Diophantine Equations over Global Function Fields II: *R*-Integral Solutions of Thue Equations

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CONTENTS

Let K be an algebraic function field over a finite field. Let L be an extension field of K of degree at least 3. Let R be a finite set of valuations of K and denote by S the set of extensions of valuations of R to L. Denote by $O_{K,R}, O_{L,S}$ the ring of R-integers of K and S-integers of L, respectively. Assume that $\alpha \in O_{L,S}$ with $L = K(\alpha)$, let $0 \neq \mu \in O_{K,R}$, and consider the solutions $(x, y) \in O_{K,R}$ of the Thue equation

$$N_{L/K}(x - \alpha y) = \mu.$$

We give an efficient method for calculating the R-integral solutions of the above equation. The method is different from that in our previous paper [Gaál and Pohst 06] and is much more efficient in many cases.

1. INTRODUCTION

Keeping the notation from our previous paper [Gaál and Pohst 06], let $k = \mathbb{F}_q$ denote a finite field with $q = p^d$ elements. The rational function field of k is $k(t)$ as usual, and K is a finite extension of $k(t)$. The integral closure of $k[t]$ in K is denoted by O_K. We assume that K is separably generated over $k(t)$ by an element z belonging to O_K and that k is the full constant field of K.

Denote by R a finite set of valuations of K containing the infinite valuations. Let L be an extension field of K of degree at least 3. Let S be the set of extensions of valuations of R to L and denote by $O_{K,R}, O_{L,S}$ the ring of R-integers of K, the ring of S-integers of L, respectively. (If R is just the set of infinite valuations of K, then $O_{K,R}$ is just O_K, the ring of integers of K.) Assume that $\alpha \in O_{L,S}$ with $L = K(\alpha)$, let $0 \neq \mu \in O_{K,R}$, and consider the solutions of the Thue equation

$$N_{L/K}(x - \alpha y) = \mu \quad \text{in} \quad x, y \in O_{K,R}. \tag{1–1}$$

The purpose of the present paper is to give an efficient method for calculating the R-integral solutions of Equation (1–1).

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Our algorithm has two goals. First, instead of just integer solutions, our algorithm calculates R-integral solutions of the equation. There may be finitely many (isolated) solutions, and there may occur finitely many parameterized families of solutions. If Equation (1–1) has infinitely many solutions (that is, such families occur), we give a method to parameterize them.

Second, this method turns out to be much more efficient than that of [Gaál and Pohst 06] in many cases. For explicit calculations in function fields both in [Gaál and Pohst 06] and here, we use KASH [Daberkow et al. 97]. In both cases, the calculations can be split into two parts:

(1) The explicit determination of certain function field elements, valuations etc. This is usually done in an interactive way and costs almost no CPU time.

(2) The test of a certain set consisting of some thousands of elements. This is done by running some loops in KASH, costing some seconds of CPU time. However, the size of this set is much smaller when using the method presented in this paper than when applying the method of [Gaál and Pohst 06].

The reason for this is the following. In [Gaál and Pohst 06], we calculated the fundamental units and represented the element $\beta = x - \alpha y$ as a power product of the fundamental units. We derived inequalities for the exponents of the fundamental units and all possible sets of exponents must then be checked.

On the other hand, in the present paper, we directly deal with the possible prime divisors of β_i/β_n (the β_i being conjugates of $\beta = x - \alpha y \in L$ over K) and use the fact that β_i/β_n determines x/y. Using an upper bound for the height of β_i/β_n, we construct all divisors that are composed of the given prime divisors and have bounded height. Calculating a basis of the corresponding Riemann-Roch space, we find out if such a divisor is principal, that is, if it can be the splitting of the element β_i/β_n into prime divisors. This simple test (performed very quickly by KASH) makes the number of possible β_i/β_n to be checked much smaller than the number of cases to be tested with the method of [Gaál and Pohst 06].

2. AUXILIARY RESULTS

In this section, we recall the "fundamental inequality," Lemma 3.1 of [Gaál and Pohst 06].

Let K be a finite extension of $k(t)$ of genus g_K. The set of all (exponential) *valuations* of K is denoted by V and the subset of infinite valuations by V_∞. For a nonzero element $f \in K$, we denote by $v(f)$ the value of f at v. For the normalized valuations $v_N(f) = v(f) \cdot \deg v$ of K, the *product formula*

$$\sum_{v \in V} v_N(f) \ = \ 0 \ \ \forall f \in K \setminus \{0\}$$

holds. The *height* of a nonzero element f of K is defined to be

$$H(f) \ := \ \sum_{v \in V} \max\{0, v_N(f)\} = \ - \sum_{v \in V} \min\{0, v_N(f)\} \ .$$

Let V_0 be a finite subset of V. Then, the nonzero elements $\gamma \in K$ satisfying $v(\gamma) = 0$ for all $v \notin V_0$ form a multiplicative group in K. These elements are called V_0-*units*. (For $V_0 = V_\infty$, the V_0-units are just the units of the ring O_K of integers of K.) We consider the unit equation

$$\gamma_1 + \gamma_2 + \gamma_3 = 0, \tag{2–1}$$

where the γ_i are V_0-units.

Remark 2.1. It suffices to assume that γ_1/γ_3 and γ_2/γ_3 are V_0-units, which makes the set V_0 smaller; see the proof of Lemma 3.1 in [Gaál and Pohst 06].

Lemma 2.2. *Let V_0 be a finite subset of V and let γ_i $(1 \le i \le 3)$ be V_0-units satisfying (2–1). Then, either $\frac{\gamma_1}{\gamma_3}$ is in K^p or its height is bounded:*

$$H\left(\frac{\gamma_1}{\gamma_3}\right) \ \le 2g_K \ - \ 2 \ + \ \sum_{v \in V_0} \deg v \ . \tag{2–2}$$

3. *R*-INTEGRAL SOLUTIONS OF THE THUE EQUATION

In this section, we detail our algorithm for determining all R-integral solutions of Equation (1–1).

Assume that (x, y) is a solution of (1–1). Denote by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ the conjugates of α over K and set $\beta_i = x - \alpha_i y$, $1 \le i \le n$. Fix indices i, j with $1 \le i < j < n$. Using the notation

$$\gamma_i = (\alpha_j - \alpha_n)\beta_i, \ \ \gamma_j = (\alpha_n - \alpha_i)\beta_j, \ \ \gamma_n = (\alpha_i - \alpha_j)\beta_n,$$

we can write Siegel's identity

$$(\alpha_i - \alpha_j)\beta_n + (\alpha_j - \alpha_n)\beta_i + (\alpha_n - \alpha_i)\beta_j = 0$$

in the form

$$\gamma_i + \gamma_j + \gamma_n = 0. \tag{3–1}$$

Denote by V_0 the set of valuations of $L_{ijn} = K(\alpha_i, \alpha_j, \alpha_n)$ containing the extensions of the valuations of R, the extensions of those valuations that have nontrivial value for μ, and all those valuations that have nontrivial value for one of the elements $(\alpha_j - \alpha_n)/(\alpha_j - \alpha_i)$ and $(\alpha_n - \alpha_i)/(\alpha_j - \alpha_i)$. Then, γ_i/γ_n and γ_j/γ_n are V_0-units. By Lemma 2.2, these fractions are either of bounded height or are pth powers in K. According to these two possibilities, in the following we shall consider two cases. In order to obtain all solutions of Equation (1–1), both possible cases must be considered. In Case I, we get finitely many (isolated) solutions (x, y). In Case II, we can get finitely many parameterized families of solutions; see Section 3.1.

Case I. Assume that γ_i/γ_n is not a pth power in L_{ijn}. Then, by applying Lemma 2.2 in the field $M = L_{ijn}$, we derive an upper bound for the height of γ_i/γ_n.

Because

$$\frac{\beta_i}{\beta_n} \;=\; \frac{\alpha_i - \alpha_j}{\alpha_j - \alpha_n}\;\frac{\gamma_i}{\gamma_n},$$

we have

$$H\left(\frac{\beta_i}{\beta_n}\right) \;\le\; H\left(\frac{\gamma_i}{\gamma_n}\right) \;+\; H\left(\frac{\alpha_i - \alpha_j}{\alpha_j - \alpha_n}\right). \qquad (3\text{–}2)$$

We are going to construct all possible elements β_i/β_n. Observe that this element is contained in $L_{in} = K(\alpha_i, \alpha_n)$. Denote by W_0 the set of valuations of L_{in} that are extensions of the valuations of R and those valuations that have nonzero values for μ. Then, β_i/β_n is a W_0-unit of L_{in}. We consider the divisors D_v corresponding to the valuations v of W_0. We form linear combinations

$$D \;=\; \sum_{v \in V_0} a_v \cdot D_v \qquad (3\text{–}3)$$

of these divisors with suitable coefficients $a_v \in \mathbb{Z}$ so that the height

$$\sum_{v \in V_0} \max(0, a_v) \cdot \deg v$$

does not exceed the bound in (3–2), and the product formula holds:

$$\sum_{v \in V_0} a_v \cdot \deg v \;=\; 0.$$

We calculate a basis of the Riemann-Roch space corresponding to the divisors D. If this space is of dimension 1, then there is an element in L_{in} that splits into divisors in the given way, and this element is determined (up to a nonzero factor in k) by the basis element of the Riemann-Roch space. Otherwise, if this space is of dimension > 1, then there is no element of K that splits into divisors in the given way, and there is no possible value of β_i/β_n corresponding to the divisor (3–3).

Following the argument in [Mason 84, page 18], from $\beta_i = x - \alpha_i y,\ \beta_n = x - \alpha_n y$, we obtain

$$\frac{x}{y} = \frac{\alpha_n\beta_i - \alpha_i\beta_n}{\beta_i - \beta_n} = \frac{\alpha_n\dfrac{\beta_i}{\beta_n} - \alpha_i}{\dfrac{\beta_i}{\beta_n} - 1} \quad ;$$

therefore, β_i/β_n determines x/y. For $y = 0$, the corresponding x can be calculated easily from (1–1). Note that if $\beta_i/\beta_n = 1$, then we again obtain $y = 0$. Finally, Equation (1–1) implies

$$y^n \cdot \prod_{h=1}^{n}\left(\frac{x}{y} - \alpha_h\right) \;=\; \mu.$$

Hence, by

$$y^n = \frac{\mu}{\prod_{h=1}^{n}\left(\dfrac{x}{y} - \alpha_h\right)},$$

we can determine the possible values of y and from x/y and y all possible values of x, as well. In order to determine the solutions of Equation (1–1) in Case I, for all possible values of x and y, we have to check if they are in $O_{K,R}$ and if (1–1) is satisfied.

Case II. Consider now the case when γ_i/γ_n is a complete pth power in K. In the prime divisor decomposition of β_i/β_n, only divisors from W_0 can occur. Since

$$\frac{\gamma_i}{\gamma_n} \;=\; \frac{\alpha_j - \alpha_n}{\alpha_i - \alpha_j}\;\frac{\beta_i}{\beta_n},$$

in this case the values of finite valuations of L_{ijn}, appearing only in $(\alpha_j - \alpha_n)/(\alpha_i - \alpha_j)$ and not being an extension of a valuation of W_0, must be divisible by p. If this is not satisfied for certain finite valuations, then this case is excluded. Otherwise, we replace pth powers of elements in the unit equation by the elements themselves and repeat the argument.

This phenomenon can indeed occur as is shown by Example 2 in Section 4.2. In such a case, we are led in a straightforward way (see Section 3.1) to an infinite parameterized family of solutions. Note that only finitely many such parameterized families of solutions can occur. The character of the parameterized families of solutions is described in Section 3.1, which also indicates why the number of such families is finite.

3.1 Infinite Families of Solutions

Now, we turn to the case when, in all possible unit equations, the solutions are pth powers. We describe how to find the corresponding parameterized families of solutions.

In this case, for $1 \le i < j \le n-1$, Equation (3–1) implies

$$-\frac{\gamma_i}{\gamma_n} = \eta_i^{p^m}, \quad -\frac{\gamma_j}{\gamma_n} = \eta_j^{p^m},$$

where m is a positive integer and η_i and η_j are V_0-units in L_{ijn} that are not pth powers, such that $\eta_i + \eta_j = 1$. These equations give rise to the infinite parametric families of solutions (see Example 2). Since there are only finitely many possibilities for η_i and η_j (there are finitely many V_0-units in L_{ijn} of bounded height), there can be at most finitely many infinite families of solutions.

We have

$$\frac{\beta_i}{\beta_n} = \frac{\alpha_j - \alpha_i}{\alpha_j - \alpha_n} \cdot \eta_i^{p^m}, \quad \frac{\beta_j}{\beta_n} = \frac{\alpha_j - \alpha_i}{\alpha_n - \alpha_i} \cdot \eta_j^{p^m},$$

and we can derive similar formulas for all the other β_h. Using $\beta_1 \dots \beta_n = \mu$, we get

$$\beta_n^n \cdot \frac{\beta_1}{\beta_n} \cdots \frac{\beta_{n-1}}{\beta_n} = \mu,$$

whence we obtain an expression for β_n^n, which can be written as a power product of the fundamental S_n-units $\varepsilon_1, \dots, \varepsilon_r$ in $L_n = K(\alpha_n)$ (S_n denotes the set of extensions of valuations of R to L_n). This can be used to decide if there are certain values of m for which the product is a complete nth power, and, if yes, which are the suitable values of m: we obtain congruence conditions for m. In this way, we determine the value of β_n, which we then use to determine $\beta_1, \dots, \beta_{n-1}$ as a product of some fixed elements of $L_1, \dots, L_{n-1}$ and a power product of $\varepsilon_1, \dots, \varepsilon_r$. Then, we can check if these expressions are indeed conjugates of β_n, and, if yes, then

$$y = \frac{\beta_j - \beta_i}{\alpha_i - \alpha_j}$$

is certainly in $O_{L,S}$. Moreover, since the conjugates of y are equal (these equations are identical to Siegel's identity), we have $y \in O_{K,R}$. Finally, x is given by

$$x = \frac{\alpha_i \beta_j - \alpha_j \beta_i}{\alpha_i - \alpha_j},$$

and, similarly as above, we have $x \in O_{K,R}$. Carrying out those calculations in Cases I and II yields all solutions of (1–1).

4. EXAMPLES

4.1 Example 1

Let $k = \mathbb{F}_5$ and let α be a root of

$$z^4 + (4t+2)z^2 + 1 = 0.$$

Let $K = k(t)$ and R be the set of infinite valuations of K together with the valuation corresponding to $t+2$. Let $L = K(\alpha)$, $\mu = 1/(t+2)^4$. Consider the solutions $x, y \in O_{K,R}$ of the equation

$$N_{L/K}(x - \alpha y) = \mu. \tag{4–1}$$

The extension set S of the set R of valuations of K to L consists of two infinite valuations $v_{\infty,1}, v_{\infty,2}$, both of degree 1, and two valuations $v_{t+2,1}, v_{t+2,2}$ extending $t+2$ to L, both of degree 2. The field L is Galois; we have $\alpha = \alpha_1 = \sqrt{t} + \sqrt{t+1}$ and its conjugates $\alpha_2 = -\sqrt{t} + \sqrt{t+1}$, $\alpha_3 = \sqrt{t} - \sqrt{t+1}$, and $\alpha_4 = -\sqrt{t} - \sqrt{t+1}$ are also contained in L. The field L has genus 0. To construct the set V_0 of valuations of L, we have to add to S the extensions $v_{t,1}, v_{t,2}$, both of degree 1, of the valuation corresponding to t and the extensions $v_{t+1,1}, v_{t+1,2}$, both of degree 1, of the valuation corresponding to $t+1$. Then, we have $\gamma_1/\gamma_4, \gamma_2/\gamma_4$ as V_0-units in L, and the application of Lemma 2.2 implies that these elements are either of height ≤ 8 or they are 5th powers. In Case I, if they are not 5th powers, then for the height β_1/β_4, we obtain the bound 10. This element β_1/β_4 may have nontrivial values only at one of $v_{\infty,1}, v_{\infty,2}, v_{t+2,1}, v_{t+2,2}$. Searching over all elements of L with this property, we obtain the solutions

$$(x,y) = \left(\frac{1}{t+2}, 0\right), \left(\frac{2}{t+2}, 0\right), \left(\frac{3}{t+2}, 0\right), \left(\frac{4}{t+2}, 0\right), \left(0, \frac{1}{t+2}\right), \left(0, \frac{2}{t+2}\right), \left(0, \frac{3}{t+2}\right), \left(0, \frac{4}{t+2}\right).$$

Case II can be excluded by considering

$$\frac{\gamma_1}{\gamma_4} = \frac{\alpha_2 - \alpha_4}{\alpha_1 - \alpha_2} \cdot \frac{\beta_1}{\beta_4}.$$

On the right-hand side, the valuations $v_{t+1,1}, v_{t+1,2}$ occur only in $(\alpha_2 - \alpha_4)/(\alpha_1 - \alpha_2)$ with value 1, hence the left-hand side can not be a 5th power. Thus, the above list consists of all R-integral solutions of Equation (4–1).

4.2 Example 2

Let $k = \mathbb{F}_5$, $K = k(t)$, $A = t^3 + t + 1$, and let $\alpha = \alpha_1$ be a root of

$$z^3 - Az^2 - (A+3)z - 1 = 0.$$

Let $L = K(\alpha)$; denote by α_2, α_3 the other roots of the polynomial. This is an analogue of a simplest cubic number field; see [Shanks 74]. The field L is cyclic, $\alpha_2 = -1/(\alpha_1+1)$, $\alpha_3 = -1/(\alpha_2+1)$. The elements α_1 and α_2 are fundamental units in L. This function field has genus 4; it has three infinite valuations $v_{\infty,1},\ v_{\infty,2},\ v_{\infty,3}$, all of degree 1. Let $S = \{v_{\infty,1},\ v_{\infty,2},\ v_{\infty,3}\}$. Let $\mu = 1$ and consider the (S-)integral solutions $x, y \in O_K$ of

$$(x - \alpha_1\, y)\,(x - \alpha_2\, y)\,(x - \alpha_3\, y) \;=\; 1. \tag{4–2}$$

In this case, $\beta_i = x - \alpha_i y$ as well as $(\alpha_2-\alpha_3)/(\alpha_1-\alpha_2)$ and $(\alpha_3-\alpha_1)/(\alpha_1-\alpha_2)$, hence $\varepsilon = -\gamma_1/\gamma_3$ and $\eta = -\gamma_2/\gamma_3$ are units of L. Consider the unit equation

$$\varepsilon + \eta = 1 \tag{4–3}$$

in units ε, η of L.

In Case I, if ε is not a 5th power, then the application of Lemma 2.2 gives the bound 9 for the height $\varepsilon = -\gamma_1/\gamma_3$. In our case, both V_0 and W_0 is just the set of infinite valuations, hence it is more economical to construct all possible units ε from the infinite valuations (instead of deriving a somewhat larger bound for the height β_1/β_3). We obtain nine solutions of Equation (4–3), shown in Table 1. The solutions are represented by α_1 and α_2.

#	ε	η
1	$4\alpha_1\alpha_2$	$4\alpha_2$
2	$4\alpha_2$	$4\alpha_1\alpha_2$
3	$4/\alpha_1$	$4/\alpha_1\alpha_2$
4	$4/\alpha_1\alpha_2$	$4/\alpha_1$
5	$4\alpha_1$	$4/\alpha_2$
6	$4/\alpha_2$	$4\alpha_1$
7	2	4
8	4	2
9	3	3

TABLE 1.

None of the occurring values $\varepsilon, \eta \in L\backslash k$ is a 5th power. The values of $\varepsilon = -\gamma_1/\gamma_3$ enable us to calculate β_1/β_3 and from that the solutions of Equation (4–2), which are

$$(x, y) = (0, 4), (4, 1), (1, 0). \tag{4–4}$$

In Case II, if both ε and η are 5th powers, but not in k, the unit equation becomes

$$\varepsilon_0^5 + \eta_0^5 = 1,$$

with some units ε_0, η_0 implying

$$(\varepsilon_0 + \eta_0)^5 = 1^5;$$

hence,

$$\varepsilon_0 + \eta_0 = 1.$$

If both ε_0 and η_0 are still 5th powers, we can repeat the argument. This implies that all further solutions of the unit equation are of the form $(\varepsilon^{5^m}, \eta^{5^m})$ for one of the solutions (ε, η) of Equation (4–3) and a positive integer m. We have

$$\begin{aligned} \frac{\beta_1}{\beta_3} &= \frac{\alpha_2-\alpha_1}{\alpha_2-\alpha_3}\cdot\varepsilon^{5^m} = 4\alpha_1\cdot\varepsilon^{5^m}, \\ \frac{\beta_2}{\beta_3} &= \frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}\cdot\eta^{5^m} = 4\alpha_1\alpha_2\cdot\eta^{5^m}. \end{aligned} \tag{4–5}$$

Further,

$$\beta_3^3 \cdot \frac{\beta_1}{\beta_3}\cdot\frac{\beta_2}{\beta_3} = 1,$$

that is,

$$\beta_3^3 \cdot \frac{(\alpha_2-\alpha_1)^2}{(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}\cdot(\varepsilon\eta)^{5^m} = 1,$$

whence

$$\beta_3^3 = (\varepsilon\eta)^{-5^m}\frac{1}{\alpha_1^2\alpha_2}. \tag{4–6}$$

Since all occuring elements are units, for all nine pairs (ε, η) of solutions of Equation (4–3), the right-hand side of (4–6) can be represented as a power product of α_1 and α_2, and it can be easily decided if it is a cube or not.

We detail the calculations only for the first solution in Table 1. In this case, we have $\varepsilon\eta = \alpha_1\alpha_2^2$; hence,

$$\beta_3^3 = \alpha_1^{-5^m-2}\cdot\alpha_2^{-2\cdot5^m-1}.$$

Here, the exponents are divisible by 3 if and only if $m = 2\ell$ with a positive integer ℓ, that is,

$$\beta_3 = \alpha_1^{(-5^{2\ell}-2)/3}\cdot\alpha_2^{(-2\cdot5^{2\ell}-1)/3}. \tag{4–7}$$

Similarly, for $m = 2\ell$ we obtain β_3 for solutions 2–6; for the other solutions, the exponents in the representation of β_3^3 are not both divisible by 3.

For the first solution of the unit equation, we have $\varepsilon = 4\alpha_1\alpha_2, \eta = 4\alpha_2$; hence, using (4–5) and (4–7), we obtain

$$\begin{aligned} \beta_1 &= (4\alpha_1)\cdot(4\alpha_1\alpha_2)^{5^{2\ell}}\cdot\alpha_1^{(-5^{2\ell}-2)/3}\cdot\alpha_2^{(-2\cdot5^{2\ell}-1)/3} \\ &= \alpha_1^{(2\cdot5^{2\ell}+1)/3}\cdot\alpha_2^{(5^{2\ell}-1)/3}, \end{aligned}$$

from which, by taking conjugates (using $\alpha_1' = \alpha_2$ and $\alpha_2' = 1/\alpha_1\alpha_2$), we obtain

$$\beta_1' = \alpha_1^{(-5^{2\ell}+1)/3}\cdot\alpha_2^{(5^{2\ell}+2)/3},$$

which is the same as what we get for β_2 from (4–5). Also, the conjugate of β_2 is just β_3. Now, if the values of β_1 and β_2 are indeed conjugates, then the value of

$$y = \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2}$$

is an integer, as is $x = \beta_1 + \alpha_1 y = (\alpha_1\beta_2 - \alpha_2\beta_1)/(\alpha_1 - \alpha_2)$. In this way, we obtain the infinite parametric family of solutions

$$\begin{aligned} x &= \frac{1}{\alpha_1 - \alpha_2} \cdot \Big(\alpha_1 \cdot \alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}+2)/3} \\ &\qquad - \alpha_2 \cdot \alpha_1^{(2\cdot 5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}-1)/3}\Big), \\ y &= \frac{1}{\alpha_1 - \alpha_2} \cdot \Big(\alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}+2)/3} \\ &\qquad - \alpha_1^{(2\cdot 5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}-1)/3}\Big). \end{aligned}$$

For solutions 2, 4, and 5 of the unit equation, we obtain $\beta_1' \neq \beta_2$; hence, we do not get a solution (x, y). For solutions 3 and 6 of the unit equation, we obtain the following infinite parametric families of solutions (x, y), respectively:

$$\begin{aligned} x &= \frac{1}{\alpha_1 - \alpha_2} \cdot \Big(\alpha_1 \cdot \alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(-2\cdot 5^{2\ell}+2)/3} \\ &\qquad - \alpha_2 \cdot \alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}-1)/3}\Big), \\ y &= \frac{1}{\alpha_1 - \alpha_2} \cdot \Big(\alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(-2\cdot 5^{2\ell}+2)/3} \\ &\qquad - \alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}-1)/3}\Big) \end{aligned}$$

and

$$\begin{aligned} x &= \frac{1}{\alpha_1 - \alpha_2} \cdot \Big(\alpha_1 \cdot \alpha_1^{(2\cdot 5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}+2)/3} \\ &\qquad - \alpha_2 \cdot \alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(-2\cdot 5^{2\ell}-1)/3}\Big), \\ y &= \frac{1}{\alpha_1 - \alpha_2} \cdot \Big(\alpha_1^{(2\cdot 5^{2\ell}+1)/3} \cdot \alpha_2^{(5^{2\ell}+2)/3} \\ &\qquad - \alpha_1^{(-5^{2\ell}+1)/3} \cdot \alpha_2^{(-2\cdot 5^{2\ell}-1)/3}\Big). \end{aligned}$$

Hence, all solutions of Equation (4–2) are given by the four isolated solutions (4–4) together with the above three parameterized families of solutions.

Remark 4.1. The algorithms were implemented in KASH, the KANT-Shell [Daberkow et al. 97]. The computations of the examples were carried out on an AMD Athlon i686 with 1733 MHz and 512-MB RAM under Suse Linux 8.0 and took just a few seconds.

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