The First Birkhoff Coefficient and the Stability of 2-Periodic Orbits on Billiards

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In this work we address the question of proving the stability of elliptic 2-periodic orbits for strictly convex billiards. Even though it is part of a widely accepted belief that ellipticity implies stability, classical theorems show that the certainty of stability relies upon more precise conditions. We present a review of the main results and general theorems and describe the procedure to fulfill the supplementary conditions for strictly convex billiards.

1. INTRODUCTION

Let α be a plane, closed, regular, and strictly convex curve. The billiard problem on α consists of describing the free motion of a point particle in the plane region enclosed by α, with unitary velocity and elastic reflections when it impacts with the boundary. The trajectories are polygons in the region.

The motion is completely determined by the point of reflection at α and the direction of motion immediately after each reflection. For instance, the arclength parameter s, which locates the point of reflection, and the tangential component of the momentum $p = \sin\theta$, where θ is the angle between the direction of motion and the normal to the boundary at the reflection point, describe the system. Good introductions to billiards can be

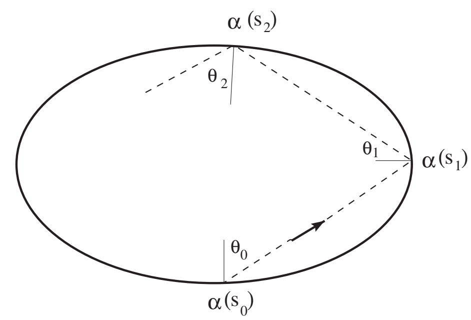

FIGURE 1. Trajectory in a convex billiard.

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FIGURE 2. Two different behaviours of maps with a linearly stable fixed point.

found in [Birkhoff 27, Chernov and Markarian 03, Hasselblat and Katok 02, Katok and Hasselblat 95, Kozlov and Treshchëv 91] or [Tabachnikov 95].

The billiard model defines a map T that for each (s, p) in the annulus $\mathcal{A} = [0, L) \times (-1, 1)$, associates the next impact and direction:

$$T: \begin{array}{ccc} \mathcal{A} & \to & \mathcal{A} \\ (s,p) & \longmapsto & (S(s,p), P(s,p)). \end{array}$$

Since the particle can travel along the same polygon in both directions, the problem is time-reversing and the inverse map T^{-1} is well defined.

The derivative of T at (s, p) is implicitly calculated and is given by the formulae:

$$\begin{aligned} \frac{\partial S}{\partial s} &= \frac{l(s,p) - R(s)\cos\theta(p)}{R(s)\cos\theta(P)} \\ \frac{\partial S}{\partial p} &= \frac{l(s,p)}{\cos\theta(p)\cos\theta(P)} \\ \frac{\partial P}{\partial s} &= \frac{l(s,p) - R(s)\cos\theta(p) - R(S)\cos\theta(P)}{R(s)R(S)} \\ \frac{\partial P}{\partial p} &= \frac{l(s,p) - R(s)\cos\theta(P)}{R(S)\cos\theta(p)} \end{aligned} \tag{1–1}$$

where S stands for $S(s, p)$ and P for $P(s, p)$, $l(s, p)$ is the distance between the two consecutive impact points $\alpha(s)$ and $\alpha(S)$, R is the radius of curvature of α, and $\cos\theta(p) = \sqrt{1 - p^2}$ is the normal component of the momentum.

If α is a C^k curve, $k \geq 2$, the billiard model gives rise to a discrete two-dimensional C^{k-1} area preserving dynamical system, whose orbits are given by

$$O(s,p) = \{T^j(s,p), j \in \mathbb{Z}\} \subset \mathcal{A}.$$

A billiard has no fixed points. However, given $n \geq 2$, Birkhoff's theorem states that T has at least two different orbits of period n which will be fixed points of T^n. The linearization of T^n at any of these fixed points, say (s, p), gives the linear area preserving map $DT^n_{(s,p)}$, which has a fixed point at the origin $(0, 0)$. According to the eigenvalues of this linear map, the fixed point (s, p) is classified as: hyperbolic if the eigenvalues are μ and $\frac{1}{\mu}$, $\mu \in \mathbb{R}$, $\mu \neq \pm 1$; elliptic if the eigenvalues are $\mu = e^{i\gamma}$ and $\overline{\mu}$, $\mu \neq \pm 1$; or parabolic if the eigenvalues are 1 or -1.

In the hyperbolic case, the Hartman-Grobman Theorem (see, for instance, [Katok and Hasselblat 95] or [Palis and de Melo 78]) assures that, on a neighbourhood of the fixed point (s, p), the dynamical behaviour of T^n is the same as the dynamical behaviour of $DT^n_{(s,p)}$ on a neighbourhood of the origin. So, (s, p) is an unstable fixed point of T^n and $\{(s, p), T(s, p), ..., T^{n-1}(s, p)\}$ is an unstable periodic orbit of T. In this case, the instability of the equilibrium of the linear map $DT^n_{(s,p)}$ implies the local instability of the periodic orbit for the complete map T.

In the elliptic case, the linear map $DT^n_{(s,p)}$ is a rotation: the origin is surrounded by closed invariant circles and is a stable equilibrium. However, this beautiful behaviour may not be inherited by the map T^n, as can be seen in the examples in Figure 2. For both of them, the fixed point is linearly elliptic. On the left side, the nonlinear map exhibits invariant closed curves surrounding the fixed point, which is then stable. For the nonlinear map on the right, no invariant curves can be observed and the fixed point seems to be unstable.[1]

Moreover, it is not even clear if the pictures in Figure 2, obtained by numerical simulations, correspond to the true behaviour of the maps. In fact, we [Dias Carneiro et al. 03] proved that any C^1 strictly convex billiard map with an elliptic 2-periodic orbit can be ap-

[1] Even more surprising is the example given by Anosov and Katok in [Anosov and Katok 70] of an ergodic area-preserving map of the disc $|z| < 1$, with an elliptic fixed point at $z = 0$. The ergodicity implies that the fixed point is unstable. This example does not represent a billiard map and we don't know if there are any billiards with this property.

proached by billiard maps with a 2-periodic orbit surrounded by closed invariant curves, i.e., with a stable orbit. We guess that this result can be extended to any period. Therefore, because of natural numerical roundoff errors, one can not be sure that the simulation corresponds to the actual billiard and not to a very close one.

As a consequence, in the elliptic case, a more careful approach is needed and higher-order terms must be taken into account to assure the local stability of periodic orbits. A classical way to handle this problem is to use the Birkhoff normal form and Moser's twist theorem [Siegel and Moser 71].

In what follows, we explain how this can be performed and applied to the billiard map in the case of 2-periodic elliptic orbits. We have employed the software Maple to calculate the necessary data and all the worksheets are available at http://www.mat.ufmg.br/comed/2004/d2004/. We then apply the results to two special classes of billiards.

Related works are [Hayli et al. 87] and [Moeckel 90]. The first author studied the stability of elliptic periodic orbits for a family of Robnik's billiards. The last author studied the generic behaviour of the first Birkhoff coefficient for one-parameter families of conservative maps.

2. NONLINEAR ANALYSIS AND THE LOCAL STABILITY OF ELLIPTIC ORBITS

Let T be an area preserving map with a n-periodic orbit $\{(0,0), T(0,0), ..., T^{n-1}(0,0)\}$. We will assume that the map is C^k with $k \geq 4$. In the case of the billiard map, this is equivalent to assuming that the curve α is at least C^5.

The map T^n can then be expanded in Taylor form up to order 3 in a neighbourhood of its fixed point $(0,0)$,

$$\begin{aligned} T^n(s,p) = (&a_{10}s + a_{01}p + a_{20}s^2 + ... + a_{03}p^3, \\ &b_{10}s + b_{01}p + b_{20}s^2 + ... + b_{03}p^3) \\ &+ \mathcal{O}(|(s,p)|^4). \end{aligned} \tag{2–1}$$

If the fixed point is elliptic with eigenvalues $\mu = \cos\gamma + i\sin\gamma$ and $\bar{\mu}$, by means of a complex linear area-preserving coordinate change which diagonalizes the linear part, the map T^n can be written as

$$\begin{aligned} z \mapsto \mu(&z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 \\ &+ c_{30}z^3 + c_{21}z^2\overline{z} + c_{12}z\overline{z}^2 \\ &+ c_{03}\overline{z}^3) + \mathcal{O}(|z|^4). \end{aligned} \tag{2–2}$$

If $\mu^j \neq 1$, $j = 1, 2, 3,$ or 4 we say that μ is nonresonant, and an analytic coordinate change brings the map into its convergent Birkhoff normal form

$$z \mapsto \mathrm{e}^{i(\gamma+\tau_1|z|^2)}z + \mathcal{O}(|z|^4) = \mu z + i\mu\tau_1 z|z|^2 + \mathcal{O}(|z|^4).$$

The first Birkoff coefficient τ_1 is given by

$$\tau_1 = \Im(c_{21}) + \frac{\sin\gamma}{\cos\gamma - 1}\left(3\,|c_{20}|^2 + \frac{2\,\cos\gamma - 1}{2\,\cos\gamma + 1}\,|c_{02}|^2\right) \tag{2–3}$$

where $\Im(c_{21})$ stands for the imaginary part of c_{21}.

The calculations leading to Equation (2–3) are standard [Hayli et al. 87, Moeckel 90] and can be easily performed using symbolic programming (see the worksheet `3NormalForm` at [Dias Carneiro et al. 04]).

By Moser's twist theorem, if the first Birkhoff coefficient τ_1 is not zero there are T^n-invariant curves surrounding the fixed point and therefore it is stable. We have that each point of the n-periodic orbit is contained in an open set, called an island, homeomorphic to a disk and invariant under T^n. Each island contains T^n-invariant curves surrounding the periodic point. So, the n-periodic orbit of T is stable.

3. ELLIPTIC 2-PERIODIC ORBITS OF CONVEX BILLIARDS

Any closed regular strictly convex C^2 curve α has at least two diameters, characterized by points with parallel tangents and equal normal lines (like the axis of an ellipse). The motion along each one of them corresponds to a 2-periodic trajectory for the billiard map associated to α.

It is easy to prove that the longest of these diameters, if isolated, corresponds to a hyperbolic orbit (see, for instance, [Katok and Hasselblat 95] or [Kozlov 00]). The other(s) can be either hyperbolic, elliptic, or parabolic. Let us suppose that one of them is elliptic and let $s = 0$ and $s = s_1$ be the arclength parameters of the trajectory. Since the motion occurs in the normal direction, the tangential component of the momentum p is zero in both of the reflection points. Then $\{(0,0), (s_1, 0)\}$ is an elliptic 2-periodic orbit of the associated billiard map $T(s,p) = (S(s,p), P(s,p))$ and $(0,0)$ is an elliptic fixed point of T^2.

Let $R_0 = R(0)$ and $R_1 = R(s_1)$ be the radii of curvature of α at $s = 0$ and $s = s_1$; and let $L = ||\alpha(0) - \alpha(s_1)||$ be the length of the trajectory. Using Equation (1–1), the linear map $DT^2_{(0,0)} = DT_{(s_1,0)}\ DT_{(0,0)}$ is given by

$$DT^2_{(0,0)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3–1}$$

where

$$A = \frac{(L-R_1)\,(L-R_0)}{R_1\,R_0} + \frac{L\,(L-R_0-R_1)}{R_0\,R_1},$$
$$B = -\frac{2\,L\,(L-R_1)}{R_1},$$
$$C = -\frac{2\,(L-R_0)\,(L-R_0-R_1)}{{R_0}^2\,R_1}, \text{ and}$$
$$D = \frac{(L-R_1)\,(L-R_0)}{R_1\,R_0} + \frac{L\,(L-R_0-R_1)}{R_0\,R_1}$$

and its eigenvalues are

$$2\,\frac{(L-R_1)\,(L-R_0)}{R_0\,R_1} - 1 \pm \frac{2\sqrt{L\,(L-R_0-R_1)\,(L-R_1)\,(L-R_0)}}{R_0\,R_1}.$$

Since the trajectory is elliptic the relations $L-R_0-R_1 < 0$ and $(L-R_0)(L-R_1) > 0$ must be fulfilled. Assuming that $4\,(L-R_0)(L-R_1) \neq R_0R_1$ and $2\,(L-R_0)(L-R_1) \neq R_0R_1$ then $\mu^j \neq 1$ for $j = 1,2,3,4$.

In the elliptic and nonresonant case, in order to investigate the stability of the fixed point, we can proceed and examine the first Birkhoff coefficient given by (2–3) . The complex coefficients c_{21}, c_{20}, and c_{02} in the formula depend on the real coefficients a_{ij} and b_{ij} of the Taylor expansion of T^2 at the origin, Equation (2–1).

The linear coefficients a_{ij} and b_{ij}, $i+j=1$, are obviously the entries of $DT^2_{(0,0)}$ and thus given by Equation (3–1). Note that $a_{10} = b_{01}$. As T is area preserving, $a_{10}^2 - a_{01}b_{10} = 1$, and, as $(0,0)$ is elliptic, $a_{01}b_{10} < 0$.

These conditions were used to write down the coordinate change leading to (2–2) and we found (see the worksheet `2Complex` at [Dias Carneiro et al. 04]):

$$\begin{aligned}\Im(c_{21}) =& \frac{a_{10}}{8}\left(-a_{21} + 3\frac{b_{10}}{a_{01}}a_{03} - 3\frac{a_{01}}{b_{10}}b_{30} + b_{12}\right)\\ &- \frac{b_{10}}{8}\left(a_{12} - 3\frac{a_{01}}{b_{10}}a_{30} - \frac{a_{01}}{b_{10}}b_{21} + 3b_{03}\right)\\ |c_{20}|^2 =& \frac{1}{16}\sqrt{-\frac{a_{01}}{b_{10}}}\left(\frac{b_{10}}{a_{01}}a_{02} + a_{20} + b_{11}\right)^2\\ &+ \frac{1}{16}\sqrt{-\frac{b_{10}}{a_{01}}}\left(\frac{a_{01}}{b_{10}}b_{20} + b_{02} + a_{11}\right)^2\\ |c_{02}|^2 =& \frac{1}{16}\sqrt{-\frac{a_{01}}{b_{10}}}\left(\frac{b_{10}}{a_{01}}a_{02} + a_{20} - b_{11}\right)^2\\ &+ \frac{1}{16}\sqrt{-\frac{b_{10}}{a_{01}}}\left(\frac{a_{01}}{b_{10}}b_{20} + b_{02} - a_{11}\right)^2\end{aligned} \tag{3–2}$$

which shows that τ_1 is linear on the real coefficients of third order and quadratic on the second order ones.

In order to explicitly calculate the first Birkhoff coefficient, all that is needed now are the second- and third-order coefficients of the Taylor expansion at $(0,0)$ of $T^2(s,p) = (S(S(s,p),P(s,p)), P(S(s,p),P(s,p)))$.

A sequence of straightforward but long computations using the chain rule gives those Taylor coefficients. To illustrate it, let us give the expression of a_{20}:

$$\begin{aligned}a_{20} =& \frac{\partial^2}{\partial s^2}S(S(s,p),P(s,p))(0,0)\\ =& \frac{\partial S}{\partial s}(0,0)\frac{\partial P}{\partial s}(0,0)\frac{\partial^2 S}{\partial s\partial p}(s_1,0)\\ &+ \frac{1}{2}\frac{\partial S}{\partial s}(s_1,0)\frac{\partial^2 S}{\partial s^2}(0,0)\\ &+ \frac{1}{2}\left[\frac{\partial P}{\partial s}(0,0)\right]^2\frac{\partial^2 S}{\partial p^2}(s_1,0)\\ &+ \frac{1}{2}\frac{\partial S}{\partial p}(s_1,0)\frac{\partial^2 P}{\partial s^2}(0,0)\\ &+ \frac{1}{2}\left[\frac{\partial S}{\partial s}(0,0)\right]^2\frac{\partial^2 S}{\partial s^2}(s_1,0).\end{aligned}$$

The first derivatives of the functions S and P are given by Formulae (1–1) and they depend on the function $l(s,p)$. So, to calculate the second and third derivatives of S and P it is necessary to evaluate the first and second derivatives of l. Let $l(s,S) = ||\alpha(S) - \alpha(s)||$. Then $l(s,p) = l(s,S(s,p))$.

By differentiating

$$l^2(s,S) = \langle \alpha(S) - \alpha(s), \alpha(S) - \alpha(s)\rangle$$

we have

$$l(s,S)\,\frac{\partial l}{\partial s}(s,S) = -\,\langle \alpha'(s), \alpha(S) - \alpha(s)\rangle \tag{3–3}$$

and so, as α' is the unitary tangent vector,

$$\frac{\partial l}{\partial s}(s,S) = -p\ .$$

Analogously,

$$\frac{\partial l}{\partial S}(s,S) = P\ .$$

These relations simply show that $-l(s,S)$ is the generating function of the billiard map.

Differentiating (3–3) with respect to s and S and using the fact that $\eta = R\,\alpha''$ is the unitary normal vector gives

$$\begin{aligned}\frac{\partial^2 l}{\partial s^2}(s,S) =& \frac{1-p^2}{l(s,S)} - \frac{\sqrt{1-p^2}}{R(s)}\\ \frac{\partial^2 l}{\partial s\,\partial S}(s,S) =& \frac{\sqrt{(1-p^2)(1-P^2)}}{l(s,S)}\ .\end{aligned}$$

The same reasoning gives

$$\frac{\partial^2 l}{\partial S^2}(s,S) = \frac{1-P^2}{l(s,S)} - \frac{\sqrt{1-P^2}}{R(S)}.$$

The chain rule now will give the first- and second-order derivatives of $l(s,p)$. To evaluate them at $(0,0)$ and $(s_1,0)$ it is useful to remember that $\alpha'(s_1) = -\alpha'(0)$, $\eta(s_1) = -\eta(0)$, and $\alpha(s_1) - \alpha(0) = L\,\eta(0)$.

Because of the recurrent structure of the formulae, the explicit calculus of the a_{ij} and b_{ij} is suitable to be implemented as a computer program (see the worksheets `0ThreeJet` and `1TaylorCoeffs` at [Dias Carneiro et al. 04]). The final expression of the Taylor expansion of T^2 is also given in the worksheet `1TaylorCoeffs`.

The second-order coefficients $a_{ij}, b_{ij}, i+j = 2$ will have linear dependence on $\frac{dR}{ds}(0) = R_0'$ and $\frac{dR}{ds}(s_1) = R_1'$ while the third-order coefficients $a_{ij}, b_{ij}, i+j = 3$ will have linear dependence on $\frac{d^2R}{ds^2}(0) = R_0''$ and $\frac{d^2R}{ds^2}(s_1) = R_1''$ and quadratic dependence on the first-order derivatives. So the first Birkhoff coefficient τ_1 will have quadratic dependence on the first derivatives of R and linear dependence on the second derivatives. The final expression of τ_1 is obtained after substitution of the a_{ij} and b_{ij} into (3–2) and then into (2–3) giving

$$\begin{aligned}\tau_1 = &-\frac{1}{8}\frac{R_0+R_1}{R_0R_1}\\ &-\frac{1}{8}\frac{L}{L-R_0-R_1}\left(\frac{L-R_1}{L-R_0}R_0'' + \frac{L-R_0}{L-R_1}R_1''\right)\\ &-\frac{1}{8}\frac{L}{(L-R_0-R_1)^2}\\ &\times\left(2\frac{L-R_1}{L-R_0}(R_0')^2 + 2\frac{L-R_0}{L-R_1}(R_1')^2 + 3R_0'R_1'\right)\\ &+\frac{1}{8}\frac{LR_0R_1}{(L-R_0-R_1)^2(4(L-R_0)(L-R_1)-R_0R_1)}\\ &\times\left(\frac{(L-R_1)^2}{L-R_0}\frac{(R_0')^2}{R_0} + \frac{(L-R_0)^2}{L-R_1}\frac{(R_1')^2}{R_1} - R_0'R_1'\right).\end{aligned}$$

The details leading to the formula are given in the worksheet `4Tau` at [Dias Carneiro et al. 04].

4. BILLIARDS WITH ISLANDS

As remarked in Section 3, a billiard on a strictly convex C^2 curve always has 2-periodic orbits and the largest one, if isolated, is hyperbolic. Unfortunately, one can not assure that at least one of the others is elliptic. In fact, there are many examples where all 2-periodic orbits are isolated and hyperbolic (see, for instance, [Dias Carneiro et al. 03] or [Kozlov 00]).

On the other hand, ellipticity is an open property, in the sense that if a billiard associated to a C^2 strictly convex curve α has an elliptic 2-periodic orbit, then any strictly convex curve sufficiently C^2-close to α generates a billiard with an elliptic 2-periodic orbit [Dias Carneiro et al. 03]. So, a large class of strictly convex billiards has elliptic 2-periodic orbits. The question is: are they stable?

In what follows, we present two classes of billiards (locally circular and symmetric) exhibiting stable 2-periodic orbits.

4.1 Locally Circular Billiards

Our first and simplest example is a 2-periodic orbit between two circles. More precisely, let α be a plane, strictly convex, closed curve parameterized by the arclength s, with the following properties:

- there are two points located by $s = 0$ and $s = s_1$ such that $\alpha'(0) = -\alpha'(s_1)$ and $\alpha(0) - \alpha(s_1) = -L\vec{\eta}(0)$, where $\vec{\eta}(0)$ is the unitary normal vector at 0.
- α is locally a circle, both near $s = 0$ and $s = s_1$, with radii R_0 and R_1 respectively.
- L, R_0, and R_1 verify $L - R_0 - R_1 < 0$, $(L-R_0)(L-R_1) > 0$, and $4(L-R_0)(L-R_1) \neq R_0R_1$, $2R_0R_1$, which are open conditions on the (L, R_0, R_1)-space.

With these properties, $\{(0,0),(s_1,0)\}$ is a nonresonant elliptic 2-periodic orbit for the billiard map T associated to α.

As α is locally circles, T is locally analytic and the first Birkhoff coefficient of the elliptic orbit can be calculated. Moreover, R' and R'' vanish at $s = 0$ and $s = s_1$. So,

$$\tau_1 = -\frac{1}{8}\left(\frac{1}{R_0} + \frac{1}{R_1}\right) \neq 0$$

and this billiard has a stable 2-periodic orbit.

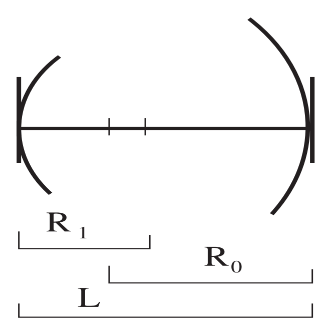

FIGURE 3. A 2-periodic orbit in a locally circular billiard.

Although extremely simple, this example shows that exchanging the curve α with the osculating circles at the impact points gives information about ellipticity, but not about stability, since τ_1 depends on the derivatives of the radius of curvature.

4.2 Ovals with a Special Symmetry

Let R be a periodic C^4 function with Fourier expansion

$$R(\varphi) = a_0 + \sum_{n=1} a_n \cos 2n\varphi$$

with $a_n > 0$ and $a_0 > \sum_{n=1} a_n$, implying that $R(\varphi) > 0, \forall\varphi$.

Let α be a curve, having R as its radius of curvature, given by

$$\begin{aligned}\alpha(\varphi) &= (x(\varphi), y(\varphi)) \\ &= \left(\int_0^{\varphi} R(\beta)\cos\beta d\beta, \int_0^{\varphi} R(\beta)\sin\beta d\beta\right).\end{aligned}$$

It is a regular, closed, and strictly convex C^5 curve.

Since R is an even function, $x(-\varphi) = -x(\varphi)$, $y(-\varphi) = y(\varphi)$, and $\overline{\alpha(0)\alpha(\pi)}$ is an axis of symmetry for α, and then $\{(0,0),(\pi,0)\}$ is a 2-periodic orbit for the associated billiard map.

We have

$$\begin{aligned}L &= ||\alpha(\pi) - \alpha(0)|| = y(\pi) \\ &= 2\,a_0 - \sum_{n=1} \frac{2\,a_n}{(2n+1)(2n-1)} \\ R(0) &= R(\pi) = R_0 = a_0 + \sum_{n=1} a_n\end{aligned}$$

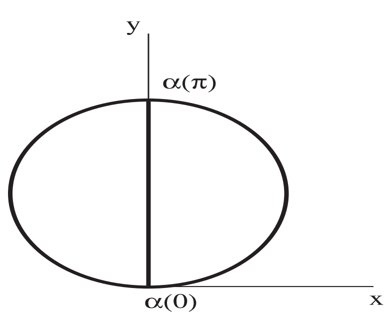

FIGURE 4. A 2-periodic orbit in a symmetric oval billiard.

and then

$$\begin{aligned}L - R(0) - R(\pi) &= L - 2R_0 \\ &= -2\sum_{n=1} a_n \left(\frac{1}{(2n+1)(2n-1)} + 1\right) \\ &< 0 \\ L - R(0) = L - R(\pi) &= L - R_0 \\ &= a_0 - \sum_{n=1} a_n \left(\frac{2}{(2n+1)(2n-1)} + 1\right).\end{aligned}$$

If the open conditions

$$(2+k)\,a_0 - \sum_{n=1} a_n \left[\frac{4}{(2n+1)(2n-1)} + (2-k)\right] \neq 0$$

hold for $k = 0, \pm1, \pm\sqrt{2}$, then $\{(0,0),(\pi,0)\}$ is elliptic and nonresonant.

Let $s = s(\varphi)$ be the arclength parameter for α. Choosing $s(0) = 0$ and $s(\pi) = s_1$ we have

$$\begin{aligned}\left.\frac{dR}{ds}\right|_{s=0} &= \left.\frac{1}{R_0}\frac{dR}{d\varphi}\right|_{\varphi=0} = 0, \\ \left.\frac{dR}{ds}\right|_{s=s_1} &= \left.\frac{1}{R_0}\frac{dR}{d\varphi}\right|_{\varphi=\pi} = 0\end{aligned}$$

$$\begin{aligned}\left.\frac{d^2R}{ds^2}\right|_{s=0} &= \left.\frac{1}{R_0^2}\frac{d^2R}{d\varphi^2}\right|_{\varphi=0} = -4\sum_{n=1} n^2 a_n < 0, \\ \left.\frac{d^2R}{ds^2}(s_1)\right|_{s=s_1} &= \left.\frac{1}{R_0^2}\frac{d^2R}{d\varphi^2}\right|_{\varphi=\pi} = -4\sum_{n=1} n^2 a_n < 0\end{aligned}$$

and the first Birkhoff coefficient is

$$\tau_1 = -\frac{1}{4R_0}\left(1 + \frac{L}{R_0(L-2R_0)}\frac{d^2R}{d\varphi^2}(0)\right) < 0.$$

So the 2-periodic orbit is stable.

In particular, this class of curves include those studied numerically by Berry in [Berry 81] and defined by $R(\varphi) = 1 + \epsilon\cos 2\varphi$ with $0 < \epsilon < 1$. If $\epsilon \neq \frac{3}{5}, \frac{3}{13}$, or $\frac{3}{41}(13 - 8\sqrt{2})$, the conditions for nonresonant ellipticity are fulfilled and the 2-periodic orbit is stable.

It is claimed in [Berry 81] that when $\epsilon = \frac{3}{5}$ (meaning parabolicity of the 2-periodic orbit) there is neutral stability. There is no specific observations for the other two values of ϵ. It would be interesting to investigate the behaviour of this and other examples at resonances.

5. FINAL REMARKS

We choose to restrict ourselves to the calculation of the first Birkhoff coefficient (τ_1) for 2-periodic elliptic and nonresonant orbits. We do not address the calculation of higher-order coefficients nor the case of orbits with larger periods.

This choice has some advantages and, of course, some limitations. Initially, it was motivated by the search for generic properties of strictly convex billiard maps. In fact, as shown in [Dias Carneiro et al. 03], knowledge of the explicit form of the first Birkhoff coefficient of 2-periodic elliptic orbits allows the proof that any C^5 strictly convex billiard map with an elliptic 2-periodic orbit is approached by billiards with a nonresonant 2-periodic elliptic orbit with $\tau_1 \neq 0$ and so with islands. However, to guarantee the existence of islands for one specific orbit, higher-order Birkhoff coefficients must be taken into account when the first one is zero. This will ask for higher-order Taylor coefficients for the iterations of the billiard map which, by increasing the recurrence level, increases the length and complication of the computations. Our program may not be very efficient in this case.

On the other hand, the analysis of k-periodic orbits is a natural and important question. Even though Moser's twist theorem applies to any period, some practical problems appear. First of all, one must localize the orbit, i.e., find its s and p parameters. For a 2-periodic orbit this is easy, since $p = 0$ at any point of the orbit. Rychlik in [Rychlik 89] gives a geometric way to handle the 3-periodic case, but it is not clear that his method can be generalized to larger periods.

Once localized, the conditions of ellipticity and nonresonance of the orbit must be fulfilled, which is feasible. Then, for the calculation of τ_1, it will be necessary to calculate the Taylor expansion, up to order 3, of as many iterates of the billiard map as the period in question. Although our approach could be employed for any given period, even at a very high computational cost, the general case is out of our scope.

So, to generalize our work more sophisticated or specifically designed software tools may be needed. Maybe, the tools proposed by Rychlik in [Rychlik 00] will fit this purpose.

At this point, it is important to note that in the calculation of τ_1 for 2-periodic orbits many handmade simplifications and cancellations were done in order to write an understandable and significant formula. The skills involved with performing this task are usually unavailable, even in sophisticated mathematical software.

Nevertheless, the simple case presented here has the advantage of being accessible by nonexperts, while still being of great interest and importance in the investigation of nonintegrable billiards.

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