The Hyperplanes of $DW(5,2)$

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CONTENTS

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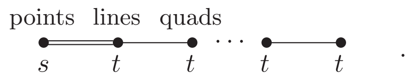
A *(geometric) hyperplane* of a geometry is a proper subspace meeting every line. We present a complete list of the hyperplane classes of the symplectic dual polar space $DW(5,2)$. Theoretical results from Shult, Pasini and Shpectorov, and the author guarantee the existence of certain hyperplanes. To complete the list, we use a backtrack algorithm implemented in the computer algebra system GAP. We finally investigate what hyperplane classes arise from which projective embeddings of $DW(5,2)$.

1. INTRODUCTION

A partial linear space is a geometry in which two points lie on at most one line. A subspace of a geometry is a point set that contains each point of a line l if it meets l in at least two points. A *geometric hyperplane*, or for short, a *hyperplane*, of a geometry Γ is a proper subspace meeting every line. We denote collinearity by $\perp$, and if P is a point, then $P^\perp$ is the set of points collinear with P including P. Moreover, if Γ is a geometry of diameter d, i.e., its collinearity graph has diameter d, then for a point P of Γ, the set of points of Γ at distance i from P, $i = 1, 2, ..., d$, is denoted by $\Gamma_i(P)$; e.g., $P^\perp = \Gamma_1(P) \cup \{P\}$.

The aim of this paper is to find up to isomorphism all hyperplanes of the dual polar space $DW(5,2)$, which is the smallest thick dual polar space of rank 3. This research has been motivated by the quest for hyperplanes of the duals of polar spaces. Throughout this paper, Δ is the dual of a finite polar space Π of finite rank n. The elements of type i of Δ are the $(n-i)$-dimensional singular subspaces of Π; e.g., the points of Δ are the maximal, i.e., $(n-1)$-dimensional, singular subspaces of Π. Incidence in Δ is symmetrized containment induced from Π. The elements of type 3 of Δ are called *quads* since the point-line residue $Res^-_\Delta(\alpha)$ of a quad α of Δ is a generalized quadrangle. The point-line residue is the dual of the generalized quadrangle $Res^+_\Pi(\alpha)$ consisting of the submaximal and maximal singular subspaces of Π containing the $(n-3)$-dimensional singular subspace α

of Π. If Δ is finite, its quads are generalized quadrangles of order (s, t) and Δ belongs to the diagram



If H is a hyperplane of a geometry Γ, an element η of Γ of type at least 2 is either contained in H or $H \cap \eta$ is a hyperplane of η. Let H be a hyperplane of the dual polar space Δ. If α is a quad of Δ not contained in H, then $\alpha \cap H$ is a hyperplane of the generalized quadrangle $Res^{-}_{\Delta}(\alpha)$. It is well known that hyperplanes of generalized quadrangles are of one of the following three types (for reference, see [Payne and Thas 84, Section 2.3.1]):

- the *perp* $P^{\perp}$ of a point,
- a full subquadrangle which is a hyperplane, or
- an *ovoid*, i.e., a set of mutually noncollinear points meeting every line.

If α is a quad of Δ such that $\alpha \cap H = P^{\perp} \cap \alpha$ for some point P of α, then α is called *singular* and the point $P \in \alpha$ with $\alpha \cap H = P^{\perp} \cap \alpha$ the *deep point of* α *with respect to* α. If $\alpha \cap H$ is a subquadrangle, we call α a *subquadrangular quad*. If $\alpha \cap H$ is an ovoid of α, then α is called *ovoidal*. We call a point R of H *deep* if $R^{\perp} \subset H$. Note that in general, a deep point with respect to some quad is not deep.

A hyperplane H of a dual polar space of rank at least 3 is called *locally singular* (or *locally subquadrangular* or *locally ovoidal*) if all quads of $\Delta \setminus H$ are singular (or subquadrangular or ovoidal, respectively). In each of these cases, H is called *locally uniform*, otherwise *locally nonuniform*.

The Singular Hyperplane. One example of a hyperplane of a dual polar space is the singular hyperplane. A dual polar space of rank n is a near $2n$-gon, i.e., a partial linear space of diameter at most n such that, for each point P and line l, there is a unique point on l nearest to P. Hence, if D is a point of a dual polar space Δ, the points of Δ at nonmaximal distance from D form a hyperplane. This hyperplane is locally singular and called the *singular hyperplane with deepest point* D. Row 1 of Table 1 contains its combinatorics.

The uniform hyperplanes of the dual polar space $DW(5, q)$ and the dual geometry of the symplectic polar space $W(5, q)$ are completely known by theoretical classification results of Shult [Shult 92], Pasini and Shpectorov [Pasini and Shpectorov 01], and Cooperstein and Pasini [Cooperstein 03]. We present them in Section 2.

For nonuniform hyperplanes, very little is known. The author has shown in [Pralle 01], that if H is a nonuniform hyperplane of a finite dual polar space, then there exists a singular quad. Moreover, he has classified in [Pralle 02] the nonuniform hyperplanes of dual polar spaces of rank 3 without subquadrangular quads. Of the three families of such hyperplanes, only one exists in $DW(5, 2)$:

The Extension of an Ovoid of a Quad. If ω is a quad of Δ and Ω is an ovoid of the generalized quadrangle $Res_{\Delta}(\omega)$, then the point set $H = \cup_{X \in \Omega} X^{\perp}$ is a hyperplane of Δ. This hyperplane contains the quad ω, the quads meeting ω are singular and the quads disjoint from ω are ovoidal. In Table 1, H appears in the fourth row. If G is the action of $Sp(6, 2)$ on $DW(5, 2)$, the stabilizer of H in G acts transitively on the points of $\Delta - H$ and has three orbits in H. This hyperplane stabilizes the ovoid Ω, the complement $\omega - \Omega$ of the quad ω, and the point set of $H \cap (\Delta - \omega)$.

Note that the singular hyperplane of a dual polar space of rank 3 with deepest point P is similar to the just described hyperplane consisting of the neighbours of an ovoid of a quad. If σ is a quad on P, then the singular hyperplane H with deepest point P consists of the points of Δ collinear with the perp $P^{\perp} \cap \sigma$ of P in the quad σ. More generally, if Γ is a dual polar space of rank n and if H_0 is a hyperplane of an element of type $n - 1$, the set of points $H = \bigcup_{X \in H_0} X^{\perp}$ is a hyperplane of Γ. Since generalized quadrangles have three different kinds of hyperplanes, there is a third hyperplane of this form in the dual polar space Δ of rank 3 if quads of Δ admit subquadrangles which are hyperplanes.

The Extension of a Subquadrangle of a Quad. If σ is a quad and Σ is a subquadrangle of σ, then $H = \bigcup_{X \in \Sigma} X^{\perp}$ is a hyperplane of Δ. This hyperplane appears in row 5 of Table 1.

To get an idea of the variety of nonuniform hyperplanes of dual polar spaces, the aim of this paper is to find up to isomorphism all hyperplanes of the smallest thick dual polar space of rank 3 that is the dual $DW(5, 2)$ of the symplectic polar space $W(5, 2)$. By means of a backtrack search with the computer algebra system GAP [Gap 00], we have constructed the whole subspace lattice containing a given subspace S_0 and reduced the lattice up to isomorphism. Since, by [Pralle 01], nonuniform hyperplanes require at least one singular quad, and since the nonuniform hyperplanes without subquadrangular quads are known by [Pralle 02], we start with a subspace S_0 meeting one quad τ in the perp of a point and one quad σ

in a subquadrangle. The algorithm generates subspaces containing S_0 such that τ and σ do not change their hyperplane intersection, i.e., τ remains singular and σ subquadrangular for all subspaces generated.

We remark that, for $DW(5,2)$ there is another approach to finding all of its hyperplanes: if a geometry Γ is projectively embeddable by a morphism $e : \Gamma \to PG(V)$, we say a geometric hyperplane H of Γ *arises from the embedding* e if there exists a hyperplane $\overline{H}$ of $PG(V)$ such that $H = e^{-1}(\overline{H} \cap e(\Gamma))$. By Ronan [Ronan 87], if Γ is embeddable and has exactly three points on every line, then all hyperplanes of Γ arise from its universal embedding $e_{un} : \Gamma \to PG(V)$. Since $DW(5,2)$ is embeddable, the hyperplanes of $DW(5,2)$ arise from its universal embedding $e_{un} : DW(5,2) \to PG(14,2)$ (for e_{un} see Li [Li 01]). Thus, the hyperplane classes of $DW(5,2)$ may be represented by the intersection of $e_{un}(DW(5,2))$ and a representative of every orbit of the hyperplanes of $PG(14,2)$ under the action of $Sp(6,2)$.

In general, geometric hyperplanes of embeddable geometries do not arise from an embedding. For such geometries, our backtrack algorithm still works to find all hyperplane classes.

The dual polar space $DW(5,2)$ admits projective embeddings in $PG(d,2)$, $7 \leq d \leq 14$, that are all quotients of the universal embedding into $PG(14,2)$. Three of them are particularly interesting. We present them in Section 5.2 and investigate what hyperplane of $DW(5,2)$ arises from each of these three embeddings. It turns out that there are indeed hyperplane classes of $DW(5,2)$ arising from its universal embedding which cannot be generalized for $q > 2$, since the universal embedding of $DW(5,2)$ is essentially different from the universal embedding of $DW(5,q)$ for $q > 2$.

Our main results are presented in Table 1, the geometric description of all hyperplane classes of $DW(5,2)$ and their embeddings. In Section 2, we present the known results about uniform hyperplanes of dual polar spaces. Section 4 is devoted to the nonuniform hyperplanes of $DW(5,2)$. The existence of two of them is explained by the theoretical results in [Pralle 01] and [Pralle 02]. The algorithm for the determination of the remaining, so far unknown, classes of hyperplanes of $DW(5,2)$ is described in Section 3. In Section 4, we present, geometrically, the newly found nonuniform hyperplanes with an aim not only to present combinatorial properties of the hyperplanes. These descriptions may serve for geometric generalizations of families of hyperplanes of which our algorithm has found only the smallest member in $DW(5,2)$. In Section 5, we focus on the embeddings of $DW(5,2)$ and its hyperplanes.

Before presenting Table 1, we note the combinatorics of finite dual polar spaces and, in particular, those of $DW(5,2)$. Let Δ be a finite dual polar space of rank n such that the point-line residues of its quads are generalized quadrangles of order (s,t). Then Δ has

- $(s+1)(st+1)\cdots(st^{n-1}+1)$ points,
- $(st+1)\cdots(st^{n-1}+1)(t^{n-1}+\ldots+t+1)$ lines,
- $(st^2+1)\cdots(st^{n-1}+1)(t^{n-1}+\ldots+t+1)(t^{n-2}+\ldots+t+1)/(t+1)$ quads,
- $t^{n-1}+\ldots+t+1$ lines per point, and
- $(t^{n-1}+\ldots+t+1)(t^{n-2}+\ldots+t+1)/(t+1)$ quads per point.

The dual polar space $DW(5,2)$ has parameters $s = t = 2$ and rank 3, thus it has 135 points, 315 lines, 63 quads, three points per line, and seven lines and seven quads per point forming a Fano plane. The point-line residues of the quads are symplectic generalized quadrangles $DW(3,2) \cong W(3,2)$ (note $W(3,q)$ is self-dual for even q, see [Payne and Thas 84, Section 3.2.1]).

If S is a subspace of Δ, we say a point P of S has *order* o *with respect to* S if there are o lines on P contained in S. In $DW(5,2)$, points can have orders $o \in \{0,...,7\}$.

In Table 1, each row contains the combinatorial properties of a hyperplane H of one of the twelve classes of hyperplanes of $DW(5,2)$. The second column contains the number of points of H, the third the number of lines contained in H. Columns $4,5,...,11$ contain the numbers of points of H of order $0,1,...,7$, respectively, i.e., if P is a point counted in column $4+i$ for $i=0,...,7$, then P has order i and $P^{\perp} \cap H$ has $2i+1$ points. In the two columns following the order of the stabilizer of H, the number of orbits of the stabilizer of H in the action G of $Sp(6,2)$ on the complement $DW(5,2) - H$ of the hyperplane H of the dual polar space $DW(5,2)$ and the number of orbits of G on H are given. The last column displays from which embedding the hyperplane arises. The hyperplanes with e_{sp} arise from the spin embedding and consequently from any embedding; those with e_{gr} do arise from the Grassmann embedding, which is the universal embedding of $DW(5,q)$ for $q \geq 3$, but do not arise from any embedding of lower dimension than the Grassmann embedding; and the hyperplanes with e_{un} arise only from the universal embedding of $DW(5,2)$ into $PG(14,2)$.

We conclude this introduction with another remark about hyperplanes of a geometry Γ with three points on

	points	lines	points incident with 0, ..., 7 lines								quads in H	singular q.	subquadr. q.	ovoidal q.	$\lvert Stab_G(H)\rvert$	Orb(H)	Orb($\Delta\setminus H$)	embeddings
1.	71	91				56				15	7	56			10752	3	1	e_{sp}
2.	63	63				63						63			12096	1	1	e_{sp}
3.	105	210							105		28		35		40320	1	1	e_{un}
4.	55	35		40		10				5	1	30		32	3840	3	1	e_{gr}
5.	87	147				6		72		9	13	18	32		2304	3	1	e_{gr}
6.	81	126					54		27		9	27	27		1296	2	1	e_{un}
7.	73	98			12		48		13		4	24	27	8	384	5	3	e_{un}
8.	71	91		2		38		30		1	3	28	24	8	192	6	3	e_{gr}
9.	65	70			30		30		5		1	35	15	12	240	4	2	e_{un}
10.	63	63		12		39		12				31	16	16	192	5	5	e_{gr}
11.	65	70	2		21		42					28	21	14	336	4	3	e_{un}
12.	57	42	8		42				7			28	7	28	1344	3	4	e_{un}

TABLE 1. Combinatorics of the 12 classes of hyperplanes of $DW(5,2)$.

every line. If H_1 and H_2 are different hyperplanes of Γ, then the complement $H := \overline{H_1 \Delta H_2}$ of their symmetric difference $H_1 \Delta H_2$ is also a hyperplane. Let $H_1, ..., H_{12}$ be hyperplanes of $DW(5,2)$ such that H_i is a representative of the hyperplane class of row i of Table 1. So, H_1 is a singular hyperplane with deepest point D, H_2 a split Cayley hexagon $H(2)$ (see Section 2), H_3 a locally subquadrangular hyperplane (see Section 2), H_4 the extension of an ovoid Ω of a quad ω, and H_5 the extension of a subquadrangle Σ of a quad σ. Then $H_6, ..., H_{12}$ may be expressed as $H_6 = \overline{H_3 \Delta H_5}$ with $\sigma \subset H_3$, $H_7 = \overline{H_1 \Delta H_3}$ with $D \in H_3$, $H_8 = \overline{H_1 \Delta H_4}$ with $D \in H_4 \setminus \omega$, $H_9 = \overline{H_3 \Delta H_4}$ with $\omega \subset H_3$, $H_{10} = \overline{H_1 \Delta H_4}$ with $D \notin H_4$, $H_{11} = \overline{H_2 \Delta H_3}$, and $H_{12} = \overline{H_1 \Delta H_3}$ with $D \notin H_3$.

2. THE UNIFORM HYPERPLANES OF $DW(5,2)$

In this section, we present the known classes of uniform hyperplanes of finite dual polar spaces. Three of them are hyperplanes of the dual symplectic polar space $DW(5,q)$ and appear in the first three rows of Table 1.

The Singular Hyperplanes. As mentioned in Section 1, for every dual polar space Δ and every point P of Δ, the points of Δ at nonmaximal distance from P form the singular hyperplane H with deepest point P. If we denote the action of $Sp(6,2)$ on Δ by G, then $Stab_G(H)$ is flag-transitive on the complement $\Delta - H$. More precisely, G fixes the deep point P and stabilizes and acts transitively on the point sets $\Delta_i(P)$, $i = 1,2,3$. The singular hyperplane is the first row of Table 1.

The Split Cayley Hexagons $H(2)$. By Shult [Shult 92] and Pralle [Pralle 02, Theorem 1], in a dual polar space Δ of rank 3, only one locally singular hyperplane exists besides the singular hyperplane. It is a split Cayley hexagon $H(K)$ and Δ is the dual of an orthogonal parabolic polar space $Q(6,K)$ (for reference, see Van Maldeghem [Van Maldeghem 98, Section 2.4]). Since $W(5,2) \cong Q(6,2)$, the split Cayley hexagon $H(2)$ also occurs in our list of hyperplanes of $DW(5,2)$. It is the second hyperplane in Table 1. The stabilizer of $H(2)$ in $Sp(6,2)$ is the Lie group $G_2(2)$. It is flag-transitive on both $H(2)$ and $\Delta - H(2)$.

The Locally Subquadrangular Hyperplanes. In [Pasini and Shpectorov 01], Pasini and Shpectorov prove that there are only two families of locally subquadrangular hyperplanes in finite dual polar spaces. One of them is an example in the dual of the Hermitian polar space $DH(6,4)$, hence it does not appear in our list of hyperplanes of $DW(5,2)$. The other family is an infinite series of locally subquadrangular hyperplanes of which the smallest member appears in our list of hyperplanes of $DW(5,2)$: if $\Pi_0 \cong Q^+(2n-1,2)$, $n \geq 3$, is a hyperplane of $\Pi = D\Delta \cong Q(2n,2)$, then the maximal singular subspaces of Π not contained in Π_0 are the points of a hyperplane of Δ. The $(n-3)$-subspaces of Π_0 are subquadrangular quads of $\Delta - H$, and the $(n-3)$-subspaces of $\Pi - \Pi_0$ are quads of Δ contained in H. For $n = 2$, this hyperplane is the third in Table 1.

No Ovoid in $DW(5,2)$. Supposing finiteness and flag-transitivity on $\Delta - H$, Pasini and Shpectorov [Pasini and

Shpectorov 01] prove the nonexistence of locally ovoidal hyperplanes or briefly, ovoids of dual polar spaces. With an earlier result of Shult (see [Pasini and Shpectorov 01, Section 2.8]), Cooperstein and Pasini [Cooperstein 03] prove the nonexistence of ovoids in $DW(5,q)$ without supposing flag-transitivity.

3. THE ALGORITHMIC APPROACH

In Sections 1 and 2, we presented the hyperplanes of dual polar spaces known by theoretical classification results. There are five of them in $\Delta = DW(5,2)$ which are given in rows 1–5 of Table 1. To find all hyperplanes of Δ nonisomorphic to these five, we have constructed the lattice of all subspaces of Δ containing an appropriate start subspace S_0 by means of a backtrack algorithm implemented in the computer algebra system GAP.

The Start Subspace S_0. As mentioned in Section 1, hyperplanes of Δ not isomorphic to one of the known and already presented hyperplanes are nonuniform and force one quad to be singular and one to be subquadrangular. Hence, the input for the algorithm is a subspace S_0 consisting of a grid Q of a quad σ and the perp $P^\perp \cap \delta$ of a point P in a quad δ which has a line in common with σ, and a set A of points of Δ such that the point set $A \cup S_0$ is a subspace consisting of a singular quad δ and a subquadrangular quad σ. The output of the algorithm is a reduced list H of all hyperplanes containing S_0 intersecting A trivially.

The backtrack algorithm was used with two different start spaces to find all hyperplanes. First, one chooses σ and δ such that $l \subset H$, and second, one supposes $l \not\subset H$. After having found in the first run all hyperplanes having a singular and a subquadrangular quad sharing a line belonging to H, the second run returns all hyperplanes such that the line of intersection of any intersecting singular and subquadrangular quad does not belong to H.

The Backtrack Algorithm. Backtrack algorithms are well known. Our algorithm has three main steps in each turn of the backtrack loop. In the first, it generates subspaces, in the second, it calculates canonical representatives for the newly generated subspaces, and in the last, it adds each of these representatives either to the backtrack or the hyperplane list if the lists do not yet contain any subspace isomorphic to this subspace.

Generating Subspaces. Let G be the automorphism group of Δ, i.e., the action of $Sp(6,2)$ on the dual polar space Δ. For every subspace S_i of the backtrack list L, if there are m orbits of the stabilizer $Stab_G(S_i)$ of S_i on the complement $\Delta \setminus S_i$, the algorithm generates the subspaces $S_{i+j} = \langle S_i, x_j \rangle$ containing S_i and a representative point x_j of the jth orbit for $j = 1, ..., m$ (in the algorithm scheme below, this is `OnePointExtensions`$(S_i, Stab_G(S_i))$).

If S_{i+j} contains a point of A, then S_{i+j} is rejected since it may not contain both a singular and a subquadrangular quad.

Canonical Representatives. To test for isomorphism before adding a subspace *cand* to a list, there are essentially two solutions. Before adding a subspace *cand* to a list, one should test whether *cand* is isomorphic to any of the list members R by searching for an isomorphism in G mapping *cand* onto R. In GAP, `RepresentativeAction`$(G, cand, R)$ returns an element in G that maps *cand* onto R, if one exists, and `fail` otherwise. The search for an isomorphism in a group is very time consuming. Therefore, before using the group action, one compares the combinatorial properties of *cand* and R. Only if they coincide, one looks for isomorphisms. However, the complexity of the algorithm is $O(n^2)$, where n is the length of the list.

The other approach is not to add *cand* to the list, but instead add a canonical representative $C(cand)$ of it. To determine representatives is expensive, but the comparison of $C(cand)$ with each element of the list is just testing equality. Hence the complexity of the algorithm is only $O(n)$. We have chosen this way and describe the calculation of a canonical representative $C(cand)$ in Section 3.1.

Adding a Candidate to a List. If a subspace S_{i+j} generated by `OnePointExtensions` is a hyperplane, then the algorithm adds $C(S_{i+j})$ to the list H of hyperplanes if it is not yet contained in H (in the scheme of the algorithm below, this function is denoted by `AddReduced`$(H, C(S_{i+j}))$). If S_{i+j} is a hyperplane, then $C(S_{i+j})$ is not added to L since hyperplanes are maximal subspaces, and the extensions of a hyperplane would be the whole point set which is not a hyperplane. Thus the algorithm has the following basic form which is a standard backtrack algorithm:

```
H := [ ];
L := [S0];
n := 1;
while n ≤ Length(L) do
  T := OnePointExtensions(L[n], StabG(L[n]));
  for S in T do
   if S ∩ A = ∅ then
```

```
      C(S) := CanonicalRepresentative(G, S);
      if IsHyperplane(S) then
       AddReduced(H, C(S));
      else AddReduced(L, C(S));
    n := n + 1;
  return(H);
```

3.1 Canonical Representative

In this section, we describe how to determine a canonical representative of a subspace S. Our implementation of Δ uses the permutation representation of $Sp(6,2)$ on the point set $\mathcal{P} = \{1, ..., 135\}$. The main tool for canonical representatives is the following: let the power set of $\mathcal{P}$ be ordered lexicographically. For a subset $X \subset \mathcal{P}$ and a subgroup U of $G = Aut(\Delta)$, set $\texttt{SmallestImage}(U, X) = \min\{X^g \mid g \in U\}$ as the smallest image of X under the action of U with respect to the lexicographic order. In our implementation, besides the smallest image of X under U with respect to the lexicographic order, the function $\texttt{SmallestImage}(U, X)$ returns the element $g \in U$ mapping X onto its smallest image and also the stabilizer of the smallest image in U.

One could apply $\texttt{SmallestImage}(\cdot, \cdot)$ to the subspace S in which we search for a canonical representative and the full automorphism group G. But the larger the set or the group is, the harder it is to find the smallest image. Therefore, we want to inspect more geometric properties of S that are hidden in the automorphism group G of Δ without determining the stabilizer $Stab_G(S)$. We first order the point set of S according to the order defined in Section 1. If o lines on a point P of S are contained in S, then P has order o. The points of S fall in one to eight subsets $S_0, ..., S_7$, where S_i is the set of points of order i in S. Since the amount of work to be done by $\texttt{SmallestImage}(\cdot, \cdot)$ depends on the set and group size, we order $S_0, ..., S_7$ increasingly by their cardinalities. Then we start with the smallest of these sets, say S_i, to determine its smallest image S'_i under G. Next, we determine the smallest image of the second smallest set, say S_j, under the action of the stabilizer $Stab_G(S'_i)$ which we know already from $\texttt{SmallestImage}(G, S_i)$. Note that already in the second step, the group is much smaller and the point set often only slightly bigger. Continuing this process, we finally get a canonical representative $C(S)$ of the subspace S.

4. THE NONUNIFORM HYPERPLANES OF $DW(5,2)$

This section is devoted to the geometric description of the hyperplanes of $\Delta = DW(5,2)$ found by our computer search. As mentioned, they are nonuniform containing both singular and subquadrangular quads.

For completeness, we recall the two nonuniform hyperplanes known by theoretical classification results and already presented already in Section 1: the extensions of an ovoid or a subquadrangle of a quad consisting of the neighbours of, respectively, an ovoid or a subquadrangle of a quad of Δ where the subquadrangle is a hyperplane of the quad.

For the ease of notation, we define $-$-lines (respectively, $+$-lines). If H is a hyperplane of a dual polar space Δ and l is a line of Δ, then l is called a *$-$-line* (respectively, *$+$-line*) *with respect to* H if l is (respectively, is not) contained in H. This terminology is motivated by the study of affine dual polar spaces, the complements of hyperplanes of dual polar spaces. A line of the affine dual polar space $\Delta - H$ is a $+$-line of Δ with respect to H, whereas a line contained in H, not in $\Delta - H$, is a $-$-line of Δ with respect to H.

We remind the reader that a point $P \in H$ is called *deep* if $P^\perp \subset H$. Moreover, a quad is called *deep* if it is contained in H.

If G denotes the action of $Sp(6,2)$ on the dual polar space Δ, then the automorphism group of the affine dual polar space $\Delta - H$ or, equivalently, of the hyperplane H is the stabilizer N of H in G. In the geometric description of the hyperplanes, we also note the orbits of N in H and $\Delta - H$.

4.1 A Subspace of H Acting as a Dual Polar Space

The hyperplane H consists of 81 points, of which 27 have order 6 and 54 have order 4. There are nine deep, 27 subquadrangular, and 27 singular quads. The combinatorics of this hyperplane are in row 6 of Table 1.

The set $\mathcal{H}$ of points of order 6 of H is a connected subspace of Δ with a line set $\mathcal{L}$ of 27 lines. Let $\mathcal{P}$ be the set of quads contained in H. With incidence inherited from the polar space $\Pi \cong W(5,2)$ dual of Δ, consider the incidence structure $\Pi_0 := (\mathcal{P}, \mathcal{L}, \mathcal{H})$ as a substructure of Π. Then, the elements of $\mathcal{H}$ are planes of Π and each such plane is incident with three points of $\mathcal{P}$ and three lines of $\mathcal{L}$ forming a triangle. The residue of a point $P \in \mathcal{P}$ in Π_0 consists of six lines and nine planes on P forming a dual grid.

More precisely, Π_0 is a short-lined polar space of order $(1,1,2)$. The 27 lines of $\mathcal{L}$ cover 36 points of Π, of which, as mentioned, the nine points of concurrency of lines of $\mathcal{L}$ correspond to the nine quads of Δ contained in H. The 27 remaining points on the lines of $\mathcal{L}$ are the subquadrangular quads of Δ. The singular quads are the 27 points of Π on no line of $\mathcal{L}$.

The group N acts transitively on $\Delta - H$ and has the two orbits $\mathcal{H}$ and $\mathcal{H} - H$ on H.

4.2 A Star-Similar Hyperplane H Missing a Deep Point

The combinatorics of the hyperplane H are in row 7 of Table 1. The orders of the 73 points of H are 2, 4, and 6. There are 13 points of order 6, 48 of order 4, and 12 of order 2. There are four deep, 27 subquadrangular, 24 singular, and eight ovoidal quads.

The 13 points of order 6 form a singular subspace $\mathcal{P}$ of Δ, i.e., there is one point of the 13, say P, such that the subspace $\mathcal{P}$ consists of the points on six of the seven lines through P. The seventh line on P, say l, is a +-line. The four deep quads contain P, and the three remaining quads on P are subquadrangular.

Through each point $X \in \mathcal{P} - \{P\}$, there are −-lines through P, four lines each consisting of X and two points of order 4 of H and one line consisting of X and two points of order 2 of H. The 12 points of order 2 of H are the points of H in the three subquadrangular quads on P which are not collinear with P.

If Q is one of the 16 points of H at distance 3 from P, then Q has order 4, and exactly one quad on Q is ovoidal, comprising the three +-lines through Q. The remaining six quads on Q are subquadrangular.

The eight ovoidal quads are those meeting (the unique +-line) l (through P) and not containing P. If ω is such an ovoidal quad, it has three points of H at distance 2 from P, which have order 4 according to the above, and it has two points at distance 3 from P, which have order 4 as mentioned in the previous paragraph. Thus, the 16 points of H at distance 3 from P belong uniquely to the eight ovoidal quads meeting $l \setminus \{P\}$ in a single point.

Since N stabilizes the sets of points of H of the same order, it fixes the point P and the point set $\Delta_1(P) \cap H$ of the 12 remaining points of H of order 6 on which it acts transitively. Since N fixes P, it stabilizes the sets $\Delta_i(P)$ of points at distance i from P for $i = 1, 2, 3$. Since there are 16 points of H at distance 3 from P of order 4, 32 points of $\Delta_2(P) \cap H$ of order 4, and 12 points of $\Delta_2(P) \cap H$ of order 2, and since N has five orbits on H, these sets are orbits of N.

The three orbits of N on $\Delta - H$ are $l \setminus \{P\}$, $\Delta_2(P) - H$, and $\Delta_3(P) - H$.

4.3 A Star-Like Hyperplane H with Ovoidal Quads

The hyperplane H has 71 points of which one is deep, 30 have order 5, 38 order 3, and two lie on just one line. This hyperplane's combinatorics may be found in row 8 of Table 1. There are three deep, 24 subquadrangular, 28 singular, and eight ovoidal quads.

Let P be the unique deep point of H. The two points of order 1 are collinear on a line l through P. There are three lines l_1, l_2, l_3 through P such that the six points of $(l_1 \cup l_2 \cup l_3) \setminus \{P\}$ have order 3. The remaining three lines through P, say g_1, g_2, g_3, consist of P and two points of order 5.

The three quads contained in H contain P. Clearly, they do not contain l since the three quads on l are singular with deep point P. Moreover, the three deep quads do not contain a common line, but intersect pairwise in three distinct lines. These intersection lines are g_1, g_2, and g_3 since the points on g_1, g_2, g_3 have order ≥ 5, which follows from the fact that each of g_1, g_2, g_3 belongs to two deep quads. Hence, each of l_1, l_2, l_3 belongs to a unique deep quad, and the remaining quads on l_1, l_2, and l_3 are singular with deep point P.

The points in the deep quads not in $l_1 \cup l_2 \cup l_3$ have order 5. These are all 30 points of H of order 5.

As in Section 4.2, the eight ovoidal quads are the quads meeting l in a single point distinct from P. They do not meet any of the deep quads, thus they meet H in points of order 3. Together, the union of the ovoidal quads meets H in $2 \cdot (4 \cdot 4 + 1) = 34$ points of which 32 have order 3, namely all except the two points on l. Moreover, these 32 points of order 3 have distance 3 from P. Together with the six points of order 3 on l_1, l_2, l_3, these are all points of order 3 of H, and we have described all points in H.

Since P is the unique deep point, N stabilizes the sets $\Delta_i(P)$ for $i = 0, 1, 2, 3$. Moreover, fixing the sets of points of the same order, N stabilizes the point sets $(l_1 \cup l_2 \cup l_3) \setminus \{P\}$, $(g_1 \cup g_2 \cup g_3) \setminus \{P\}$, and $l \setminus \{P\}$. Hence N has at least four orbits on $\Delta_0(P) \cup \Delta_1(P)$. The 24 remaining points of order 5 are the points of the deep quads not collinear with P, hence they belong to $\Delta_2(P)$. Since the ovoidal quads contain points of H at distance 3 from P and since we know N has six orbits on H, one of them is $\Delta_2(P) \cap H$, which is the set of 24 points of order 5 at distance 2 from P, and the other is $\Delta_3(P) \cap H$, which is the set of 32 points of order 3 at distance 3 from P.

Since $P^{\perp} \subset H$ and since N stabilizes the line l on P, N stabilizes the points of $\Delta - H$ collinear with a point on l and it stabilizes the sets $\Delta_2(P)$ and $\Delta_3(P)$. Hence, the three orbits of N on $\Delta - H$ are the points of $\Delta_2(P) - H$ collinear with a point on l, the points of $\Delta_2(P) - H$ collinear with no point on l, and $\Delta_3(P) - H$.

4.4 An Almost-Deep Ovoid in a Deep Quad

The hyperplane H has 65 points of which five have order 6, 30 order 4, and 30 order 2. This hyperplane is given in row 9 of Table 1. There are one deep quad, 15 subquadrangular, 35 singular, and 12 ovoidal quads.

Let δ denote the deep quad. The five points $P_1, ..., P_5$ of order 6 form an ovoid Ω of δ. The remaining ten points of δ have order 4. For $i = 1, ..., 5$, denote the unique $+$-line through P_i by l_i. The 15 subquadrangular quads are the quads on the lines $l_1, ..., l_5$. Thus, on each line h of δ, there exists exactly one subquadrangular quad. The third quad on h is singular with deep point $h \cap \Omega$. Hence, there are exactly 15 singular quads meeting δ.

First, we consider the points of H not in δ collinear with a point of Ω and second, those of H collinear with a point X of $\delta \setminus \Omega$. Let h_1, h_2, h_3 be the three $-$-lines on P_1 not in δ. By the above, the three quads containing the $+$-line l_1 are subquadrangular, hence each of them is spanned by l_1 and one of h_1, h_2, h_3. The other quads on each of h_1, h_2, h_3 are singular with P_1 as the deep point. Thus the points on h_1, h_2, h_3 distinct from P_1 have order 2. Similarly, the points collinear with P_i, $i = 2, ..., 5$, not on l_i and not contained in δ have order 2. Together, these are the 30 points of H of order 2.

Now, let X be a point of $\delta \setminus \Omega$. Then X has order 4 and there is a unique $-$-line g on X not contained in δ. The three quads on g are subquadrangular since they have two $-$-lines on X, namely g and the line of intersection with δ. Thus the two points of $g \setminus \{X\}$ have order 4. Similarly, each of the ten points of $\delta \setminus \Omega$ is collinear with exactly two points of H not in δ that have order 4. Together, these are the 20 remaining points of H of order 4.

For the quads disjoint from δ, consider a point $R \in (h_1 \cup h_2 \cup h_3) \setminus \{P_1\}$. By the above, it has order 2 and the two $-$-lines are contained in a subquadrangular quad meeting δ. Thus two of the four quads on R disjoint from δ are singular, whereas the other two quads do not contain any $-$-line through R, hence they are ovoidal. Since $(h_1 \cup h_2 \cup h_3) \setminus \{P_1\}$ consists of six points and since each quad meets $(h_1 \cup h_2 \cup h_3) \setminus \{P_1\}$ in exactly one point, there are $6 \cdot 2 = 12$ ovoidal and singular quads disjoint from δ. The remaining eight quads disjoint from δ meeting $P_1^{\perp}$ in a point on the $+$-line l_1 are singular.

The four orbits of N on H are the ovoid Ω of the unique deep quad δ, the points of $\delta - \Omega$, the set of points of H collinear with points of Ω, and the set of points of H collinear with a point of $\delta - \Omega$ but not in δ. The two orbits of N on $\Delta - H$ are the points of $\Delta - H$ collinear with a point of Ω and the remaining points of $\Delta - H$.

4.5 A Tangential Hyperplane H of the Polar Space Π

The 63 points of the hyperplane H have orders 1, 3, and 5. There are 12 points on just one line, 39 points of order 3, and 12 points of order 5. H contains no deep quad, 31 quads are singular, 16 quads are subquadrangular, and 16 are ovoidal. The combinatorics of H are noted in row 10 of Table 1.

The points of order 1 are mutually noncollinear, and the same holds for the points of order 5.

There exists a unique singular quad α with deep point D such that all points of $\alpha \cap H$ have order 3.

If $\sigma \neq \alpha$ is a quad on a $-$-line h through D, then σ is singular since h is a line in $\sigma \cap H$ containing the point D that belongs to no other $-$-line in σ apart from h. Thus, its deep point R is one of $h \setminus \{D\}$. If σ' is the third singular quad on h, its deep point is the remaining point on h.

Let m be a $+$-line of α through R. Then the quads containing m are singular, since they intersect the singular quad σ with deep point R in $-$-lines. Thus, all 31 quads meeting α including α are singular.

Considering the polar space $\Pi \cong W(5, 2)$ dual of Δ, the singular quads are the points collinear with the point α, hence the tangential hyperplane $\alpha^{\perp}$. The remaining 16 ovoidal and 16 subquadrangular quads are the points of the affine space $\mathcal{A} := PG(5, 2) \setminus \alpha^{\perp}$.

Through each point of $\Delta_1(D) \cap \alpha$ pass two $-$-lines not in α. On each of these twelve lines of H not contained in α and meeting α, there lies one point of order 1 and one point of order 5. The remaining 32 points of H not in α have order 3 and distance 3 from D.

Let l be one of the two $-$-lines through R not in α. Let P be the point of order 5 on l and Q be the point of order 1 on l. The four quads containing Q but not l are ovoidal. The four quads through P not containing l are subquadrangular, since they intersect the two singular quads on l with deep point P in $-$-lines and the one singular quad on l with deep point R in $+$-lines.

Finally, let X be a point of $H \setminus \alpha$ with $\pi_\alpha(X) \notin H$, i.e., X is at distance 3 from D. Then X has order 3. There are three subquadrangular quads and one ovoidal quad disjoint from α on X (recall that the three quads on X meeting α are singular).

In the polar space Π, X is a totally isotropic plane of the affine space $\mathcal{A}$ intersecting the hyperplane $\alpha^{\perp}$ of $PG(5, 2)$ in a line not through the point α. The three subquadrangular quads through X are three of the four affine points, say S_1, S_2, S_3, of the plane X, and the single ovoidal quad on X is the fourth affine point O of X.

Clearly, the three affine lines of X joining S_1, S_2, and S_3 belong to H, whereas the three affine lines joining O with one of S_1, S_2, S_3 are not contained in H.

Since α is the unique singular quad all of whose points have order 3, N stabilizes α and fixes its deep point D. Hence, N stabilizes the sets $\Delta_i(D)$ for $i = 0, ..., 3$ and their intersections with α. On H, N has the five orbits $\{D\}$, $\Delta_1(D) \cap H = \alpha \cap H - \{D\}$, the set $\Delta_3(D) \cap H$ of the remaining 32 points of order 3 of H, and the two sets of points of H of order 1; these partition $\Delta_2(D) \cap H$. Similarly on $\Delta - H$, N has the orbits $\Delta_2(D) \cap \alpha$, $\Delta_2(D) - (H \cup \alpha)$, and $\Delta_3(D) - H$, and the set $\Delta_1(D) - \alpha$ falls in two orbits. The set $\Delta_1(D) - \alpha$ consists of the points on the four $+$-lines through D not belonging to α. N acts transitively on these lines and has two orbits on their points, i.e., each of the two orbits of N in $\Delta_1(D) - \alpha$ consist of one point on each of the four $+$-lines through D.

4.6 Two Isolated Points

The combinatorics of the hyperplane H are collected in row 11 of Table 1. H has two points on no $-$-line, say P_1 and P_2 at distance 3 from each other, 21 points of order 2, and 42 points of order 4. There are 28 singular, 21 subquadrangular, and 14 ovoidal quads of which the ovoidal quads are those on P_1 and P_2.

Let ω be a (ovoidal) quad on P_1, and let X be the point $\pi_\omega(P_2)$. There are three points of $\omega \cap H$ at distance 2 from both P_1 and P_2, namely those collinear with X. Each of these three points is contained in a unique ovoidal quad on P_2. Hence, there are $3 \cdot 7 = 21$ points of H each belonging to two ovoidal quads, hence the points have order at most 2. Thus, the points of H at distance 2 from both P_1 and P_2 are precisely the 21 points of order 2. Note they are mutually noncollinear.

The previously unconsidered fifth point Q of $\omega \cap H$ has distance 3 from P_2, and ω is the only ovoidal quad on Q. Q is one of the remaining 42 points of H of order 4. Thus, the four lines on Q not in ω are contained in H. The six quads through Q distinct from ω are subquadrangular, since each of them contains one $+$-line of ω and two of the $-$-lines through Q not in ω.

Since in each quad on P_1, respectively P_2, there is exactly one point of H at distance 3 from P_2, respectively P_1, there are 14 such points in H. So far, we have taken into account only points of H at distance at most 2 from one of P_1 or P_2. In Δ, there are exactly 28 points at distance 3 from both P_1 and P_2. These are the remaining 28 points of H of order 4. If R is one of them, then there are four singular and three subquadrangular quads containing R, and R is the deep point of one of these singular quads.

The four orbits of N on H are $\{P_1, P_2\}$, the 21 points of order 2 forming the set $\Delta_2(P_1) \cap \Delta_2(P_2)$, the 14 points of order 4 forming the set $(\Delta_2(P_1) \cap \Delta_3(P_2)) \cup (\Delta_2(P_2) \cap \Delta_3(P_1)) \cap H$, and the remaining 28 points of H of order 4 building the set $\Delta_3(P_1) \cap \Delta_3(P_2)$.

On $\Delta - H$, N has the three orbits

$$(\Delta_1(P_1) \cup \Delta_3(P_2)) \cup (\Delta_1(P_2) \cup \Delta_3(P_1)),$$

$$(\Delta_1(P_1) \cup \Delta_2(P_2)) \cup (\Delta_1(P_2) \cup \Delta_2(P_1)),$$

and

$$((\Delta_2(P_1) \cup \Delta_3(P_2)) \cup (\Delta_2(P_2) \cup \Delta_3(P_1))) - H.$$

4.7 Subquadrangular Quads through a Point

The hyperplane H consists of eight isolated points pairwise at distance 2, 42 points of order 2, and seven points of order 6. There exists a point P of Δ not in H such that the points of order 6 form the set $P^\perp \cap H$. No quad is contained in H, there are 28 ovoidal, 28 singular, and seven subquadrangular quads. The combinatorics may be found in the last row of Table 1.

The seven subquadrangular quads are the quads on P. The three points $P^\perp \cap H \cap \sigma$ of order 6 belonging to one quad σ on P build a triad of the generalized quadrangle σ and an ovoid of the grid $\sigma \cap H$.

If ω is an ovoidal quad, then it contains two isolated points and three points of order 2.

Since the seven points of order 6 have pairwise distance 2, they partition the set of 42 lines of H. Thus, each line of H lies in a quad with P. Since the quads on P are the subquadrangular quads and P does not belong to H, the subquadrangular quads also partition the line set of H. Moreover, each line of H consists of one point of order 6 and two points of order 2. The points on lines of Δ through points of H of order 6 not belonging to $P^\perp$ are the $7 \cdot (6 \cdot 2)/2 = 42$ points of H of order 2.

On a point of order 2 of H on two $-$-lines l_1, l_2, there are one subquadrangular and six singular quads with deep points the four points of order 6 on l_1 and l_2.

Note for each pair of an isolated point Q and a point R of order 6, Q and R have distance 3.

Since N has three orbits on H and H contains points of three different orders, the points of H of the same order form an orbit.

Since the points of order 6 of H have the unique center P, P is fixed by N and N stabilizes the sets $\Delta_i(P)$ for $i = 0, 1, 2, 3$. Since N has four orbits on $\Delta - H$, the sets

$\Delta_0(P) = \{P\}$, $\Delta_1(P) - H, \ldots, \Delta_3(P) - H$ are exactly the orbits of N on $\Delta - H$.

5. HYPERPLANES ARISING FROM AN EMBEDDING

A linear embedding of a geometry Γ is an injective mapping $e : \Gamma \to PG(V)$ into the projective space $PG(V)$ of a vector space V such that

- $e(X) \leq e(Y)$ if and only if $X \leq Y$ for all elements $X, Y \in \Gamma$,
- $e(X) = \langle \{e(P) \mid P \text{ point of } X\} \rangle$, and
- $V = \langle \{e(P) \mid P \text{ point of } \Gamma\} \rangle$.

An embedding $e_{un} : \Gamma \to PG(V)$ is called *universal* if for any other embedding $e : \Gamma \to PG(W)$ there exists a homomorphism $\varphi : PG(V) \to PG(W)$ such that $e = \varphi \circ e_{un}$.

A hyperplane H of an embeddable geometry Γ *arises from the embedding* $e : \Gamma \to PG(V)$, if there exists a hyperplane h of $PG(V)$ such that $H = e^{-1}(h \cap e(\Gamma))$. A very interesting question about a hyperplane is whether or not it arises from an embedding. For instance, the singular hyperplanes of dual polar spaces arise from embeddings. The dual polar space $\Delta = DW(2n-1, 2)$, $n \geq 2$, has projective embeddings into projective spaces of dimension d with $2^n - 1 \leq d \leq \frac{(2^n+1)(2^{n-1}+1)}{3} - 1$ (see [Pasini 03, Section 9.1]), whereas for $q > 2$, $DW(2n-1, q)$ has a unique embedding in $PG(\binom{2n}{n} - \binom{2n}{2n-2} - 1, q)$ (see [Cooperstein 98]). We present the embeddings in Section 5.1 and investigate in Section 5.2 from which embedding the hyperplanes of $DW(5, 2)$ arise.

5.1 The Embeddings of $DW(5, 2)$

From the projective embeddings of $DW(5, 2)$, three are of particular interest:

- the universal embedding $e_{un} : DW(5, 2) \to PG(14, 2)$,
- the Grassmann embedding $e_{gr} : DW(5, 2) \to PG(13, 2)$, and
- the spin-embedding $e_{sp} : DW(5, 2) \to PG(7, 2)$.

The universal embedding of a linear point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ with three points on each line maps $\mathcal{P}$ into the projective space $PG(\tilde{V})$ over the factor space $\tilde{V} = W/U$ where W is the $\mathbb{F}_2$-vector space with basis the point set $\mathcal{P}$ and with U the subspace of W generated by all vectors $a_1 + a_2 + a_3$ if $\{a_1, a_2, a_3\}$ form a line. For $DW(5, 2)$, $\tilde{V}$ has projective dimension 15 [Li 01]. Obviously, this construction works only for $q = 2$.

Let $V = \mathbb{F}_2^6$ be the vector space with the alternating form f defining the polar space $\Pi \cong W(5, 2)$ dual of Δ.

The universal embedding of the symplectic dual polar space $DW(5, q)$ with $q > 2$ is induced by the embedding of the Grassmannian of planes of $PG(5, q)$ [Cooperstein 98]. It clearly is also an embedding for $q = 2$. The points of $\Delta \cong DW(5, q)$ are the planes of Π, hence they are certain points of the Grassmannian of planes of $PG(5, q)$. The lines of the Grassmannian of planes are the pairs $\{A, B\}$ for subspaces $A < B \leq PG(V)$ with $\dim(A) = 1 = \dim(B) - 2$. The embedding of $DW(5, q)$ induced by the embedding of the Grassmannian of planes of $PG(5, q)$ in $PG(\bigwedge^3 V) = PG(19, q)$ maps the planes of $W(5, q)$ in a 13-dimensional subspace of $PG(\bigwedge^3 V)$ (see, for instance, [Pralle 02, Section 2]).

The spin-embedding e_{sp} of $DW(5, q)$ exists only for even q since it is a consequence of the isomorphism $W(5, q) \cong Q(6, q)$ for q even. Considering $Q(6, 2)$ as a nondegenerate hyperplane of the orthogonal space $Q^+(7, 2)$ of rank 4, each singular plane of $Q(6, 2)$ belongs to exactly one member of each of the two classes $\mathcal{M}_1$ and $\mathcal{M}_2$ of singular maximal subspaces of $Q^+(7, 2)$. Conversely, each element of $\mathcal{M}_1$ and $\mathcal{M}_2$ intersects the hyperplane $Q(6, 2)$ of $Q^+(7, 2)$ in a singular plane of $Q(6, 2)$. A triality of $Q^+(7, 2)$ is a morphism of order 3 that maps the points of $Q^+(7, 2)$ onto one of the two classes of maximal subspaces, say $\mathcal{M}_1$, $\mathcal{M}_1$ onto $\mathcal{M}_2$, $\mathcal{M}_2$ onto the points, and the singular lines onto the singular lines, and that preserves incidence. The product of these embeddings is the spin-embedding of $W(5, 2)$ into $Q^+(7, 2)$. It can also be established through the Grassmann embedding e_{gr} by factorizing a suitable six-dimensional subspace of the codomain $\overline{V} = \mathbb{F}_2^{14}$.

If $e_{un} : \Delta = DW(5, 2) \to \tilde{V}$ is the universal embedding, $\tilde{V}$ has a one-dimensional subspace N such that every 2-subspace of $\tilde{V}$ containing N meets $e_{un}(\Delta)$ in at most one point. The codomain $\overline{V}$ of the Grassmann embedding is the factor space $\tilde{V}/N$. Thus, if H is a hyperplane arising from e_{gr}, it arises from e_{un} as well. Similarly, as the codomain of the spin-embedding is a factor space of the codomain of e_{gr}, a hyperplane arising from e_{sp} also arises from e_{gr} and e_{un}.

As mentioned, the Grassmann embedding is the universal embedding of $DW(5, q)$ for $q > 2$ [Cooperstein 98]. The dimension of the universal embedding is bigger than that of the Grassmann embedding only for $q = 2$. Hence, hyperplanes of our list in Table 1 arising from the universal embedding e_{un} but not from the Grassmann embedding will not generalize as hyperplanes arising from an embedding for $q > 2$.

5.2 Hyperplane Properties Induced by the Embeddings

By [Ronan 87, Corollary 2 of Theorem 1], if an embeddable geometry has three points on every line, then all its hyperplanes arise from its universal embedding.

Proposition 5.1. *All hyperplanes of $DW(5,2)$ arise from e_{un}.*

In the following, we prove the assertions in the last column of Table 1, i.e., from which of the three embeddings e_{sp}, e_{gr}, e_{un} of least dimension the hyperplane classes arise.

Proposition 5.2. *If H is a hyperplane arising from the spin embedding e_{sp}, then its points may have orders 3 or 7.*

Proof: For a point P of Δ, the spin-embedding $e_{sp} : \Delta \to Q^+(7,2)$ maps the projective plane $Res_\Delta(P)$ onto a projective plane of the quotient $PG(6,2) \cong PG(7,2)/e_{sp}(P)$ which is singular with respect to the quadric $Q(6,2)$ induced by $Q^+(7,2)$ on $PG(7,2)/e_{sp}(P)$. Considering a hyperplane of $PG(7,2)$ in $PG(7,2)/e_{sp}(P)$, the hyperplane intersects the plane π in a line or contains it. Thus in Δ, either three or seven lines through P belong to H. □

Corollary 5.3. *Only the locally singular hyperplanes of $DW(5,2)$ listed in Table 1 in rows 1 and 2 arise from the spin embedding.*

Proof: The singular hyperplane of $DW(5,2)$ with deepest point P consists of the planes of $W(5,2)$ that have a point in common with the plane P of $W(5,2)$. Under the spin-embedding e_{sp}, these planes are mapped onto the tangential hyperplane $e_{sp}(P)^\perp$ of $Q^+(7,2)$.

The split Cayley hexagon $H(2)$ may be represented by a hyperplane section of $O^+(7,2)$ [Van Maldeghem 98, Theorem 2.4.10]. Hence, this hyperplane (row 2 in Table 1) arises from e_{sp}.

Because of the conditions on the point orders according to Proposition 5.2, by Table 1 none of the other hyperplanes of $DW(5,2)$ arise from e_{sp}. □

Proposition 5.4. *If H is a hyperplane arising from the Grassmann embedding e_{gr}, its points may have orders 1, 3, 5, or 7.*

Proof: Let P be a point of H. The Grassmann embedding e_{gr} induces an embedding $e_P : Res_\Delta(P) \to PG(5,2)$ of the projective plane $Res_\Delta(P) = PG(2,2)$ of lines and quads containing P into $PG(5,2)$, which is the Veronesean embedding of $PG(2,2)$ [Pasini 03, Theorem 9.3 and Section 6.1]. The hyperplane sections of the Veronesean variety are conics of $PG(2,2)$. A conic of $PG(2,2)$ consists of either one point, three points (a line or a nondegenerate conic), five points (two lines), or all seven points. Hence, there are either one, three, five, or seven lines through P in H. □

Corollary 5.5. *Only the hyperplane classes of $DW(5,2)$ listed in Table 1 in rows 4, 5, 8, and 10 arise from the Grassmann embedding e_{gr} but not from e_{sp}.*

Proof: By Propositions 5.2 and 5.4, the listed hyperplane classes are the only ones with appropriate point orders. To check that they arise from e_{gr}, we have implemented the Grassmann embedding $e_{gr} : DW(5,2) \to PG(13,2)$ by means of the computer algebra program GAP [Gap 00]. As one could also prove theoretically by means of the descriptions of the hyperplane classes in Section 4, it turns out that the images of the mentioned hyperplanes under e_{gr} span only hyperplanes of $PG(13,2)$. □

In particular, the hyperplane classes not mentioned in Corollaries 5.3 or 5.5 do not generalize as hyperplane classes of $DW(5,q)$ for $q > 2$ since the universal embedding of $DW(5,q)$ for $q > 2$ is the Grassmann embedding $e_{gr} : DW(5,q) \to PG(13,q)$.

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