. $[\Sigma] = Q$ with Q a fixed class in $H_2(X, \mathcal{L}_K; \mathbb{Z}) \cong \mathbb{Z}$.
3. $[\partial\Sigma] = \gamma$, where γ generates the free part of the homology group $H_1(\mathcal{L}_K, \mathbb{Z}) \cong \mathbb{Z}\gamma$ (modulo torsion).

Now we are ready to define the moduli spaces that appear in the geometric interpretation of the triply graded theory, cf. (1–5). Let Σ be an embedded Riemann surface that satisfies the three conditions and let $A \in \Omega^1(\Sigma)$ be a flat $U(1)$ gauge connection on Σ,

$$F_A = 0.$$

We define $\mathcal{M}_{g,Q}(X, \mathcal{L}_K)$ to be moduli "space" of the embedded Riemann surfaces Σ with a gauge connection A, modulo the gauge equivalence, $A \to A + df$, where $f \in \Omega^0(\Sigma)$. Assuming that the dependence on X and $\mathcal{L}_K$ is clear from the context, we often refer to this moduli space simply as $\mathcal{M}_{g,Q}$. The cohomology groups $H^k(\mathcal{M}_{g,Q})$ are labeled by three integers: the degree k, the genus g, and the relative homology class $Q \in H_2(X, \mathcal{L}_K; \mathbb{Z}) \cong \mathbb{Z}$. These are the three gradings of our triply graded theory.

Remark 4.1. Since in general $\mathcal{M}_{g,Q}$ may be singular and noncompact, one needs to be careful about the definition of $H^k(\mathcal{M}_{g,Q})$. This problem is familiar in the closely related context of Gromov–Witten theory, where instead of embedded Riemann surfaces with a flat connection one "counts" stable holomorphic maps (possibly with boundary). In Gromov–Witten theory, there is a way to define cohomology classes and intersection theory on the moduli spaces of stable maps (see [Katz and Liu 02, Li and Song 02, Graber and Zaslow 02] for some recent work on the mathematical formulation and calculation of the open Gromov–Witten invariants). Similarly, the physical interpretation of the sl(N) knot homology [Gukov et al. 05] suggests that, at least in the present case, there should exist a suitable definition of $\mathcal{M}_{g,Q}$ such that the cohomology groups $H^k(\mathcal{M}_{g,Q})$ can be identified with the triply graded knot homology groups.

Example 4.2. (The unknot.) In this case, the only nontrivial holomorphic curves are holomorphic disks wrapped on the northern and the southern hemispheres of $\mathbb{C}\mathbf{P}^1 \subset X$. Their moduli spaces are isolated points, $\mathcal{M}_{g,Q} \cong \text{pt}$ for $g = 0$ and $Q = \pm 1$, which correspond to the two terms a and a^{-1} in the unreduced superpolynomial for the unknot $\bar{\mathcal{P}}(a, q, t) = a - a^{-1}$.

4.1 Genus Expansion and Symmetry

Now let us look more closely at the structure of the moduli space $\mathcal{M}_{g,Q}$, assuming that it is well-defined. Let Σ be a nondegenerate Riemann surface of genus g. The moduli space of gauge equivalence classes of flat $U(1)$ connections $A \in \Omega^1(\Sigma)$ is isomorphic to a $2g$-dimensional torus,

$$\mathrm{Hom}(\pi_1(\Sigma); U(1))/U(1) \cong T^{2g}.$$

Therefore, $\mathcal{M}_{g,Q}$ has the structure of a fibration

$$\begin{array}{ccc} T^{2g} & \rightarrow & \mathcal{M}_{g,Q} \\ & & \downarrow \\ & & \mathcal{M}_{g,Q}^{\text{geom}} \end{array} \qquad (4\text{–}3)$$

where $\mathcal{M}_{g,Q}^{\text{geom}}$ is the moduli space of embedded Riemann surfaces (4–2) that satisfy the three conditions given earlier. In many cases, the fibration structure (4–3) can be recognized directly in the structure of the superpolynomial written in terms of the variables a, t, and y, where $y = (qt^{1/2} + q^{-1}t^{-1/2})^2$. In particular, the contribution of an isolated Riemann surface with genus g and relative homology class Q looks like [Gukov et al. 05]

$$a^Q t^r (qt^{1/2} + q^{-1}t^{-1/2})^{2g},$$

where the last factor is the familiar Poincaré polynomial of T^{2g}. In general, the superpolynomial $\bar{\mathcal{P}}(K)$ should have the structure

$$\bar{\mathcal{P}}(K) = \sum_{g,Q,i} \hat{D}_{Q,g,i} a^Q t^i (qt^{1/2} + q^{-1}t^{-1/2})^{2g}, \qquad (4\text{–}4)$$

where $\hat{D}_{Q,g,i} \in \mathbb{Z}$ encode the geometry of the fibration (4–3). We refer to the expansion (4–4) as the genus expansion. It is natural to expect a similar structure also in the case of the reduced superpolynomial $\mathcal{P}(K)$. Notice that in the reduced case, the expansion of the form (4–4) is equivalent to the existence of the symmetry

$$\phi : \mathcal{H}_{i,j,*}(K) \rightarrow \mathcal{H}_{i,-j,*}(K) \qquad (4\text{–}5)$$

that we discussed earlier, in Section 3. In the geometric interpretation, this symmetry follows from the fibration structure (4–3).

For the genus expansion of the reduced superpolynomial, let us also define the *holomorphic genus*, $g_h(K)$, to be the maximum value of g that occurs in the sum (4–4). It has a clear geometric meaning as the maximum genus of the holomorphic Riemann surface (4–2) that satisfies the three conditions. With this definition, $2g_h(K)$ is equal to the maximum power of q that appears in the reduced superpolynomial. The conjectured relation with knot Floer homology suggests the following bound:

$$g_3(K) \leq g_h(K), \qquad (4\text{–}6)$$

where $g_3(K)$ is the Seifert genus of K.

4.2 Relation to Gromov–Witten Invariants

Let us conclude this section by noting that taking the Euler characteristic in the triply graded knot homology $\mathcal{H}_*(K)$ translates into taking the Euler characteristic in $H^*(\mathcal{M}_{g,Q})$. On the other hand, the invariants $\chi(\mathcal{M}_{g,Q})$, which in the physics literature are called "integer BPS invariants," contain the information about all-genus open Gromov–Witten invariants of $(X, \mathcal{L}_K)$ [Ooguri and Vafa 00, Labastida et al. 00]. The relation between the open Gromov–Witten invariants and the integer BPS invariants is highly nontrivial. For example, the genus-counting parameter u in the open Gromov–Witten theory is related to the variable q that we use via the following change of variables (also familiar in the context of the closed Gromov–Witten theory [Maulik et al. 05]):

$$q = e^{iu}. \qquad (4\text{–}7)$$

Via this relation, all the information about the relative Gromov–Witten theory of $(X, \mathcal{L}_K)$ can be compactly recorded in a finite set of nonzero integer BPS invariants. One can use this relationship both ways. In particular, one can find the Euler characteristic $\chi(\mathcal{M}_{g,Q})$ by computing the open Gromov–Witten invariants, say via the localization technique [Graber and Zaslow 02, Katz and

Liu 02, Li and Song 02]. It would be interesting to extend the existing techniques to compute the dimensions of the individual cohomology groups $H^k(\mathcal{M}_{g,Q})$.

5. EXAMPLES AND PATTERNS

We now describe the superpolynomials associated with some specific knots with 10 or fewer crossings. Although we lack a definition for the triply graded theory and are unable to compute the sl(N) homology in general, we can still make intelligent guesses as to the form of the superpolynomial, based on Conjecture 3.1 and the known values of $\widehat{\mathrm{HFK}}$ and HKR_2. These example illustrate the internal consistency of the structure proposed in Conjecture 3.1. Once we have looked at these examples, we explore some patterns observed there in more detail in Sections 5.3–5.6.

5.1 Thin Knots

In both knot Floer homology and sl(2) Khovanov homology, the smallest knots exhibit the following simple behavior: If we plot the homological grading versus the polynomial grading, all the generators line up along a single line. Moreover, this line always has the same slope, which corresponds to the appropriate δ-grading being constant (see Section 3.3 for definitions). Such knots are called *thin* (with respect to either $\widehat{\mathrm{HFK}}$ or HKR_2). In the triply graded case, we can define thinness analogously:

Definition 5.1. A knot K is *$\mathcal{H}$–thin* if all generators of $\mathcal{H}(K)$ have the same δ-grading.

For an $\mathcal{H}$–thin knot, the t-grading of a term of $\mathcal{P}(K)$ is determined by the a- and q- gradings. Thus, there can be no cancellation when we specialize $\mathcal{P}(K)$ to $P(K)$, and so $\mathcal{P}(K)$ is completely determined by its HOMFLY polynomial and the common δ-grading of its generators. Noting that the common δ-grading is equal to $-S(K)/2$, the precise relationship between $\mathcal{P}(K)$ and $P(K)$ is concisely expressed by

$$\mathcal{P}_K(a,q,t) = (-t)^{-S(K)/2} P_K(at, iqt^{1/2}).$$

If K is thin, the dimension of $\mathcal{H}(K)$ is equal to the determinant of K. Moreover, all differentials other than d_1 and d_{-1} automatically vanish, since these differentials lower the δ-grading. Finally, the fact that d_1 and d_{-1} anticommute and each has one-dimensional homology implies that $\mathcal{H}(K)$ can be decomposed as the direct sum of a number of "squares" with Poincaré polynomial $a^i q^j t^k(1+a^{-2}q^2t^{-1})(1+a^{-2}q^{-2}t^{-3})$ and a single "sawtooth" summand isomorphic to $\mathcal{H}(T_{2,k})$ for some value of k. It follows that

$$\mathcal{P}_K(a,q,t) = \mathcal{P}_{T_{2,k}}(a,q,t) + (1+a^{-2}q^2t^{-1})(1+a^{-2}q^{-2}t^{-3})Q(a,q,t),$$

where Q is a polynomial with positive coefficients. We thus obtain a restriction on the HOMFLY polynomial of a thin knot: if $T_{2,k}$ is a torus knot whose signature is equal to $S(K)$, the polynomial

$$\frac{P(K) - P(T_{2,k})}{(1-a^{-2}q^2)(1-a^{-2}q^{-2})}$$

must be alternating.

As with $\widehat{\mathrm{HFK}}$ and HKR_2, we expect that some classes of simple knots are $\mathcal{H}$–thin. In particular, we make the following conjecture:

Conjecture 5.2. *If K is a two-bridge knot, then K is $\mathcal{H}$–thin, and $S(K) = \sigma(K)$.*

Since two-bridge knots are alternating and hence thin for $\widehat{\mathrm{HFK}}$ and HKR_2 [Ozsváth and Szabó 03a, Lee 02a], it is easy to check that Conjecture 5.2 holds for $N = 0, 1, 2$. Thus, to prove it one needs to show that for $N \geq 3$,

$$\mathrm{KhR}_N(K)(q,t) = (-t)^{-\sigma(K)/2} P(K)(q^N t, iqt^{1/2}). \quad (5\text{–}1)$$

Most of Conjecture 5.2 has been proved in [Rasmussen 05b], where it is shown that (5–1) holds for all $N \geq 5$. The proof uses only elementary properties of Khovanov and Rozansky's original definition, in particular the skein exact sequence. The approach has difficulties for $N = 3$ or 4, and this portion of Conjecture 5.2 remains open. All knots with fewer than 8 crossings are two-bridge. Their superpolynomials (assuming the conjecture) are shown in Table 2.

It is well known [Ozsváth and Szabó 03a], [Lee 02a] that alternating knots are thin with respect to both $\widehat{\mathrm{HFK}}$ and HKR_2. However, the analogous statement for $\mathcal{H}$–thinness cannot be true. To see why, we introduce the notion of a knot having an *alternating* HOMFLY polynomial. We say that $P(K)$ is alternating if the sign of the coefficient of $a^{2i}q^{2j}$ is $\pm(-1)^j$, where the factor of $\pm$ is the same for all coefficients. It is not difficult to see that if K is $\mathcal{H}$–thin, then $P(K)$ is alternating. On the other hand, there are examples of alternating knots whose HOMFLY polynomials are not alternating, the smallest being 11^a_{263} (numbering from *Knotscape* [Hoste and Thistlethwaite 99]).

Knot	$\mathcal{P}$
3_1	$a^2q^{-2}+a^2q^2t^2+a^4t^3$
4_1	$a^{-2}t^{-2}+q^{-2}t^{-1}+1+q^2t+a^2t^2$
5_1	$a^4q^{-4}+a^4t^2+a^6q^{-2}t^3+a^4q^4t^4+a^6q^2t^5$
5_2	$a^2q^{-2}+a^2t+a^2q^2t^2+a^4q^{-2}t^2+a^4t^3+a^4q^2t^4+a^6t^5$
6_1	$a^{-2}t^{-2}+q^{-2}t^{-1}+2+q^2t+a^2q^{-2}t+a^2t^2+a^2q^2t^3+a^4t^4$
6_2	$q^{-2}t^{-2}+a^2q^{-4}t^{-1}+a^2q^{-2}+q^2+2ta^2+a^4q^{-2}t^2+a^2q^2t^2$ $+a^2q^4t^3+a^4t^3+a^4q^2t^4$
6_3	$a^{-2}q^{-2}t^{-3}+a^{-2}t^{-2}+q^{-4}t^{-2}+q^{-2}t^{-1}+a^{-2}q^2t^{-1}+3$ $+a^2q^{-2}t+q^2t+a^2t^2+q^4t^2+a^2q^2t^3$
7_1	$a^6q^{-6}+a^6q^{-2}t^2+a^8q^{-4}t^3+a^6q^2t^4+a^8t^5+a^6q^6t^6+a^8q^4t^7$
7_2	$a^2q^{-2}+a^2t+a^4q^{-2}t^2+a^2q^2t^2+2a^4t^3+a^6q^{-2}t^4+a^4q^2t^4$ $+a^6t^5+a^6q^2t^6+a^8t^7$
7_3	$a^{-4}q^4+a^{-8}q^{-2}t^{-7}+a^{-6}q^{-4}t^{-6}+a^{-6}q^{-2}t^{-5}+a^{-8}q^2t^{-5}+2a^{-6}t^{-4}$ $+a^{-4}q^{-4}t^{-4}+a^{-6}q^{-2}t^{-3}+a^{-6}q^2t^{-3}+a^{-4}t^{-2}+a^{-6}q^4t^{-2}+a^{-4}q^2t^{-1}$
7_4	$a^{-2}q^2+a^{-8}t^{-7}+a^{-6}q^{-2}t^{-6}+2a^{-6}t^{-5}+2a^{-4}q^{-2}t^{-4}+a^{-6}q^2t^{-4}$ $+2a^{-4}t^{-3}+a^{-2}q^{-2}t^{-2}+2a^{-4}q^2t^{-2}+2a^{-2}t^{-1}$
7_5	$a^4q^{-4}+a^4q^{-2}t+2a^4t^2+a^6q^{-4}t^2+2a^6q^{-2}t^3+a^4q^2t^3+2a^6t^4$ $+a^4q^4t^4+a^8q^{-2}t^5+2a^6q^2t^5+a^8t^6+a^6q^4t^6+a^8q^2t^7$
7_6	$2a^2q^{-2}+q^2+q^{-2}t^{-2}+t^{-1}+a^2q^{-4}t^{-1}+3a^2t+2a^4q^{-2}t^2$ $+2a^2q^2t^2+2a^4t^3+a^2q^4t^3+2a^4q^2t^4+a^6t^5$
7_7	$a^{-4}t^{-4}+2a^{-2}q^{-2}t^{-3}+2a^{-2}t^{-2}+q^{-4}t^{-2}+2q^{-2}t^{-1}+2a^{-2}q^2t^{-1}$ $+4+a^2q^{-2}t+2q^2t+2a^2t^2+q^4t^2+a^2q^2t^3$

TABLE 2. Reduced superpolynomial for prime knots with up to 8 crossings.

Conversely, knots with alternating HOMFLY polynomials need not be $\mathcal{H}$–thin. The knot 9_{42} (numbering from Rolfsen [Rolfsen 76]) is a good example of this phenomenon. It has HOMFLY polynomial

$$P(9_{42})=a^{-2}q^{-2}+a^{-2}q^2-q^{-4}-1-q^4+a^2q^{-2}+a^2q^2,$$

which is certainly alternating. If we assume that $\mathcal{H}(9_{42})$ is thin and try to endow it with differentials satisfying Conjecture 3.1, however, we arrive at a contradiction. The requirement that d_1 and d_{-1} have one-dimensional homology and anticommute with each other quickly leads to the dot diagram shown on the left-hand side of Figure 5. However, in that diagram both d_1 and d_{-1} do not square to zero. The problem is resolved by postulating the presence of an additional two generators at the center of the diagram, as shown on the right-hand side of Figure 5. The resulting diagram correctly predicts $\widehat{\mathrm{HFK}}(9_{42})$ and $\mathrm{HKR}_2(9_{42})$.

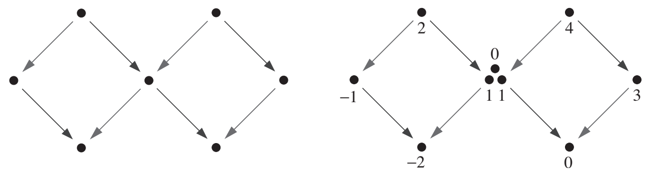


FIGURE 5. Two possible dot diagrams for the knot 9_{42}. The left-hand diagram assumes that $\mathcal{H}(9_{42})$ is thin and arrives at a contradiction: $d_1^2 \neq 0$. The right-hand diagram corrects this problem by introducing a pair of additional generators.

5.2 Thick Knots

Some knots are easily identified as being $\mathcal{H}$–thick. In particular, if a knot is thick with respect to either $\widehat{\mathrm{HFK}}$ or HKR_2, it is necessarily $\mathcal{H}$–thick as well. The knots with fewer than 11 crossings that fit this criterion are

$$8_{19},\ 9_{42},\ 10_{124},\ 10_{128},\ 10_{132},\ 10_{136}, 10_{139},\ 10_{145},\ 10_{152},\ 10_{153},\ 10_{154},\ 10_{161}.$$

We have already described the first two of these in Figures 3 and 5. In Section 8, we give dot diagrams illustrating what we believe are the superpolynomials of the 10-crossing knots in the list above. For most of these knots, our reasons for asserting that this is the superpolynomial are purely internal: it seems difficult to produce another diagram satisfying all the hypotheses of Conjecture 3.1. In addition, there are skein-theoretic arguments that support our calculations for 8_{19} and 10_{128}, although these currently fall short of a complete proof. In both of these cases, the skein-theoretic calculation gave the answer we had previously guessed based on our conjecture, and we view this as at least some evidence that our calculations are on the right track.

The interesting examples provided by these thick knots allow us to probe the rich structure of the triply graded theory. Even the simple thick knots we considered exhibit some very different types of behavior. Some thick knots, such as 9_{42}, 10_{132}, 10_{136}, and 10_{145}, have "invisible" generators that cannot be seen from the HOMFLY polynomial. Others, like 8_{19} and 10_{124}, have no invisible generators, but have nontrivial d_{-2}, d_0, and d_2. Many exhibit both features. There are cases, like 10_{145}, where the gradings in the superpolynomial are such that d_2 might conceivably be nontrivial, but the requirement that the differentials anticommute prohibits it.

Although the sample of knots we consider here is admittedly small, a number of interesting patterns may be observed from it. The rest of this section is devoted to describing a few of these.

5.3 Dimension of $\widehat{\mathrm{HFK}}$ and HKR_2

It is an interesting and rather puzzling fact that the knot Floer homology and sl(2) Khovanov homology of a given knot often have the same dimension [Rasmussen 05a]. Indeed, explaining this was one of our motivations for considering a triply graded theory. At first glance, however, the triply graded theory we have described does not seem to help all that much. One case in which it does provide insight is for those knots for which d_2 and d_0 both vanish (thin knots, but also some thick examples such as 9_{42}). In this case, the correspondence is obvious: the dimensions of $\widehat{\mathrm{HFK}}$ and HKR_2 are both equal to that of $\mathcal{H}$. However, there are many knots for which d_2 and d_0 are nontrivial but the two dimensions still agree. To consider an extreme example, our proposal for $\mathcal{H}(10_{128})$ has dimension 27, while the dimensions of $\widehat{\mathrm{HFK}}$ and HKR_2 are both 13.

The fact that the correspondence still holds in such cases suggests that we should look for an explanation of why the part of $\mathcal{H}$ killed by d_2 should have the same dimension as the part killed by d_0. Examining the diagrams in Section 8, a rather striking pattern comes to light: for knots with $S \geq 0$, any dot that has a nonzero image under one of d_2, d_0, and d_{-2} must have a nonzero image under the other two as well! (For $S < 0$, the requirement is reversed: any generator that is in the image of one differential is in the image of the other two as well.) Although we don't have any explanation for this phenomenon, it seems clear that if we understood it, we would be well on the way to understanding why $\widehat{\mathrm{HFK}}$ and HKR_2 have the same dimension for so many knots.

5.4 Braid Index and Estimates on S

It is well known that the minimum braid index of a knot is bounded by the difference between the maximum and minimum exponents of a in its HOMFLY polynomial. The same principle applies to the superpolynomial. More generally, we have the following proposition:

Proposition 5.3. *Let $a_{\max}(\mathcal{P}(K))$ and $a_{\min}(\mathcal{P}(K))$ be the maximum and minimum powers of a appearing in $\mathcal{P}(K)$. Then for any planar diagram D of K,*

$$w(D) - c(D) + 1 \leq a_{\min}(\mathcal{P}(K)) \leq a_{\max}(\mathcal{P}(K)) \leq w(D) + c(D) - 1,$$

where $w(D)$ is the writhe of D and $C(D)$ the number of components in its oriented resolution.

The analogue of this theorem for the HOMFLY polynomial was proved by Morton in [Morton 86]. As we now describe, Morton's argument carries through to the setting of superpolynomials. Since we don't have a definition of $\mathcal{P}(K)$, this statement can be taken in two ways. The first is that, like the $\mathrm{sl}(N)$ homology, the triply graded theory should satisfy a skein exact triangle. Morton's proof is purely skein-theoretic, and it is not hard to see that it carries over to any theory that has a skein exact triangle. The other point of view is that this is a limiting statement about the $\mathrm{sl}(N)$ homology as $N \to \infty$. In particular, using the skein exact triangle one can show that

$$\begin{aligned} &N\,(w(D) - c(D) + 1) - E \\ &\quad \le q_{\min}(\mathrm{KhR}_N(K)) \le q_{\max}(\mathrm{KhR}_N(K)) \qquad (5\text{–}2) \\ &\quad \le N\,(w(D) + c(D) - 1) + E, \end{aligned}$$

where $|E|$ is uniformly bounded independent of N. If Conjecture 1.3 holds, we have

$$\lim_{N\to\infty} \frac{1}{N}\, q_{\min}(\mathrm{KhR}_N(K)) = a_{\min}(\mathcal{P}(K))$$

and similarly for $a_{\max}(\mathcal{P}(K))$. The proposition then follows by taking the limit of (5–2) as $N \to \infty$.

In the same paper, Morton asked whether there might be a connection between $a_{\min}(P(K))$ and the bound on the genus of a knot provided by Bennequin's inequality. Since Bennequin's inequality actually provides a lower bound for the four-ball genus g_* of K [Rudolph 95], one might ask whether the same is true for $a_{\min}(P(K))$:

$$2g_*(K) \overset{??}{\ge} a_{\min}(P(K)). \qquad (5\text{–}3)$$

Although it is true in many examples, this inequality is false in general. For knots with fewer than 11 crossings, the knot $K = 10_{132}$ is the only counterexample; there $g_*(K) = 1$, but $a_{\min}(P(K)) = 4$. A brief inspection of the proposed dot diagram for 10_{132} in Section 8 suggests an explanation for what has gone wrong: $a_{\min}(\mathcal{P}(K)) = 2$, but the terms with lowest degree in a are not visible in the HOMFLY polynomial.

If we replace $a_{\min}(P(K))$ by $a_{\min}(\mathcal{P}(K))$ in (5–3), we expect that the resulting inequality will be true. Indeed, it is clear from the definition that $a_{\min}(\mathcal{P}(K)) \le S(K) \le a_{\max}(\mathcal{P}(K))$. If $S(K)$ provides a lower bound for the four-ball genus of K (which seems quite likely), $a_{\min}(\mathcal{P}(K))$ will do so as well. Continuing in this vein, we can combine Proposition 5.3 with the previous inequality to obtain the following estimate for S:

$$w(D) - c(D) + 1 \le S(K) \le w(D) + c(D) - 1,$$

where D is any planar diagram of K. Zoltán Szabó pointed out to us that using the work of Livingston [Livingston 04], it is not difficult to see that s and τ satisfy similar estimates. We sketch the proof of this fact for τ; the argument for s is the same.

Suppose K has a planar diagram D, and let $n_\pm(K)$ denote the number of positive and negative crossings. If we change all the negative crossings to positive, we obtain a new knot K^+, and [Livingston 04] and [Rudolph 99] tell us that

$$2\tau(K^+) = n_+(D) + n_-(D) - c(D) + 1.$$

To get back to K, we must change $n_-(D)$ crossings from positive to negative, which can lower τ by at most $n_-(D)$. Thus

$$\begin{aligned} 2\tau(K) &\ge n_+(D) - n_-(D) - c(D) + 1 \\ &= w(D) - c(D) + 1. \end{aligned}$$

Similarly, changing all of D's positive crossings to negative, we see that

$$2\tau(K) \le w(D) + c(D) + 1.$$

5.5 d_1 and the Unreduced Homology

Although we have focused on reduced homology, we expect that our work also has relations with the unreduced theory. In general, the unreduced homology $\overline{\mathrm{HKR}}_N(K)$ is related to $\mathrm{HKR}_N(K)$ by a spectral sequence that has E_1 term equal to $\mathrm{HKR}_N(K) \otimes \mathbb{Q}[X]/(X^N)$. When $N = 2$, the differential in this spectral sequence seems to be related to the Lee/Turner differential on HKR_2. For example, if K is thin, the presence of the Lee/Turner differential implies that

$$\mathrm{KhR}_2(K) = q^{s(K)} + (1 + q^2 t)\,\mathrm{KhR}_2'(K), \qquad (5\text{–}4)$$

where $\mathrm{KhR}_2'(K)$ is a polynomial with positive coefficients. The unreduced homology can also be expressed in terms of KhR_2':

$$\overline{\mathrm{KhR}}_2(K) = (q+q^{-1})q^{s(K)} + (q^{-1}+q^3 t)\,\mathrm{KhR}_2'(K). \qquad (5\text{–}5)$$

This suggests that the differential on the E_1 term of the spectral sequence is determined by the relation $d_{E_1}(a) = X d_{1*}(a)$, where d_{1*} denotes the Lee/Turner differential.

The analogue for the superpolynomial is that for *any* knot K we have

$$\mathcal{P}(K) = \left(\frac{a}{q}\right)^{S(K)} + (1 + ta^2q^{-2})Q_+(a,q,t), \qquad (5\text{–}6)$$

where $Q_+(a,q,t)$ is a polynomial with positive coefficients. This follows immediately from the existence of the canceling differential d_1 given by Conjecture 3.1. (The reason that the standard canceling differential on HKR_2 does not always force (5–4) is that, unlike d_1 on $\mathcal{H}_*$, it is not necessarily homogeneous in its behavior with respect to the grading.) When K is thin, we expect that the differential in the spectral sequence will again be determined by d_{1*}: $d_{E_{N-1}}(a) = X^{N-1} d_{1*}(a)$. This suggests the following analogue of (5–5):

$$\overline{\mathrm{KhR}}_N(K) = q^{(N-1)S(K)} \left(\frac{q^N - q^{-N}}{q-q^{-1}}\right) + (q^{-1} + q^{2N-1}t)\left(\frac{q^{N-1} - q^{-N+1}}{q - q^{-1}}\right) \times Q_+(a = q^N, q, t).$$

Expressing this in terms of the unreduced superpolynomial, we get

$$\bar{\mathcal{P}}(K) = (a - a^{-1})\left(\frac{a}{q}\right)^{S(K)} + (q^{-1} + a^2 q^{-1} t)(aq^{-1} - a^{-1}q) Q_+(a,q,t). \tag{5–7}$$

Let us illustrate the structure of the unreduced superpolynomial with the following example.

Example 5.4. (The figure-eight knot.) Since the figure-eight knot 4_1 is $\mathcal{H}$–thin, its reduced superpolynomial is easy to determine. The result is presented in Table 2. It has the expected structure (5–6) with $S(4_1) = 0$ and

$$\mathcal{P}'(4_1) = \frac{1}{a^2t^2} + q^2 t. \tag{5–8}$$

Substituting this into (5–7), we obtain the unreduced superpolynomial for the figure-eight knot:

$$\bar{\mathcal{P}}(4_1) = a - a^{-1} + (q^{-1} + a^2q^{-1}t)(aq^{-1} - a^{-1}q)(a^{-2}t^{-2} + q^2t). \tag{5–9}$$

It is easy to check that specializing to $t = -1$ and $a = q^2$ we reproduce, respectively, the correct expressions for the unnormalized HOMFLY polynomial and the sl(2) Khovanov homology. Moreover, substituting (5–9) into (1–2), we obtain the following prediction for the unreduced sl(N) homology:

$$\overline{\mathrm{KhR}}_N(4_1) = \sum_{i=0}^{N-1} q^{2i-N+1} + (1 + q^{2N}t)(q^{-2N}t^{-2} + q^2t)\sum_{i=0}^{N-2} q^{2i-N+1}. \tag{5–10}$$

5.6 d_{-1} and Three-Step Pairings

As discussed in Section 3.4, Conjecture 3.1 requires that $\mathcal{H}$ admit two distinct canceling differentials: d_1 and d_{-1}. This implies that HKR_N should admit a second canceling differential as well. We end this section by describing some empirical evidence that supports the idea that HKR_2 admits an additional canceling differential.

To begin with, we show that the unique term that is not canceled by d_{-1} has grading $(aqt)^{S(K)}$. This is because d_{-1} is interchanged with d_1 by the symmetry ϕ: the uncanceled term for d_1 is $a^{S(K)}q^{-S(K)}t^0$, which is taken to $a^{S(K)}q^{S(K)}t^n$ by ϕ, and n can then be computed by using that ϕ preserves the δ-grading. We thus have the following analogue of (5–6):

$$\mathcal{P}(K) = (aqt)^{S(K)} + (1 + a^2q^2t^3)Q_-(a,q,t), \tag{5–11}$$

where $Q_-(a,q,t)$ is a polynomial with positive coefficients.

If K is $\mathcal{H}$-thin, we can substitute $a = q^2$ to obtain the following prediction for the sl(2) Khovanov homology of K:

$$\mathrm{KhR}_2(K) = (q^3t)^{S(K)} + (1 + q^6t^3)Q_-(a = q^2, q, t). \tag{5–12}$$

Independent of this, given a HKR_2–thin knot K, we have

$$\mathrm{KhR}_2(K) = (-t)^{-S(K)/2} J(K)(q^2 = -q^2t),$$

where $J(K)$ is the Jones polynomial $P(K)(a = q^2, q)$. Combining this with the fact that $J(K)(q^2) - 1$ is divisible by $1 - q^6$ (see, e.g., Proposition 12.5 of [Jones 87]), it is not difficult to see that (5–12) holds for some polynomial $Q_-(q,t)$. It is not clear that this polynomial should have positive coefficients, as predicted by (5–11), but for thin knots with fewer than 12 crossings, we have checked that this is the case. More generally, we have the following:

Definition 5.5. We say a knot K has a *three-step pairing* on KhR_2 if for some $m, n \in \mathbb{Z}$, we have

$$\mathrm{KhR}_2(K) = q^m t^n + (1 + q^6t^3)Q_-(q,t),$$

where Q_- is a polynomial with positive coefficients.

A knot that admits a three-step pairing has an obvious candidate for the canceling differential induced by d_{-1}, though of course a canceling differential need not force a three-step pairing. Such knots are surprisingly common. In addition to the thin knots mentioned above, we checked some 5,000 knots with fewer than 16 crossings that happen to be $(1,1)$ knots and found that all of them had three-step pairings. A number of these knots are complicated enough that they do not satisfy (5–4), which makes this all the more remarkable.

6. TORUS KNOTS

Let $T_{n,m}$ be a torus knot of type (n, m), where n and m are relatively prime integers, $n < m$. In this section, we propose an explicit expression for the superpolynomial for all torus knots of type $(2, m)$ and $(3, m)$, and discuss its structure for general torus knots $T_{n,m}$. We consider reduction to the sl(N) knot homology and to the knot Floer homology, and show that our predictions are consistent with the known results. The differentials d_N play an important role in this discussion.

Let us begin by recalling the expression for the HOMFLY polynomial of a torus knot $T_{n,m}$.

6.1 HOMFLY Polynomial

The explicit expression for $P(T_{n,m})$ was found by Jones [Jones 87]:

$$P(T_{n,m}) = \frac{a^{m(n-1)}[1]_q}{[n]_q} \sum_{\beta=0}^{n-1} (-1)^{n-1-\beta} \frac{q^{-m(2\beta-n+1)}}{[\beta]_q![n-1-\beta]_q!} \times \prod_{\substack{j=\beta-n+1 \\ j\neq 0}}^{\beta} \left(q^j a - q^{-j} a^{-1}\right), \tag{6–1}$$

where $[n]_q = q^n - q^{-n}$ is the "quantum dimension" of n written in a slightly unconventional normalization, and

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q \quad \text{with} \quad [0]_q! = 1. \tag{6–2}$$

One can manipulate the expression (6–1) into the following form, which will be useful to us below:

$$P(T_{n,m}) = (aq)^{(n-1)(m-1)} \frac{1-q^{-2}}{1-q^{-2n}} \times \sum_{\beta=0}^{n-1} q^{-2m\beta} \left(\prod_{i=1}^{\beta} \frac{a^2 q^{2i} - 1}{q^{2i} - 1} \right) \left(\prod_{j=1}^{n-1-\beta} \frac{a^2 - q^{2j}}{1 - q^{2j}} \right). \tag{6–3}$$

Assuming that all the terms in the superpolynomial $\mathcal{P}(T_{n,m})$ are "visible" in the HOMFLY polynomial, one might hope to obtain $\mathcal{P}(T_{n,m})$ by inserting powers of $(-t)$ in the expression for $P(T_{n,m})$. In order to do this, it is convenient to simplify (6–1) further and write it as a sum of terms without denominators. For example, for $n = 2$ and $m = 2k+1$, we obtain

$$\begin{aligned} P(T_{2,2k+1}) &= \frac{a^{2k+1}}{(q^2 - q^{-2})} \Big[-a(q^{2k} - q^{-2k}) \\ &\qquad + a^{-1}\left(q^{2k+2} - q^{-2k-2}\right) \Big] \\ &= -a^{2k+2} \sum_{i=1}^{k} q^{4i-2k-2} + a^{2k} \sum_{i=0}^{k} q^{4i-2k}, \end{aligned} \tag{6–4}$$

where in the first two lines we combined the terms with the same power of a. Similarly, for $(3, m)$ torus knots, we obtain

$$\begin{aligned} P(T_{3,3k+1}) = a^{6k} \sum_{j=0}^{k} \sum_{i=0}^{3j} q^{6j-4i} - a^{6k+2} \sum_{j=1}^{k} \sum_{i=0}^{6j-2} q^{6j-2i-2} \\ + a^{6k+4} \sum_{j=0}^{k-1} \sum_{i=0}^{3j} q^{6j-4i} \end{aligned} \tag{6–5}$$

and

$$\begin{aligned} P(T_{3,3k+2}) = a^{6k+2} &\sum_{j=0}^{k} \sum_{i=0}^{3j+1} q^{6j-4i+2} \\ &- a^{6k+4} \sum_{j=0}^{k} \sum_{i=0}^{6j} q^{6j-2i} \\ &+ a^{6k+6} \sum_{j=0}^{k-1} \sum_{i=0}^{3j+1} q^{6j-4i+2}. \end{aligned} \tag{6–6}$$

In general, $P(T_{n,m})$ has the following structure, which follows directly from (6–3):

$$P(T_{n,m}) = \sum_{J=0}^{n-1} a^{(m-1)(n-1)+2J} P^{(J)}(q), \tag{6–7}$$

where each $P^{(J)} \in \mathbb{Z}[q, q^{-1}]$ can be written in terms of $n-1$ repeated sums; cf. (6–4)–(6–6).

6.2 The Structure of the Superpolynomial

We wish to find an explicit form of the superpolynomial for torus knots $T_{n,m}$, that has all the right properties to be the Poincaré polynomial of the triply graded homology theory $\mathcal{H}$. Before we proceed to a more detailed analysis, let us make a few general remarks about the expected structure of the superpolynomial for torus knots $T_{n,m}$. Simple examples of torus knots of type $(2, m)$ and $(3, m)$ already appeared in Sections 3 and 5. In these examples, all the terms in the reduced superpolynomial $\mathcal{P}(T_{n,m})$ are "visible" in the HOMFLY polynomial. We will assume that this is also the case for more general torus knots. In particular, this means that the structure of the superpolynomial $\mathcal{P}(T_{n,m})$ is similar to that of (6–7):

$$\mathcal{P}(T_{n,m}) = \sum_{J=0}^{n-1} a^{(m-1)(n-1)+2J} \mathcal{P}^{(J)}(q,t), \qquad (6\text{–}8)$$

where

$$\mathcal{P}^{(J)} \in \mathbb{Z}_{\geq 0}[q, q^{-1}, t].$$

Notice that only nonnegative powers of t appear in $\mathcal{P}^{(J)}(q,t)$. Moreover, the examples of $T_{2,m}$ and $T_{3,m}$ torus knots studied below suggest that only even (respectively odd) powers of t appear in $\mathcal{P}^{(J)}(q,t)$ for even (respectively odd) values of J, and the maximal degree of t does not exceed $(m-1)(n-1)+J$.

The structure of the superpolynomial $\mathcal{P}(T_{n,m})$ should also be consistent with the action of the differentials d_1 and d_{-1}. In particular, it should be consistent with (5–6) and (5–11):

$$\mathcal{P}(T_{n,m}) = a^S q^{-S} + (a^2 q^{-2} t + 1) Q_+(a,q,t) \qquad (6\text{–}9)$$

and

$$\mathcal{P}(T_{n,m}) = (aqt)^S + (a^2 q^2 t^3 + 1) Q_-(a,q,t), \qquad (6\text{–}10)$$

where for a torus knot $T_{n,m}$,

$$S(T_{n,m}) = (n-1)(m-1) \qquad (6\text{–}11)$$

and $Q_\pm \in \mathbb{Z}_{\geq 0}[a,q,t]$. Similarly, the unreduced superpolynomial should have the structure

$$\bar{\mathcal{P}}(T_{n,m}) = \left(\frac{a}{q}\right)^S (a - a^{-1}) + (a^{-1} + at)\bar{\mathcal{P}}'(a,q,t), \qquad (6\text{–}12)$$

cf. (5–7), where $\bar{\mathcal{P}}' \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}, t^{\pm 1}]$.

We believe that for any torus knot $T_{n,m}$ there exists an explicit expression for the superpolynomial with all the required properties. We were able to find such an expression for all torus knots of type $(2,m)$ and $(3,m)$, and to obtain some partial results for arbitrary torus knots $T_{n,m}$.

6.3 Torus Knots $T_{2,2k+1}$

The $(2, 2k+1)$ torus knots are in many respects the simplest of all knots. There are several different ways to determine their superpolynomials (reduced and unreduced), all of which lead to the same result. One reason for this—which was already used for simple examples of $(2,2k+1)$ torus knots in [Gukov et al. 05] and in Sections 3 and 5 here—is that all the terms in the sl(2) homology of $T_{2,2k+1}$ are "visible" in the HOMFLY polynomial. In particular, for torus knots of type $(2, 2k+1)$, Conjectures 1.2 and 1.3 hold for all values of $N \geq 2$. This nice property can be used to determine the superpolynomial of $T_{2,2k+1}$ either by combining the information about the HOMFLY polynomial and the sl(2) homology, or by comparing the sl(2) and sl(3) knot homologies, or in some other way.

For example, the HOMFLY polynomial of $T_{2,2k+1}$ is given by (6–4), namely

$$P(T_{2,2k+1}) = -a^{2k+2} \sum_{i=1}^{k} q^{4i-2k-2} + a^{2k} \sum_{i=0}^{k} q^{4i-2k},$$

while the sl(2) Khovanov homology is

$$\mathrm{KhR}_2(T_{2,2k+1}) = q^{2k} t^0 + q^{2k+4} t^2 + q^{2k+6} t^3 + \cdots + q^{6k+2} t^{2k+1}.$$

If we substitute $a = q^2$ and compare terms, it is easy to guess the formula in the following proposition:

Proposition 6.1. *The reduced superpolynomial* $\mathcal{P}(T_{2,2k+1})$ *has the form* (6–8), *that is,*

$$\mathcal{P}(T_{2,2k+1}) = a^{2k}\mathcal{P}^{(0)} + a^{2k+2}\mathcal{P}^{(1)}, \qquad (6\text{–}13)$$

where

$$\mathcal{P}^{(0)} = \sum_{i=0}^{k} q^{4i-2k} t^{2i} \quad \text{and} \quad \mathcal{P}^{(1)} = \sum_{i=1}^{k} q^{4i-2k-2} t^{2i+1}. \qquad (6\text{–}14)$$

Of course, $T_{2,2k+1}$ is a two-bridge knot, and therefore a particular case of Conjecture 5.2. This is a very useful family of examples to have in mind, however, so it is worth considering them in greater detail. Note that we have stated the formula above as a proposition. As usual, this is to be interpreted as a statement about KhR_N for $N \gg 0$. Its proof follows immediately from the proof of Conjecture 5.2 given in [Rasmussen 05b].

Let us check that $\mathcal{H}(T_{2,2k+1})$ satisfies the conditions of Conjecture 3.1. First, observe that $\mathcal{H}(T_{2,2k+1})$ is thin: all generators have δ-grading $-k$. For $i \neq 0$, d_i lowers the δ-grading by $|i|$, while d_0 lowers the δ-grading by 1. Thus d_1 and d_{-1} must be the only nontrivial differentials. Their action is illustrated in Figure 6. From the figure, it is obvious that the symmetry property holds. Finally, if we substitute $a = 1/t$, the reduced superpolynomial specializes to $\mathrm{HFK}(T_{2,2k+1})$:

$$\mathrm{HFK}(T_{2,2k+1}) = q^{-2k} t^{-2k} + q^{-2k} t^{-2k} (1 + q^{-2} t^{-1}) \sum_{i=1}^{k} q^{4i} t^{2i}. \qquad (6\text{–}15)$$

FIGURE 6. Dot diagram for the superpolynomial of $T_{2,2k+1}$.

We remark that the vanishing of d_N for $N \neq 1, -1$ is really quite special. As we shall see in the next section, the situation is qualitatively different for torus knots $T_{n,m}$ with $n > 2$, where any differential d_N can potentially be nontrivial for a fixed value of n and sufficiently large m.

Now let us turn to the unreduced superpolynomial of $T_{2,2k+1}$. The unnormalized HOMFLY polynomial of $T_{2,2k+1}$ can be easily obtained from (6–4) by multiplying it by $\bar{P}(\text{unknot}) = (a - a^{-1})/(q - q^{-1})$:

$$\bar{P}(T_{2,2k+1}) = \frac{1}{q - q^{-1}} \Big[- a^{2k+3} \sum_{i=1}^{k} q^{4i-2k-2} + a^{2k+1} \sum_{i=0}^{2k} q^{2i-2k} - a^{2k-1} \sum_{i=0}^{k} q^{4i-2k} \Big]. \tag{6–16}$$

On the other hand, the unreduced sl(2) homology of $T_{2,m}$ is known to be given by [Khovanov 99]

$$\overline{\text{KhR}}_2(T_{2,2k+1}) = (q + q^{-1}) q^{2k} + \sum_{i=1}^{k} q^{4i+2k-1} t^{2i} + \sum_{i=1}^{k} q^{4i+2k+3} t^{2i+1}. \tag{6–17}$$

Now one can use the conjectured relation (1–2) to find the superpolynomial $\bar{\mathcal{P}}(T_{2,2k+1})$. Namely, multiplying both (6–16) and (6–17) by $(q - q^{-1})$, we obtain two expressions, which are supposed to be specializations of $\bar{\mathcal{P}}(T_{2,2k+1})$ to $t = -1$ and $a = q^2$, respectively:

$$(q - q^{-1}) \bar{P}(T_{2,2k+1}) = (a - a^{-1}) \left(\frac{a}{q} \right)^{2k} + a^{2k} (a q^{-2} - a^3 q^{-2} - a^{-1} + a) \sum_{i=1}^{k} q^{4i-2k}$$

and

$$(q - q^{-1}) \overline{\text{KhR}}_2(T_{2,2k+1}) = (q^2 - q^{-2}) q^{2k} + (1 + q^4 t - q^{-2} - q^2 t) \sum_{i=1}^{k} q^{4i+2k} t^{2i}.$$

By matching the corresponding terms in these two expressions, we arrive at the formula contained in the following proposition, which is a special case of (5–7):

Proposition 6.2. *For a torus knot $T_{2,2k+1}$, the unreduced superpolynomial $\bar{\mathcal{P}}(T_{2,2k+1})$ is given by*

$$\bar{\mathcal{P}}(T_{2,2k+1}) = (a - a^{-1}) \left(\frac{a}{q} \right)^{2k} + a^{2k} (a^2 q^{-2} - 1)(a^{-1} + at) \sum_{i=1}^{k} q^{4i-2k} t^{2i}. \tag{6–18}$$

As a mathematical statement, this is to be interpreted in terms of Conjecture 1.2. In other words, it says that for $N > 1$, the sl(N) knot homology of $T_{2,2k+1}$ is given by

$$\overline{\text{KhR}}_N(T_{2,2k+1}) = q^{(2k-1)(N-1)} \times \Big[\sum_{i=0}^{N-1} q^{2i} + (1 + q^{2N} t) \sum_{i=1}^{k} \sum_{j=0}^{N-2} q^{4i+2j} t^{2i} \Big]. \tag{6–19}$$

Again, this formula can be confirmed by direct calculation. Perhaps the easiest approach is to start from Proposition 6.1 and use the spectral sequence relating reduced and unreduced homology. All the differentials in this spectral sequence vanish for dimensional reasons except for d_{N-1}, which is potentially nonzero on k different elements. To verify the nontriviality of d_{N-1}, one can use Gornik's theorem [Gornik 04] that there is a differential on $\text{HKR}_N(K)$ whose homology is supported in dimension zero. This cannot be the case unless all components of d_{N-1} that can be nonzero actually are nonzero.

6.4 Torus Knots $T_{3,m}$

In this and the following section, we consider torus knots of type $(3, m)$, and we will mainly discuss the reduced theory. We start by summarizing our prediction for the superpolynomial of $T_{3,m}$:

Conjecture 6.3. *For a torus knot $T_{3,m}$, the reduced superpolynomial $\mathcal{P}(T_{3,m})$ has the form (6–8):*

$$\mathcal{P}(T_{3,m}) = a^{2m-2} \mathcal{P}^{(0)} + a^{2m} \mathcal{P}^{(1)} + a^{2m+2} \mathcal{P}^{(2)}, \tag{6–20}$$

where for $m = 3k + 1$,

$$\mathcal{P}^{(0)} = \sum_{j=0}^{k} \sum_{i=0}^{3j} q^{6j-4i} t^{4k+2j-2i},$$

$$\mathcal{P}^{(1)} = \sum_{j=1}^{k} \sum_{i=0}^{6j-2} q^{6j-2i-2} t^{4k+2j-2\lfloor i/2 \rfloor+1}, \tag{6–21}$$

where $\lfloor x \rfloor$ *denotes the integer part of* x,

$$\mathcal{P}^{(2)} = \sum_{j=0}^{k-1} \sum_{i=0}^{3j} q^{6j-4i} t^{4k+2j-2i+4},$$

whereas for $m = 3k + 2$,

$$\mathcal{P}^{(0)} = \sum_{j=0}^{k} \sum_{i=0}^{3j+1} q^{6j-4i+2} t^{4k+2j-2i+2},$$

$$\mathcal{P}^{(1)} = \sum_{j=0}^{k} \sum_{i=0}^{6j} q^{6j-2i} t^{4k+2j-2\lfloor i/2 \rfloor+3}, \tag{6–22}$$

$$\mathcal{P}^{(2)} = \sum_{j=0}^{k-1} \sum_{i=0}^{3j+1} q^{6j-4i+2} t^{4k+2j-2i+6}.$$

Below we summarize some checks of (6–20)–(6–22):

1. If we set $t = -1$, we recover the correct expression for the normalized HOMFLY polynomial (6–5)–(6–6).

2. It is easy to verify that (6–20)–(6–22) have the structure of (6–9) and (6–10), where $S(T_{3,m}) = 2(m-1)$.

3. The general result (6–20)–(6–22) is consistent with our computations of $\mathcal{P}(T_{3,m})$ for small values of m (see examples in Sections 3 and 5).

4. Taking homology with respect to d_2 gives the correct result for $\mathrm{KhR}_2(T_{3,m})$.

5. Taking homology with respect to d_0 gives the correct result for $\mathrm{HFK}(T_{3,m})$.

The first three checks are fairly straightforward. We verify the properties (4) and (5) in the following two sections, where we also give the definitions of d_2 and d_0. Another consistency check is that $\mathcal{P}(T_{3,m})$ has the expected symmetry ϕ. Indeed, using the explicit form of the superpolynomial in (6–13) and (6–20), it is easy to verify the following result:

Proposition 6.4. *For* $n = 2$ *and* $n = 3$, *there is an involution*

$$\phi : \mathcal{H}_{i,j,*}(T_{n,m}) \to \mathcal{H}_{i,-j,*}(T_{n,m}). \tag{6–23}$$

In other words, for torus knots $T_{2,m}$ and $T_{3,m}$, the reduced superpolynomial $\mathcal{P}(T_{n,m})$ can be written as a polynomial in a, t, and $y = (q^{-1}t^{-1/2} + qt^{1/2})^2$, in agreement with the genus expansion structure (4–4).

6.5 Reduction to KhR

As we explained in Section 3, the reduction to the $\mathrm{sl}(N)$ knot homology involves taking cohomology with respect to the differentials d_N and specializing to $a = q^N$. Unlike the case of $(2, m)$ torus knots discussed earlier in this section, the triply graded theory of $T_{3,m}$ is complicated enough that any differential d_N can be potentially nonzero if m is sufficiently large. In order to see this, we recall that d_N is graded of degree $(-2, 2N, -1)$ for $N \geq 1$. In particular, since it lowers the a-grading by 2 units and t-grading by 1 unit, it should necessarily involve the terms from $\mathcal{P}^{(1)}$ in (6–20).

First, let us consider the case $m = 3k + 1$. It is convenient to split the sum over i in the expression (6–21) for $\mathcal{P}^{(1)}(T_{3,3k+1})$ into a sum over even and odd values of i, and rewrite the result as

$$\mathcal{P}^{(1)}_{+}(T_{3,3k+1}) = a^{6k+2} \sum_{j=1}^{k} \sum_{i=0}^{3j-1} q^{6j-4i-2} t^{4k+2j-2i+1}, \tag{6–24}$$

$$\mathcal{P}^{(1)}_{-}(T_{3,3k+1}) = a^{6k+2} \sum_{j=1}^{k} \sum_{i=0}^{3j-2} q^{6j-4i-4} t^{4k+2j-2i+1}. \tag{6–25}$$

Now we want to study what happens to these terms under the action of d_N. Notice that here we tacitly identify the elements of the homology groups $\mathcal{H}$ with the corresponding terms in the superpolynomial. For example, in this terminology, a nontrivial action of the graded differential d_N is described by a multiplication by $a^{-2}q^{2N}t^{-1}$. Applying this to (6–24)–(6–25) and rearranging the sum, we obtain

$$a^{6k} \sum_{j=N}^{k+N-1} \sum_{i=N-1}^{3j+1-2N} q^{6j-4i} t^{4k+2j-2i}, \tag{6–26}$$

$$a^{6k} \sum_{j=N-1}^{k+N-2} \sum_{i=N-2}^{3j+2-2N} q^{6j-4i} t^{4k+2j-2i}. \tag{6–27}$$

In this form, it is easy to recognize some of the terms from $\mathcal{P}^{(0)}(T_{3,3k+1})$. Indeed, comparing the range of the summation in (6–24) and (6–25) with that in (6–21), we conclude that d_N can be potentially nontrivial for torus knots $T_{3,3k+1}$ with $k \geq N - 1$.

Similarly, we find that the terms in the expressions (6–24) and (6–25) can potentially be in the image of d_N

acting on the following terms in $\mathcal{P}^{(2)}(T_{3,3k+1})$:

$$a^{6k+4}\sum_{j=2-N}^{k+1-N}\sum_{i=2-N}^{3j+2N-2} q^{6j-4i}t^{4k+2j-2i+4}, \qquad (6\text{–}28)$$

$$a^{6k+4}\sum_{j=1-N}^{k-N}\sum_{i=1-N}^{3j+2N-1} q^{6j-4i}t^{4k+2j-2i+4}. \qquad (6\text{–}29)$$

Again, comparing these expressions with (6–21), we conclude that d_N has to be trivial, unless $k \geq N-1$.

Summarizing, we find that for torus knots $T_{3,3k+1}$, all differentials d_N with $N \leq k+1$ can potentially be nontrivial. Notice, in particular, that there are terms in $\mathcal{P}^{(1)}(T_{3,3k+1})$ that have the right grading to be in the image of d_N as well as to map under d_N to some other terms in $\mathcal{P}^{(0)}$. Unfortunately, in this case, the structure of our triply graded theory alone does not uniquely determine the action of d_N for general N. For $N=2$, we find that d_2 acts on the terms

$$a^{6k+2}\sum_{j=1}^{k+2-N}\sum_{i=0}^{3j-2} q^{6j-4i-4}t^{4k+2j-2i+1} \qquad (6\text{–}30)$$
$$+\, a^{6k+4}\sum_{j=0}^{k+1-N}\sum_{i=0}^{3j} q^{6j-4i}t^{4k+2j-2i+4}$$

in $\mathcal{P}_-^{(1)}(T_{3,3k+1})$ and $\mathcal{P}^{(2)}(T_{3,3k+1})$ and maps them to the corresponding terms in $\mathcal{P}^{(0)}(T_{3,3k+1})$ and $\mathcal{P}_+^{(1)}(T_{3,3k+1})$. Indeed, subtracting all these terms from $\mathcal{P}(T_{3,3k+1})$ and specializing to $a=q^2$, we obtain

$$\begin{aligned}\mathcal{P}_2(T_{3,3k+1}) &= \mathcal{P}(q^2,q,t) \qquad (6\text{–}31)\\ &\quad -(1+t)q^{12k}\sum_{j=1}^{k}\sum_{i=0}^{3j-2} q^{6j-4i}t^{4k+2j-2i}\\ &\quad -(1+t^{-1})q^{12k+8}\sum_{j=0}^{k-1}\sum_{i=0}^{3j} q^{6j-4i}t^{4k+2j-2i+4}\\ &= (1+q^4t^2+q^6t^3+q^{10}t^5)\sum_{i=0}^{k-1} q^{6k+6i}t^{4i}\\ &\quad + q^{12k}t^{4k}, \qquad (6\text{–}32)\end{aligned}$$

which agrees with the values of $\mathrm{KhR}_2(T_{3,3k+1})$ computed by Shumakovitch and Bar-Natan. (Although we will not prove it here, this formula is almost certainly true in general; using [Bar-Natan 05c], it can be easily checked for $k<100$, for example.)

For $T_{3,3k+2}$, the analysis is similar. Again, we obtain several possibilities for how d_N might act on various terms in the superpolynomial $\mathcal{P}(T_{3,3k+2})$:

$$a^{6k+6}\sum_{j=0}^{k+1-N}\sum_{i=0}^{3j+1} q^{6j-4i+2}t^{4k+2j-2i+6} \qquad (6\text{–}33)$$
$$\to a^{6k+4}\sum_{j=N-1}^{k}\sum_{i=N-2}^{3j-2N+2} q^{6j-4i}t^{4k+2j-2i+3},$$

$$a^{6k+6}\sum_{j=0}^{k-N}\sum_{i=0}^{3j+1} q^{6j-4i+2}t^{4k+2j-2i+6} \qquad (6\text{–}34)$$
$$\to a^{6k+4}\sum_{j=N}^{k}\sum_{i=N-1}^{3j-2N} q^{6j-4i-2}t^{4k+2j-2i+3},$$

$$a^{6k+4}\sum_{j=0}^{k-N+1}\sum_{i=0}^{3j} q^{6j-4i}t^{4k+2j-2i+3} \qquad (6\text{–}35)$$
$$\to a^{6k+2}\sum_{j=N-1}^{k}\sum_{i=N-1}^{3j-2N+2} q^{6j-4i+2}t^{4k+2j-2i+2},$$

$$a^{6k+4}\sum_{j=0}^{k-N+2}\sum_{i=0}^{3j-1} q^{6j-4i-2}t^{4k+2j-2i+3} \qquad (6\text{–}36)$$
$$\to a^{6k+2}\sum_{j=N-2}^{k}\sum_{i=N-2}^{3j+3-2N} q^{6j-4i+2}t^{4k+2j-2i+2}.$$

By analogy with $(3,3k+1)$ torus knots, one might expect that in the present case d_2 acts as in (6–33) and (6–36). In other words, one might expect that d_2 acts on the following terms in $\mathcal{P}(T_{3,3k+2})$:

$$a^{6k+6}\sum_{j=0}^{k+1-N}\sum_{i=0}^{3j+1} q^{6j-4i+2}t^{4k+2j-2i+6} \qquad (6\text{–}37)$$
$$+\, a^{6k+4}\sum_{j=0}^{k-N+2}\sum_{i=0}^{3j-1} q^{6j-4i-2}t^{4k+2j-2i+3}.$$

Indeed, this leads to the following result for the sl(2) homology:

$$\mathrm{KhR}_2(T_{3,3k+2}) = (1+q^4t^2+q^6t^3+q^{10}t^5) \qquad (6\text{–}38)$$
$$\times \sum_{i=0}^{k} q^{6k+2+6i}t^{4i} - q^{12(k+1)}t^{4k+5},$$

which again agrees with the calculated value.

Remark 6.5. As we pointed out earlier, our prediction for $\mathcal{H}(T_{3,m})$ enjoys a symmetry (6–23), which means that the superpolynomial $\mathcal{P}(T_{3,m})$ can be written as a polynomial in a, t, and $y=(q^{-1}t^{-1/2}+qt^{1/2})^2$, in agreement with the genus expansion structure. What is more surprising is that d_N acts in a way that respects this structure! Indeed, it easy to verify that both expressions in (6–30) and (6–37) can be written in terms of the variables a, t, and y.

6.6 Reduction to HFK

We find that for all $(3, m)$ torus knots, the differential d_0 acts on the same terms as d_2. Indeed, following the same steps as in (6–31), we obtain for $m = 3k + 1$,

$$\begin{aligned}\mathcal{P}_0(T_{3,3k+1}) &= \mathcal{P}(a = t^{-1}, q, t)\\ &\quad - (1+t^{-1}) \sum_{j=1}^{k} \sum_{i=0}^{3j-2} q^{6j-4i-4} t^{-2k+2j-2i-1}\\ &\quad - (1+t^{-1}) \sum_{j=0}^{k-1} \sum_{i=0}^{3j} q^{6j-4i} t^{-2k+2j-2i}\\ &= t^{-2k} \Big[1 + \sum_{i=1}^{k} \big(q^{6i} t^{2i} + q^{6i-2} t^{2i-1} + q^{-6i+2} t^{-4i+1}\\ &\qquad + q^{-6i} t^{-4i} \big) \Big],\end{aligned}$$

and for $m = 3k + 2$,

$$\begin{aligned}\mathcal{P}_0(T_{3,3k+2}) &= t^{-2k-1} \Big[\Big(q^2 t + 1 + \frac{1}{q^2 t} \Big)\\ &\quad + \sum_{i=1}^{k} \big(q^{6i+2} t^{2i+1} + q^{6i} t^{2i} + q^{-6i} t^{-4i} + q^{-6i-2} t^{-4i-1} \big) \Big].\end{aligned}$$

In both cases, this agrees with the known expressions for $\mathrm{HFK}(T_{3,m})$.

6.7 Partial Results for $T_{n,m}$

Hoping to extend the above results to all torus knots $T_{n,m}$, one would like to have a more direct way of deriving the superpolynomial from the general expression (6–3) for the HOMFLY polynomial. For example, our expression (6–13)–(6–14) for the reduced superpolynomial of $T_{2,m}$ can be obtained directly from the general formula (6–3) for the HOMFLY polynomial by inserting powers of $(-t)$ and expanding the denominator in a power series:

$$\begin{aligned}\mathcal{P}(T_{2,m}) &= (-aqt)^{m-1} \left(\frac{1 - q^{-2} t^{-2}}{1 - q^{-4} t^{-2}} \right) \qquad (6\text{–}39)\\ &\quad \times \left[\frac{1 + a^2 q^{-2} t}{1 - q^{-2} t^{-2}} + q^{-2m} (-t)^{2-m} \frac{a^2 + q^{-2} t^{-3}}{1 - q^{-2} t^{-2}} \right].\end{aligned}$$

Observe that the two terms inside the square brackets correspond to the $\beta = 0$ and $\beta = 1$ terms in (6–3). Similarly, for $n = 3$, one has three terms in (6–3), which correspond to $\beta = 0$, 1, and 2. Comparing the structure of these terms with the corresponding terms in the superpolynomial (6–20), we find that again, certain parts of the superpolynomial can be obtained directly from the HOMFLY polynomial. Namely, these are the terms that correspond to $\beta = 0$ and $\beta = 2$. They have a form similar to that of the $\beta = 0$ and $\beta = 1$ terms in (6–39) and suggest that for a general torus knot $T_{n,m}$, certain parts of the superpolynomial are also given by a simple modification of the terms with $\beta = 0$ and $\beta = n - 1$ in the HOMFLY polynomial (6–3). Namely, up to an overall power of a, q, and t, the contribution of the $\beta = 0$ term to the superpolynomial looks like

$$\prod_{j=1}^{n-1} \frac{1 + a^2 q^{-2j} t}{1 - t^{-2} q^{-2(j+1)}}, \qquad (6\text{–}40)$$

and the contribution of the $\beta = n - 1$ term looks like

$$\prod_{j=1}^{n-1} \frac{a^2 + q^{-2j} t^{-3}}{1 - t^{-2} q^{-2(j+1)}}, \qquad (6\text{–}41)$$

where the terms in the denominator are understood to be expanded in a power series. We analyze the contribution of the $\beta = 0$ term more carefully in the following section.

7. STABLE HOMOLOGY OF TORUS KNOTS

Although we were unable to produce a general formula for the superpolynomial of $T_{m,n}$, we can make a prediction about its behavior as $m \to \infty$. To be precise, let us define

$$\mathcal{P}_s(T_{m,n}) = (a^{-1} q)^{(m-1)(n-1)} \mathcal{P}(T_{m,n}). \qquad (7\text{–}1)$$

This has the effect of translating the dot diagram for $\mathcal{P}(T_{m,n})$ in such a way that the leftmost dot is always at the origin of the (a, q) coordinate system. We then let

$$\mathcal{P}_s(T_n) = \lim_{m \to \infty} \mathcal{P}_s(T_{m,n}).$$

Assuming that the limit exists, we refer to $\mathcal{P}_s(T_n)$ as the *stable superpolynomial* of T_n. For example, when $n = 2$, the calculations of Section 6.3 show that the stable superpolynomial is given by

$$\mathcal{P}_s(T_2) = (1 + a^2 q^2 t^3) \sum_{i=0}^{\infty} q^{4i} t^{2i}.$$

As a dot diagram, this would be represented by an up-and-down chain of dots, starting at the origin of coordinates and carrying on indefinitely to the right. This is illustrated in Figure 7.

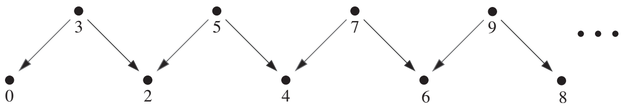


FIGURE 7. Dot diagram for the stable superpolynomial of T_2, obtained as the limit of the dot diagrams in Figure 6.

Conjecture 7.1. *For all n, the limit of (7–1) exists and is given by*

$$\mathcal{P}_s(T_n) = \frac{(1+a^2q^2t^3)(1+a^2q^4t^5)\cdots(1+a^2q^{2n-2}t^{2n-1})}{(1-q^4t^2)(1-q^6t^4)\cdots(1-q^{2n}t^{2n-2})}, \tag{7–2}$$

where terms in the denominator are understood to be expanded as a series in positive powers of q and t.

Of course, we should verify that if we substitute $t = -1$, our prediction for the stable superpolynomial reduces to the stable HOMFLY polynomial of T_n.

Lemma 7.2. *If $\mathcal{P}_s(T_n)$ is the expression given in (7–2) then*

$$\mathcal{P}_s(T_n)\big|_{t=-1} = \lim_{m\to\infty}(qa^{-1})^{(m-1)(n-1)}P(T_{m,n}).$$

Proof: Using the formula given in (6–3) together with the symmetry $P_K(a,q) = P_K(a,q^{-1})$, we see that

$$(qa^{-1})^{(m-1)(n-1)}P(T_{m,n}) = \frac{1-q^2}{1-q^{2n}} \tag{7–3}$$
$$\times \sum_{\substack{\beta+\gamma=n-1\\ \beta,\gamma\geq 0}} q^{2m\beta}\Big(\prod_{i=1}^{\beta}\frac{a^2-q^{2i}}{1-q^{2i}}\Big)\Big(\prod_{j=1}^{\gamma}\frac{1-a^2q^{2j}}{1-q^{2j}}\Big).$$

As $m \to \infty$, all terms of the sum will contribute higher and higher powers of q, with the exception of the term for which $\beta = 0$. We thus obtain

$$P_s(T_n) = \lim_{m\to\infty}(qa^{-1})^{(m-1)(n-1)}P(T_{m,n}), \tag{7–4}$$
$$= \frac{1-q^2}{1-q^{2n}}\Big(\prod_{j=1}^{n-1}\frac{1-a^2q^{2j}}{1-q^{2j}}\Big), \tag{7–5}$$

which agrees with the expression obtained by substituting $t = -1$ in (7–2). □

Observe that our conjectured expression for the stable superpolynomial has the minimum size dictated by the stable HOMFLY polynomial. Indeed, it is easy to see from equation (7–2) that the homological grading of any term in $\mathcal{P}_s(T_n)$ is congruent to half its a-grading modulo 2. Thus if we substitute $t = -1$, all terms with a given power of a will have the same sign. In contrast, the sl(2) Khovanov homology of a torus knot is usually much larger than the minimum size predicted by its Jones polynomial.

7.1 Origin of the Conjecture

Conjecture 7.1 was derived from the following geometric ansatz, which is in many ways more revealing. Here, $\mathcal{H}_s(T_n)$ is the homology group with Poincaré polynomial $\mathcal{P}_s(T_n)$.

Conjecture 7.3. *$\mathcal{H}_s(T_n)$ is the smallest complex satisfying the following properties:*

(i) $\mathcal{P}_s(T_n) \in \mathbb{Z}[a,q,t]$.

(ii) *$\mathcal{H}_s(T_n)$ contains $\mathcal{H}_s(T_{n-1})$ as a subcomplex.*

(iii) *$\mathcal{H}_s(T_n)$ is acyclic with respect to $d_{-1}, d_{-2}, \ldots, d_{-n+1}$.*

(iv) *The homology of $\mathcal{H}(T_n)$ with respect to d_1 is one-dimensional and generated by the monomial 1 appearing in $\mathcal{P}_s(T_n)$.*

To illustrate how (7–2) is derived from these properties, consider the simplest case, when $n = 2$. We begin the stable superpolynomial with the term 1, which generates the homology with respect to d_1. By property (i), $d_{-1}(1) = 0$. Thus for the homology with respect to d_{-1} to vanish, we must add a term $a^2q^2t^3$. Next, we must kill this new term under d_1. If it is in the image of d_1, anticommutativity of d_1 and d_{-1} will force 1 to be in the image of d_{-1}, which violates property (iv). Thus we are forced to add a third term q^4t^2 that is in the image of $a^2q^2t^3$ under d_1.

At this point, all the hypotheses are satisfied, with the exception of the fact that q^4t^2 is not killed by d_{-1}. Thus we are in the same situation in which we began, only shifted over by a factor of q^4t^2. Repeating the arguments above, we see that we must add $a^2q^6t^5 + q^8t^4$, then $a^2q^{10}t^7 + q^{12}t^6$, and so on indefinitely. Thus the stable superpolynomial has the form

$$\begin{aligned}\mathcal{P}_s(T_2) &= 1 + (a^2q^2t^3 + q^4t^2)\sum_{i=0}^{\infty}(q^4t^2)^i\\ &= 1 + \frac{a^2q^2t^3+q^4t^2}{1-q^4t^2}\\ &= \frac{1+a^2q^2t^3}{1-q^4t^2}.\end{aligned}$$

The general case is not much different. By property (ii), we start out with $\mathcal{H}_s(T_{n-1})$, which we may inductively assume satisfies properties (i)–(iv), except that it is not acyclic with respect to d_{-n+1}, which is triply graded of degree $(-2, -2n+2, -2n+1)$, and so in order to kill $\mathcal{H}(T_{n-1})$ we must add another copy of it shifted up by $(2, 2n-2, 2n-1)$. The result is acyclic with respect to d_{-i} for $0 < i \leq n-1$, but has the wrong homology with respect to d_1. To rectify this, we add another copy of $\mathcal{H}_s(T_{n-1})$, shifted by $(-2, 2, -1)$ relative to the second copy. We are now back where we started, but shifted over by $(0, 2n, 2n-2)$. Repeating, we see that $\mathcal{H}_s(T_n)$ has the general form shown in Figure 8, where the blocks labeled $A_{i,n}$ and $B_{i,n}$ each represent an appropriately shifted copy of $\mathcal{H}_s(T_{n-1})$.

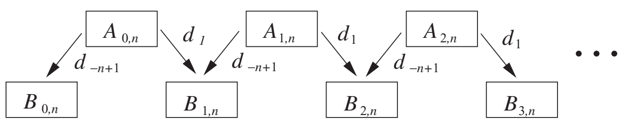


FIGURE 8. Schematic diagram of the stable complex for T_n. Although we've drawn each $A_{i,n}$ and $B_{i,n}$ as a finite box, they actually extend indefinitely to the right of the diagram.

We compute

$$\begin{aligned}\mathcal{P}_s(T_n) &= \mathcal{P}_s(T_{n-1})\Big(1 + \big(a^2q^{2n-2}t^{2n-1} + q^{2n}t^{2n-2}\big) \\ &\qquad \times \sum_{i=0}^{\infty}(q^{2n}t^{2n-2})^i\Big) \\ &= \mathcal{P}_s(T_{n-1})\left(1 + \frac{a^2q^{2n-2}t^{2n-1} + q^{2n}t^{2n-2}}{1 - q^{2n}t^{2n-2}}\right) \\ &= \mathcal{P}_s(T_{n-1})\left(\frac{1 + a^2q^{2n-2}t^{2n-1}}{1 - q^{2n}t^{2n-2}}\right),\end{aligned}$$

which clearly gives the formula of (7–2).

7.2 Reduction to HFK

Currently, it is difficult to compute $\mathrm{KhR}_N(T_{m,n})$ for values of m, n, and N that are all larger than 2, so we have no way to check Conjecture 7.1 directly. As an indirect check, however, we can compare the homology of $\mathcal{H}_s(T_n)$ with respect to d_0 and d_2 to what is known about the knot Floer homology and sl(2) Khovanov homology of torus knots.

The stable knot Floer homology of T_n is easily calculated from its stable Alexander polynomial. When we substitute $a = 1$ into the formula for the stable HOMFLY polynomial in (7–5), all the terms in the product cancel, and we are left with

$$\Delta_s(T_n) = \frac{1-q^2}{1-q^{2n}} = (1-q^2)\sum_{i=0}^{\infty} q^{2ni}.$$

Using Ozsváth and Szabó's calculation of $\widehat{\mathrm{HFK}}$ for torus knots in [Ozsváth and Szabó 04b], it follows that

$$\mathrm{HFK}_s(T_n) = (1+q^2t)\sum_{i=0}^{\infty} q^{2ni}t^{2(n-1)i}. \tag{7–6}$$

We want to define a differential d_0 on $\mathcal{H}_s(T_n)$ that anticommutes with the other d_i's and whose homology is given by the expression above. As in the construction of $\mathcal{H}_s(T_n)$, we proceed inductively. When $n = 2$, d_0 is necessarily trivial. For general n, we refer to the schematic diagram of $\mathcal{H}_s(T_n)$ in Figure 8. By the induction hypothesis, we can assume that we have already constructed the differential d_0 on each block.

To describe the part of d_0 that goes between blocks, observe that $\mathcal{H}_s(T_{n-1})$ has a subcomplex C_{n-1} obtained by omitting $A_{0,n-1}$ and $B_{0,n-1}$ from the analogous diagram for $\mathcal{H}_s(T_{n-1})$. There is a chain map $\psi : \mathcal{H}_s(T_{n-1}) \to \mathcal{H}_s(T_{n-1})$ that shifts the entire complex over one unit to the right and that defines an isomorphism from $\mathcal{H}_s(T_{n-1})$ to C_{n-1}. We define the component of d_0 that maps $A_{i,n}$ to $B_{i,n}$ to be given by ψ, and assume that all other components of d_0 between the blocks are trivial.

First, we should check that d_0 has the correct grading. The grading of $A_{i,n}$ is shifted by a factor of $(2, 2n-2, 2n-1)$ relative to that of $B_{i,n}$, while the grading of C_{n-1} is shifted by

$$(2, 2n-4, 2n-3) + (-2, 2, -1) = (0, 2n-2, 2n-4).$$

Thus d_0 shifts the grading by $(-2, 0, -3)$, as it should.

It follows easily from the definition that d_0 anticommutes with the other differentials. Thus it remains only to check that it has the correct homology. To see this, note that with respect to d_0, $\mathcal{H}_s(T_n)$ decomposes as a direct sum of complexes $D_{i,n}$, where as a group, $D_{i,n} = A_{i,n} \oplus B_{i,n}$. Since $D_{i,n}$ is just $D_{0,n}$ shifted over by a factor of $(q^{2n}t^{2n-2})^i$, we see that the Poincaré polynomial of the homology with respect to d_0 is

$$\mathcal{P}_0(T_n) = \mathcal{P}(D_{0,n})\sum_{i=0}^{\infty} q^{2ni}t^{2(n-1)i}.$$

On the other hand, it follows from the definition of ψ that $H_*(D_{i,n}, d_0) \cong H_*(D_{i,n-1}, d_0)$, so

$$\mathcal{P}(D_{0,n}) = \mathcal{P}(D_{0,n-1}) = \cdots = \mathcal{P}(D_{0,2}) = 1 + a^2q^2t^3.$$

Finally, we substitute $a = 1/t$ to obtain

$$\mathcal{P}_0(T_n) = (1+q^2t)\sum_{i=0}^{\infty} q^{2ni}t^{2(n-1)i},$$

which agrees with (7–6).

7.3 Reduction to KhR_2

As a final check on Conjecture 7.1, we use it to make some predictions about the sl(2) Khovanov homology of torus knots. Although there is not a huge amount of data with which to compare our predictions, what there is provides some of the most convincing evidence for our conjectures. Our results match perfectly with the known computations, which had previously seemed quite difficult to explain.

To predict $\mathrm{KhR}_{2,s}(T_n)$, we must understand the action of d_2 on $\mathcal{H}_s(T_n)$. As in the previous sections, we proceed inductively, starting with $n = 2$. In this case, d_2 must vanish for dimensional reasons, and we obtain the formula for the stable Khovanov homology simply by substituting $a = q^2$ and $n = 2$ into (7–2):

$$\begin{aligned}\mathrm{KhR}_{2,s}(T_2) &= (1+q^6t^3)(1+q^4t^2+q^8t^4+\cdots)\\ &= 1+q^4t^2+q^6t^3+q^8t^4+q^{10}t^5+\cdots.\end{aligned}$$

The Khovanov homology of $T_{2,m}$ is given by

$$\mathrm{KhR}_2(T_{2,m}) = q^{m-1}(1+q^4t^2+q^6t^3+\cdots+q^{2m}t^m).$$

After shifting by q^{1-m}, this agrees with the stable homology up through terms of degree q^{2m}. In general, we expect that $q^{-mn}Kh(T_{m,n})$ should also agree with $Kh_s(T_n)$ in degrees up to q^{2m}. Indeed, if we substitute $a = q^2$, the lowest-degree term appearing in the expression (7–3) for $P(T_{n,m})$ that does not come from the term where $\beta = 0$ is q^{2m+2}.

Next, we consider the case $n = 3$. Referring to Figure 8, we observe that since d_{-2} lowers the δ-grading by 1 and d_1 preserves it, the δ-grading of all terms in $A_{i,3}$ is $i+1$, while the δ-grading of $B_{i,3}$ is i. Now, d_2 lowers δ by 1, so the only possible components of d_2 go from $A_{i,3}$ to $B_{i,3}$, from $A_{i+1,3}$ to $A_{i,3}$, and from $B_{i+1,3}$ to $B_{i,3}$. In particular, $\mathcal{F}_k = \bigoplus_{i<k} D_{i,3}$ defines a filtration with respect to d_2. We compute using the spectral sequence associated with this filtration. The differential on the E_0 term is given by the restriction of d_2 to $D_{i,3}$. We hypothesize that $d_2 : A_{i,3} \to B_{i,3}$ is nontrivial and compute its image. Now, $A_{i,3}$ is isomorphic to $B_{i,3}$, but shifted in grading by $(2,4,5)$, and d_2 shifts grading by $(-2,4,-1)$. Thus the image of $A_{i,3}$ under d_2 will be isomorphic to $B_{i,3}$, but shifted by $(0,8,4)$, and the homology in $D_{i,3}$ will be generated by the first four terms in $B_{i,3}$. The Poincaré polynomial of the E_1 term is given by

$$\begin{aligned}\mathcal{P}(E_1) &= \sum_{i=0}^{\infty}\mathcal{P}(D_{i,3})\\ &= \mathcal{P}(D_{0,3})\sum_{i=0}^{\infty} q^{6i}t^{4i} \qquad (7\text{–}7)\\ &= \frac{(1+a^2q^2t^3+q^4t^2+a^2q^6t^5)}{1-q^6t^4}.\end{aligned}$$

This is illustrated in Figure 9. For dimensional reasons, there can be no further differentials. Substituting $a = q^2$ in (7–7), we obtain

$$\mathrm{KhR}_{2,s}(T_3) = \frac{(1+q^4t^2+q^6t^3+q^{10}t^5)}{1-q^6t^4}. \qquad (7\text{–}8)$$

This expression agrees with the pattern observed from direct computation. For example, Figure 10 shows $\mathrm{KhR}_2(T_{3,8})$, courtesy of Shumakovitch [Shumakovitch 04]. As expected, the homology agrees with (7–8) up through powers of q^{30} (here $30 = 14 + 2\cdot 8$).

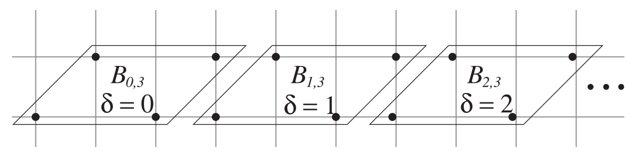


FIGURE 9. What's left in $\mathcal{H}(T_3)$ after taking homology with respect to d_2. Four generators from each $B_{i,3}$ survive.

In comparing these figures, it is convenient to label generators by their δ-grading, since this tells us on which diagonal they lie. For example, the first four generators of Figure 9 have δ-grading zero. They correspond to the four generators on the highest occupied diagonal in Figure 10. The next four generators have $\delta = 1$ and lie on the next diagonal, and so forth.

The case $n = 4$ is somewhat more complicated. To simplify things, we assume that as in the previous case, $\mathcal{F}_k = \bigoplus_{i<k} D_{i,4}$ defines a filtration with respect to d_2. Thus we are again faced with the problem of determining the component of d_2 that maps $A_{i,4}$ to $B_{i,4}$. The situation is illustrated in Figure 11. Possible differentials are indicated by arrows. If we assume that these are all nontrivial in rational homology (in integral homology, they are most likely given by multiplication by 2), we arrive at the following expression for the Poincaré polynomial of $D_{0,4}$:

$$\mathcal{P}(D_{0,4}) = (1+a^2q^2t^3)\left[1+q^4t^2+\frac{q^6t^4(1+a^2q^4t^5)}{1-q^6t^4}\right].$$

	0	1	2	3	4	5	6	7	8	9	10	11
32												1
30										1	1	
28												
26								1	1			
24						1	1					
22												
20				1	1							
18			1									
16												
14	1											

FIGURE 10. The reduced Khovanov homology of $T_{3,8}$. Here the horizontal axis corresponds to t, and the vertical axis to q.

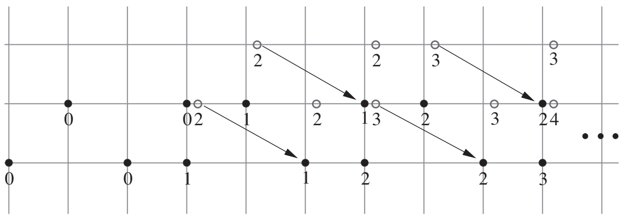


FIGURE 11. Component of d_2 from $A_{0,4}$ (hollow circles) to $B_{0,4}$ (solid circles). Possible differentials (which we assume are all nonvanishing) are shown by arrows. The labels beneath each generator show the value of δ.

As before, it is easy to see there can be no further differentials, so summing up the contributions from all $D_{i,4}$, we get the following prediction:

$$\mathrm{KhR}_{2,s}(T_4) = \frac{1+q^6t^3}{1-q^8t^6}\left[1+q^4t^2+\frac{q^6t^4(1+q^8t^5)}{1-q^6t^4}\right].$$

For comparison, Figure 12 shows the Khovanov homology of $T_{4,7}$, again computed by [Shumakovitch 04].

We leave it to the reader to check that the part of the homology in degrees less than or equal to $18+2\cdot 7=32$ agrees with the expression above.

As a final test, we compare with Bar-Natan's calculation of $\overline{\mathrm{KhR}}_2(T_{5,9})$ [Bar-Natan 05b]. Rather than computing a general formula for $n=5$, we simply write out enough of the complex to give us the stable homology up to powers of q^{24}. The results of the calculation are illustrated in Figure 14. The top half of the figure shows potential differentials between $A_{0,5}$ (hollow circles) and $B_{0,5}$ (solid circles). Again, we assume that all these differentials induce nontrivial maps on rational cohomology.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
38													1	1
36												2	1	
34														
32										2	1			
30								1	2					
28						1								
26							1							
24				1	1									
22			1											
20														
18	1													

FIGURE 12. The reduced Khovanov homology of $T_{4,7}$. Here the horizontal axis corresponds to t, and the vertical axis to q.

	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62
-10													1		1	1
-11										1		2	2		2	1
-12									2		1	3		1	1	
-13						1		2	2		3	1		1		
-14					1		1	2		1	1					
-15				1			1									
-16	1		1	1		1										

FIGURE 13. The reduced Khovanov homology of $T_{5,9}$, derived from [Bar-Natan 05b]. To save space, we plot generators versus their (q,δ)-gradings, rather than (t,q) as in the previous figures.

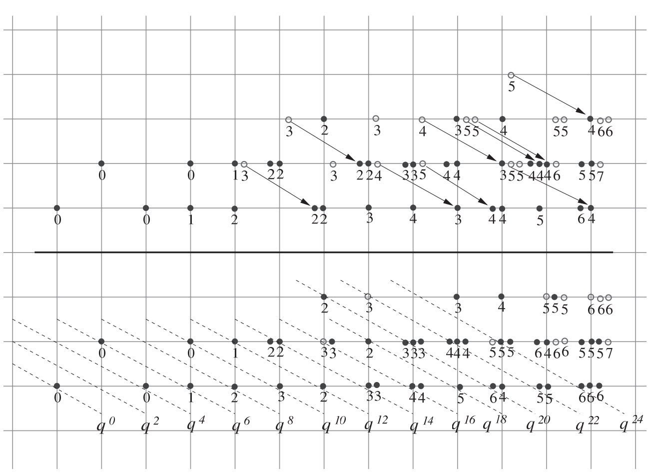
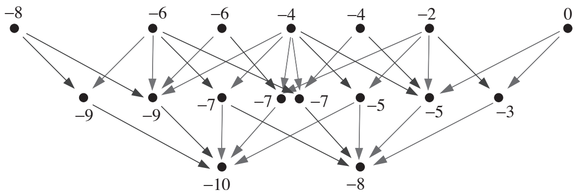


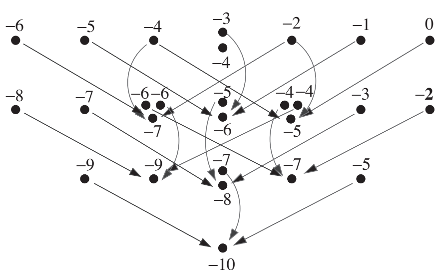
FIGURE 14. Computing the stable Khovanov homology of T_5. Generators are labeled by their δ-grading.

The bottom half of the figure shows the generators of $\mathcal{H}_s(T_5)$ that survive after taking homology. The dashed lines indicate their q-grading after we substitute $a = q^2$. By way of comparison, Figure 13 shows what we expect is the reduced Khovanov homology of $T_{5,9}$, based on Bar-Natan's calculation of the unreduced homology. As expected, the two agree up through powers of q^{50} $(50 = 32 + 2 \cdot 9)$.

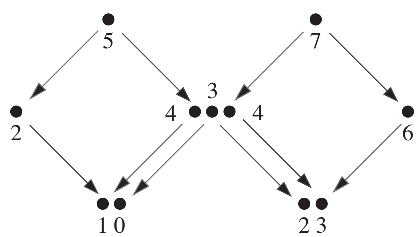
8. DOT DIAGRAMS FOR TEN-CROSSING KNOTS



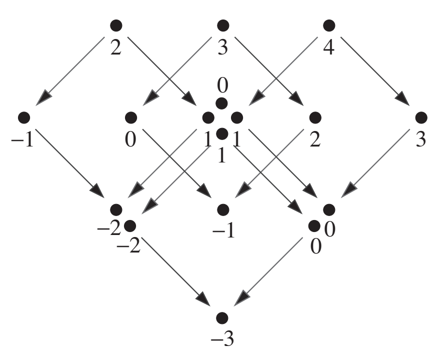
$10_{124} = \mathrm{mirror}(T_{3,5}) \colon\ a_{\min} = -12$



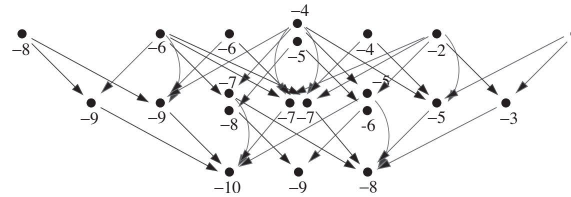
$10_{128} \colon\ a_{\min} = -12$
(d_1 and d_{-1} are not shown.)



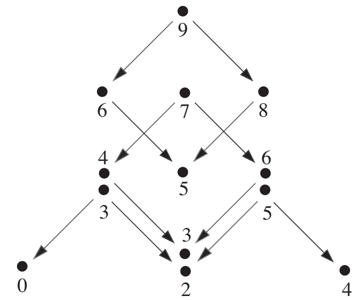
$10_{132} \colon\ a_{\min} = 2$



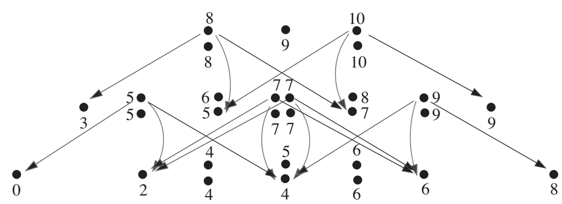
$10_{136} \colon\ a_{\min} = -4$



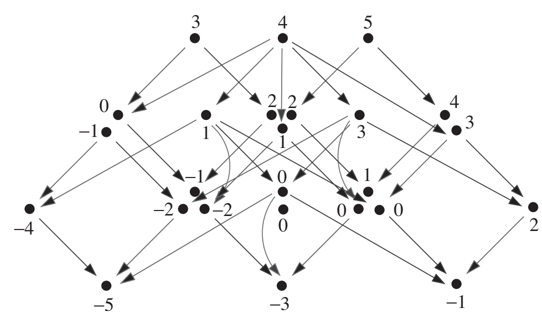
$10_{139} \colon\ a_{\min} = -12$



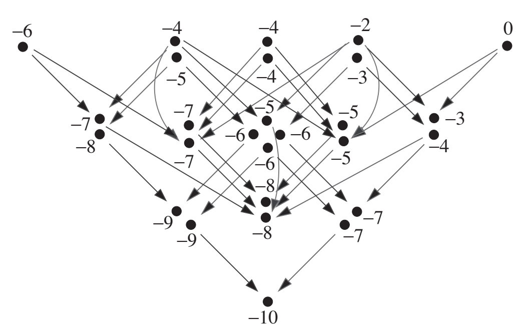
$10_{145} \colon\ a_{\min} = 4$



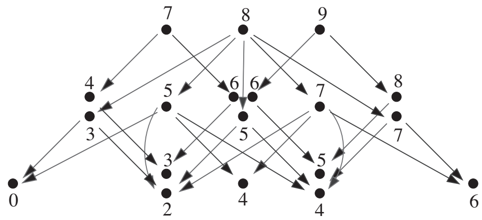
$10_{152} \colon\ a_{\min} = 8$
(d_1 and d_{-1} are not shown.)



$10_{153} \colon\ a_{\min} = -2$



$10_{154} \colon\ a_{\min} = -12$



10_{161}: $a_{\min} = 6$

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