Fibred and Virtually Fibred Hyperbolic 3-Manifolds in the Censuses

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Continuing the work of Dunfield, we determine the fibred status of all the unknown hyperbolic 3-manifolds in the cusped census. We then find all the fibred hyperbolic 3-manifolds in the closed census and use this to find over 100 examples each of closed and cusped nonfibred virtually fibred census 3-manifolds, including the Weeks manifold. We also show that the corank of the fundamental group of every 3-manifold in the cusped and in the closed census is 0 or 1.

1. INTRODUCTION

A famous open question by Thurston asks if every finite volume hyperbolic 3-manifold is virtually fibred, that is, it has a finite cover that is fibred over the circle. A finite volume hyperbolic 3-manifold (which we assume throughout to be orientable) is either closed or is the interior of a compact 3-manifold with boundary a finite union of tori, which we call the cusps. Let us treat this as two separate questions, one about closed and one about cusped 3-manifolds. A reason put forward (for instance in [Kapovich 01], [Lackenby 02]) as to why this question may not be true is that there are very few known examples of nonfibred hyperbolic 3-manifolds that are virtually fibred. However, we have data available in the form of the Callahan-Hildebrand-Weeks census of nearly 5,000 cusped hyperbolic 3-manifolds and the Hodgson-Weeks census of nearly 11,000 closed hyperbolic 3-manifolds which should make a good testing ground. Computer programs run by Dunfield [Dunfield 03] show that over 87% of the 3-manifolds in the cusped list are fibred, suggesting that nonfibred virtually fibred cusped hyperbolic 3-manifolds are not so easy to come by because fibred examples are so common.

This of course would not apply to closed 3-manifolds M, since, if M has finite homology then it is not fibred, and this is the case for nearly all 3-manifolds in the closed census (although recently [Dunfield and Thurston 03a] showed with a mammoth computation that they all have a finite cover with positive first Betti number). In this paper, we will find over 100 examples in the closed census of nonfibred virtually fibred 3-manifolds, including ten from the 30 with smallest volume. All these examples are arithmetic and the first is the Weeks manifold, which is the one of minimum volume in the census and conjectured to be the minimum volume hyperbolic 3-manifold overall. Also, one of the nonfibred virtually fibred examples has positive first Betti number, which is the first known case of such a closed 3-manifold.

In order to do this we determine the fibred 3-manifolds in the cusped and closed censuses. Our starting point is Dunfield's list [Dunfield 03] which used two programs to work out the fibred and nonfibred 3-manifolds in the cusped census, with 169 exceptions which were left as unknown. We find the fibred status of all of these unknowns: in fact five are fibred and 164 are not. After this we examine the 128 3-manifolds with positive first Betti number in the closed census and prove that 87 are fibred and 41 are not, thus providing the complete list of closed fibred 3-manifolds in the census. We then utilise the data given in the program Snap and recent work of Goodman, Heard, and Hodgson [Goodman et al. 03], to find other hyperbolic 3-manifolds which are commensurable with these fibred ones, so are virtually fibred.

All of our techniques only require knowledge of the fundamental group of the 3-manifolds, as we can utilise a result [Stallings 61] of Stallings. In particular, we can apply the Bieri-Neumann-Strebel (BNS) invariant and the Alexander polynomial to these fundamental groups. In Section 2, we give a brief description of the BNS invariant and demonstrate how it can sometimes be used to determine the fibred status of a hyperbolic 3-manifold, using a result of K. S. Brown [Brown 87]. We summarise the Alexander polynomial in Section 3.

In Section 4, we examine the unknown cusped 3-manifolds, by first applying the BNS invariant and the Alexander polynomial and then working directly with the fundamental group. Next, in Section 5, we use this information and the knowledge of commensurability classes of cusped hyperbolic 3-manifolds to find nonfibred virtually fibred cusped hyperbolic 3-manifolds. In Section 6, we obtain closed census fibred hyperbolic 3-manifolds from cusped ones. We do not quite pick up all closed fibred 3-manifolds from the census in this way, so then we use the Alexander polynomial to demonstrate that most of the remaining 3-manifolds in the closed census with positive first Betti number are not fibred, with those that remain shown to be fibred directly, using finite covers. In Section 7, we then obtain closed nonfibred virtually fibred hyperbolic 3-manifolds which are all arithmetic.

The corank of a finitely generated group is the largest integer n for which the group has a homomorphism onto the free group of rank n. To finish we quickly show, in Section 8, that all closed and cusped census 3-manifolds have corank 0 or 1.

In Section 9, we describe five tables: the first has the Alexander polynomials of the unknown cusped census 3-manifolds and the second gives cusped nonfibred virtually fibred hyperbolic census 3-manifolds. The third displays all the closed fibred census 3-manifolds. Table 4 lists all remaining closed census 3-manifolds with positive first Betti number, so these are exactly the nonfibred 3-manifolds in the closed census with positive Betti number, and Table 5 contains the closed nonfibred virtually fibred census 3-manifolds that we found.

We take as our input data, the two censuses which come with SnapPea, the related data in Snap and with [Goodman et al. 03], the presentations of fundamental groups from SnapPea as given in [Dunfield and Thurston 03b], and the list [Dunfield 03] of fibred 3-manifolds in the cusped census. From then on, we only work with a fundamental group presentation and operate either by hand or by using a program that can determine, and provide presentations for, all subgroups of a given small index of a finitely presented group, such as Magma or GAP.

2. THE BIERI-NEUMANN-STREBEL INVARIANT

If G is a finitely generated group with G' the commutator subgroup then let $\beta_1(G)$ be the first Betti number of G, that is, the number of free summands in the abelianisation $\overline{G} = G/G'$. Assuming that $b = \beta_1(G) > 0$, there exist homomorphisms of G onto $\mathbb{Z}$ and the BNS invariant gives us information on when their kernels are finitely generated. This is done in [Bieri et al. 87] by identifying nonzero homomorphisms of G into $\mathbb{R}$, up to multiplication by a positive constant, with the sphere S^{b-1}. The BNS invariant of G is an open subset Σ of S^{b-1}, with a homomorphism χ of G onto $\mathbb{Z}$ having finitely generated kernel if and only if χ is in both Σ and $-\Sigma$. If $G = \pi_1 M$ for M the fundamental group of a compact 3-manifold then it is shown that $\Sigma = -\Sigma$. In general it can be difficult to find Σ but in a paper by K. S. Brown [Brown 87], an algorithm is given to determine whether or not χ is in Σ in the case where G is a 1-relator group. If G has at least three generators then $\Sigma = \emptyset$ so the interesting case is when we have a 2-generator, 1-relator group. But compact orientable irreducible 3-manifolds with nonempty toroidal boundary always have a presentation with one less relator than the number of generators and in the cusped census of 3-manifolds many (over 4000 out of 4815) have 2-generator 1-relator fundamental groups.

The connection with fibred 3-manifolds dates back to a theorem of Stallings [Stallings 61] which states that if M is compact, orientable, and irreducible with $\pi_1 M$ possessing a surjection to $\mathbb{Z}$ with finitely generated kernel then M is fibred over the circle with the kernel being the fundamental group of the fibre. Conversely, if M is compact, orientable, and fibred then of course $\pi_1 M$ has this property and M will be irreducible except for $S^2 \times S^1$. In fact, as [Hempel 76, Chapter 11] makes clear, if irreducibility is removed from the hypothesis of Stallings' result then the conclusion still holds provided that M has no sphere boundary components (which we could cap off) and no fake 3-cells (for which we could invoke the Poincaré conjecture). In any case, we are interested in hyperbolic 3-manifolds and these are always irreducible.

Thus the Brown algorithm will determine whether or not most 3-manifolds are in the cusped census fibre. This is what Dunfield did, using a computer program to work through the 3-manifolds M that came with such a presentation and with $\beta_1(M) = 1$. The efficiency of the algorithm can be judged by the fact that the total running time was about a minute. We outline how it works: assume that $G = \langle a, b | r(a, b) \rangle$ with r reduced and cyclically reduced. First, suppose $\beta_1(G) = 1$ so that there is one homomorphism χ from G onto $\mathbb{Z}$ (up to sign), with $\chi(a) = m$ and $\chi(b) = n$ (where m and n can instantly be found by abelianising). Assume first that $m, n \neq 0$, then we work through the relation, drawing a path which starts at height 0 and rises or falls according to the value under χ of each successive letter in r. When we finish, we must again be at height 0 and we regard this as being back at the starting point, having gone round in a circle. Then χ has finitely generated kernel if and only if the path reaches both its maximum and its minimum only once.

However one generator, say a, could have zero exponent sum which happens if and only if $\chi(b) = 0$, and then the criterion is slightly different: after all there cannot now be a unique maximum. However, in practice, this case turns out to be easier to work with, so we will make a definition: let us say throughout, that a presentation of a group $\Gamma = \langle g_1, \ldots, g_m | r_1, \ldots, r_k \rangle$ with $\beta_1(\Gamma) = b \leq m$ is in standard form with respect to $g_1, \ldots, g_b$ if each of these has zero exponent sum in each relation r_i. Then these elements generate the infinite part of $\overline{\Gamma}$ with all other generators being of finite order in $\overline{\Gamma}$. Now if $G = \langle a, b | r \rangle$ is in standard form, we have that ker χ is finitely generated if and only if the maximum and minimum occur twice, which will be either end of a single flat path.

Given a compact orientable irreducible 3-manifold with n cusps, we have by using the Mayer-Vietoris sequence that $\beta_1(M) \geq n$ so that this process can only work on 1-cusped 3-manifolds. But now suppose that our 2-generator 1-relator group G has $\beta_1(G) = 2$. Then there are an infinite number of homomorphisms from G onto $\mathbb{Z}$ and here Brown's algorithm works in the following way. We draw the (reduced and cyclically reduced) relation on a two-dimensional grid, and as it has zero exponential sum in both a and b we finish at the origin. We then consider the convex hull C in $\mathbb{R}^2$ of this path and regard a homomorphism from G onto $\mathbb{Z}$ as a directional vector, with slope n/m for $\chi(a) = m, \chi(b) = n$. Then the homomorphisms with finitely generated kernel are those with slope lying between (but not including) the slope of the outward pointing normals of two successive edges of C, provided that the joining vertex, which will be a vertex of the path, has only been passed through once when the path has been traced out; as well as the vertical homomorphism if and only if C has a horizontal side of length 1 at the top that is passed through only once; and similarly for the horizontal homomorphism. In fact, a homomorphism is really represented by two vectors with the same slope, pointing in opposite directions, and both of these must satisfy the above conditions, but for a 3-manifold group the conditions on each of the two vectors will be true or false together because C has rotational symmetry of order 2.

2.1 Example

Let us demonstrate this process. We look for 1-cusped 3-manifolds in the census with $\beta_1(M) > 1$ so that we have a variety of homomorphisms to work with. The orientable cusped census is divided into three parts, with those hyperbolic 3-manifolds having a decomposition into five or less ideal tetrahedra being numbered in order of increasing volume and with the prefix **m**. Then those cusped 3-manifolds with a decomposition of six and seven ideal tetrahedra are ordered similarly with the prefix **s** and **v** respectively. (The m list includes nonorientable cusped hyperbolic 3-manifolds with the appropriate decomposition, whereas the s and v sections consist entirely of orientable 3-manifolds, as two separate lists are made for the nonorientable ones.) We find only s789, v1539, and v3209, all with homology $\mathbb{Z} + \mathbb{Z}$. SnapPea gives a 3-

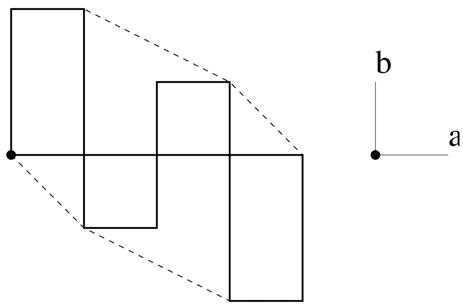

FIGURE 1. Drawing of the relation that forms convex hull C.

generator presentation for the fundamental group of two of them, but we obtain

$$\pi_1(\text{v1539}) = \langle a, b | a^4 B^2 A b^3 A B^2 A b^3 A B^2 \rangle$$

with $(m, l) = (Ab, B^3 a^5 B^2)$ a basis for the fundamental group of the cusp.

This example will be important in Section 6. Drawing out the relation to form the convex hull C as in Figure 1 and using Brown's algorithm reveal that all but the three homomorphisms (ignoring signs) $\chi(a) = 1, \chi(b) = 0$; $\chi(a) = 1, \chi(b) = 1$; and $\chi(a) = 1, \chi(b) = 2$ have finitely generated kernel.

Thus we see that determining the fibred status of cusped hyperbolic 3-manifolds with a 2-generator 1-relator fundamental group presentation presents no problem, but for a closed orientable irreducible 3-manifold M we have that every presentation of $\pi_1 M$ has at least as many generators as relators. Thus it would appear here that Brown's algorithm is now of no use, however we make an obvious yet useful point: suppose we have a 2-generator group $G = \langle a, b | r_1, \ldots, r_m \rangle$ then any 2-generator group Γ of the form $\langle a, b | r \rangle$ where r is one of the r_i (or even just a consequence of $r_1, \ldots, r_m$) surjects onto G. If we have a finitely generated kernel K of a homomorphism χ from Γ onto $\mathbb{Z}$, which can be determined by Brown's algorithm, then the image of K in G is still finitely generated, so the only issue is whether χ factors through G and this is easily solved by looking at the abelianisations of Γ and G. In particular, if we have a surjection from any $\Gamma = \pi_1 M$ to any $G = \pi_1 N$ where M and N are both compact orientable irreducible 3-manifolds with $\beta_1(N) = \beta_1(M)$ then M is fibred implies that N is too.

An obvious method to obtain fundamental group surjections from 3-manifolds to other 3-manifolds is through the use of Dehn surgery, where we attach a solid torus to a component of the boundary of a cusped 3-manifold M. If the cusp has generators m and l in $\pi_1 M$ then (p, q) Dehn filling for coprime integers p, q with $q \geq 0$ means that we attach the curve $m^p l^q$ to the compressible curve in the solid torus, thus adding this relation to $\pi_1 M$ and reducing the number of cusps by one. If we start with a 1-cusped hyperbolic 3-manifold M with $\beta_1(M) = 1$ then there will be a unique Dehn surgery forming a closed 3-manifold N with $\beta_1(N) = 1$ (we might call this curve the longitude, in analogy with a knot in S^3 where this is the only simple closed curve on the boundary homologous to 0) and thus if M is fibred and N is irreducible then N is also fibred, as the relevant homomorphism $\chi : \pi_1 M \to \mathbb{Z}$ factors through N. In fact, here we do not need to know that N is irreducible, as seen by picturing this geometrically, because we are just performing Dehn filling along the boundary slope of the fibre of M. This observation will be used in Section 6, but to conclude this section let us apply this to our example $M = \text{v1539}$. Performing (p, q) Dehn surgery with the above basis for the cusp means that the only homomorphism χ that factors through $\pi_1 N$ is $\chi(a) = \chi(b) = 1$ (unless $(p, q) = (5, 1)$ in which case they all do) which is one of the three exceptional homomorphisms so this does not tell us that N is fibred. However we can use the Dehn filling relation instead to give us Theorem 2.1.

Theorem 2.1. *There exist infinitely many closed hyperbolic fibred 2-generator 3-manifolds with bounded volume.*

Proof: We take v1539(p,1) and consider $\Gamma = \langle a, b | m^p l \rangle$ which surjects onto its fundamental group, with $\beta_1(\Gamma)$ also equal to 1 if $p \neq 5$. Taking the homomorphism $\chi(a) = \chi(b) = 1$, we draw out the relation as in Figure 2, where we have cancellation along the dotted lines if $p > 0$ but we still have a unique maximum and minimum, hence a finitely generated kernel. We then apply Thurston's Dehn surgery theorem to obtain hyperbolicity, hence irreducibility which gives us the fibred property, along with the fact that these closed 3-manifolds have volume accumulating to that of v1539. □

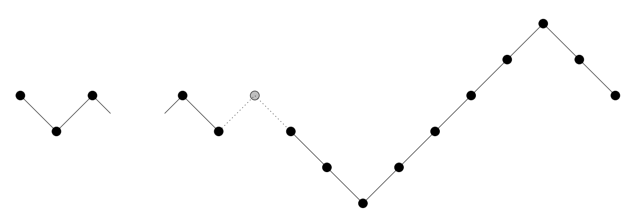

FIGURE 2. Drawing of the relation described in Theorem 2.1.

3. THE ALEXANDER POLYNOMIAL

Historically, the Alexander polynomial was first introduced for knots in S^3 but it can be defined for any finitely presented group. Although it is not able to give us so much information as the BNS invariant, it has the advantage that it is straightforward to work out from any finite presentation of a group using Fox's free differential calculus. Therefore, we give a brief description adopting the approach of Fox in [Crowell and Fox 63].

Let the finitely presented group G be $\langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$ in terms of generators and relators, and let its free abelianisation be $ab(G)$, which will be isomorphic to $\mathbb{Z}^b$ where $b = \beta_1(G)$. If F_n is the free group of rank n with free basis $x_1, \ldots, x_n$ then a derivation of the integral group ring $\mathbb{Z}[F_n]$ is a map from $\mathbb{Z}[F_n]$ to itself satisfying

$$\begin{aligned} D(v_1 + v_2) &= Dv_1 + Dv_2, \\ D(v_1 v_2) &= (Dv_1)\tau(v_2) + v_1 Dv_2 \end{aligned}$$

where τ is the trivialiser: namely the ring homomorphism from $\mathbb{Z}[F_n]$ to $\mathbb{Z}$ with $\tau(x) = 1$ for all $x \in F_n$. It is a fact that for each free generator x_j there exists a unique derivation D_j, also written $\partial/\partial x_j$, such that $\partial x_i/\partial x_j = \delta_{ij}$. To calculate the "partial derivative" $\partial w/\partial x_j$ for any $w \in F_n$ we can use the formal rules

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \qquad \frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1},$$
$$\frac{\partial (w_1 w_2)}{\partial x_j} = \frac{\partial w_1}{\partial x_j} + w_1 \frac{\partial w_2}{\partial x_j},$$

where generally w_2 will be the last letter in the word $w = w_1 w_2$. Let γ be the natural map from $\mathbb{Z}[F_n]$ to $\mathbb{Z}[G]$ and let α be the same from $\mathbb{Z}[G]$ to $\mathbb{Z}[ab(G)]$. Then the Alexander matrix A of the presentation is the $m \times n$ matrix with entries

$$a_{ij} = \alpha\gamma \left(\frac{\partial r_i}{\partial x_j} \right).$$

We define the kth elementary ideal $E_k(A)$ to be the ideal of $\mathbb{Z}[ab(G)]$ generated by the $(n-k) \times (n-k)$ minors of A if $0 < n - k \leq m$, thus, under this notation, k is the number of columns that are deleted in forming the minors. Finally, we define the Alexander polynomial Δ_G to be the generator (up to units) of the smallest principal ideal containing $E_1(A)$. To calculate it we can choose a basis $(t_1, \ldots, t_b)$ for $ab(G)$, apply the free differential calculus as above and then form our matrix by evaluating. From here we can determine the minors and their highest common factor. Of course this would be of little use if it depended on the presentation of G, but that it is invariant can be seen directly, as shown in [Crowell and Fox 63, VII 4.5], by observing that applying a Tietze transformation to a presentation does not change the elementary ideals. Alternatively we have a topological definition of the Alexander polynomial, as described in [McMullen 02, Section 2] or [Dunfield 01, Section 3]: if X is a finite CW-complex with $\pi_1 X = G$ and $f : \tilde{X} \to X$ is the regular cover corresponding to the homomorphism α from G to $ab(G)$ then, taking $p \in X$, the Alexander module of X over the group ring $\mathbb{Z}[\text{ab}(G)]$ is $H_1(\tilde{X}, f^{-1}(p); \mathbb{Z})$. The connection between the two approaches is that by taking a free resolution of this module, we obtain the Alexander matrix as above (or rather under our notation it is the transpose of A). The Alexander polynomial Δ_G is only defined up to units, thus we can think of Δ_G as a Laurent polynomial in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_b^{\pm 1}]$ up to multiplication by $\pm t_1^{k_1} \cdots t_b^{k_b}$. Of course the actual coefficients depend on this basis. Sometimes there will be a natural choice, such as for a b-component link in S^3 where we would take meridians about each link. However we might not in general have this luxury, although we can always make a change of basis if necessary by putting $t_i = s_1^{k_{i1}} \cdots s_b^{k_{ib}}$ with the vectors $(k_{i1}, \ldots, k_{ib})$ making up an element of $GL(b, \mathbb{Z})$.

The utility of the Alexander polynomial for us here is the well known result, derived later, that if we have a compact 3-manifold M with $\beta_1(M) = 1$ then its Alexander polynomial $\Delta_M(t)$, in this case a Laurent polynomial defined up to units and with $\Delta_M(1/t)$ equal to $\Delta_M(t)$ times a unit, is monic if M is fibred. We also have by Dunfield a suitable generalisation of this for the case $\beta_1(M) \geq 2$ which we will use later. Theorem 5.1 of [Dunfield 01] states that if the Alexander polynomial Δ_M has no terms with coefficients that are ± 1 then M is not fibred. More precisely, let N be the Newton polytope of Δ_M, that is the convex hull in $\mathbb{R}^b$ of the points $(k_1, \ldots, k_b)$ where $t_1^{k_1} \cdots t_b^{k_b}$ is a (nontrivial) term of Δ_M. If none of the vertices of N have coefficient ± 1 in Δ_M then the Bieri-Neumann-Strebel invariant Σ of $\pi_1 M$ is empty and so there are no homomorphisms onto $\mathbb{Z}$ with finitely generated kernel.

4. THE UNKNOWN CUSPED 3-MANIFOLDS

When Dunfield ran his programs on the 4,815 3-manifolds in the cusped census to see which were fibred, he first set up the computer to apply Brown's algorithm to any 3-manifold M with a 2-generator 1-relator presentation and with $\beta_1(M) = 1$. As we have seen in Section 2, this is guaranteed to terminate and give a definite yes/no answer. The program took about a minute in total to complete the 4,105 examples given to it, 3,653 of which were fibred and 452 of which were not.

The other algorithm that was applied was Lackenby's idea of taut ideal triangulations. We will not be using this because our emphasis is on methods that only require knowledge of the fundamental group; we note only that this process will not tell us that the 3-manifold is nonfibred but it has no restriction as above on the number of generators or relators. When this was applied to the cusped census it produced 541 further fibred 3-manifolds, as well as confirming a lot of the 3-manifolds already known to be fibred by Brown's algorithm. There were some 3-manifolds that it did not work for, and the running time was a lot longer.

Thus this leaves 169 cusped 3-manifolds whose status is unknown. In this section we will determine whether or not these are fibred. As any unknown 3-manifold has already passed through the two algorithms above, we proceed by a variety of listed methods involving fewer and fewer 3-manifolds. We work on the assumption that they are most likely to be nonfibred, because a fibred 3-manifold has had two chances already to be detected, and then only at the very end do we admit the possibility that what remains might be fibred.

4.1 Use Other Data

In [Callahan et al. 99], all knots in S^3 appearing in the m or s part of the census are determined and listed, helpfully with the genus of their fibre or an x if they are nonfibred. We might as well annotate Dunfield's list to provide a fuller description of such 1-cusped 3-manifolds. We find ourselves marking an unknown 3-manifold on 3 occasions: m372 is the nonfibred knot 9_{46} in the Alexander-Briggs/Rolfsen-Bailey tables (for alternative names we have 3,3,2 1- in Conway notation or $9n5$ in the Dowker-Thistlethwaite ordering used in Knotscape, where n denotes a nonalternating knot), s879 is a nonfibred knot with 11 crossings (equivalently 5,3,2 1- or $11n139$), and s704 is the fibred knot 10_{140} (equivalently 4,3,2 1- or $10n29$) with genus 2. (This last one is somewhat lucky—very few of the remaining 3-manifolds are fibred.)

4.2 Any Other 2-Generator Groups?

In the course of our study, we found one 3-manifold M with a 2-generator 1-relator presentation and with $\beta_1(M)=1$ which was listed as unknown. This is v3036 with presentation

$$a^3b^3AbAb^3a^3b^3AbAb^4AbAb^3$$

which we see is in standard form with respect to a. On applying Brown's algorithm, we reach the top after the middle a^3 term whence we have b^3, so this is not fibred.

We also find two 2-generator 1-relator 3-manifolds M with $\beta_1(M)=2$ and with status unknown, for which we can use the extended version of Brown's algorithm. We can quickly check that these are the only unknowns of this form because the cusped census collects 3-manifolds with the same number of cusps together. But $\beta_1(M)$ is at least the number of cusps and we know that there are only three cases where M has one cusp but $\beta_1(M)=2$, with these listed as fibred. Therefore we work down the table of 2-cusped 3-manifolds, all of which happen to have $\beta_1(M)=2$, and look them up in Dunfield's list. We know that either they will be proved fibred using taut foliations or they will be unknown. In fact we find that it is the former in all but four cases: v2943, v3379, v3384, v3396. The last two have homology $\mathbb{Z}_5+\mathbb{Z}+\mathbb{Z}$ and $\mathbb{Z}_3+\mathbb{Z}+\mathbb{Z}$ respectively so are not 2 generator, but we find

$$\pi_1(\text{v2943}) = \langle a,b| \; abAB^2AbaBAba^3bABab^2aBAbaBA^3B\rangle,$$

$$\pi_1(\text{v3379}) = \langle a,b| \; abABa^3BAbaBAbaB^2abABabA^3baBAbaBAb^2AB\rangle,$$

neither of which are fibred, seen by drawing out the relation and noting that all vertices of the convex hull are passed through more than once.

Moreover there are only three 3-cusped 3-manifolds M, all of which are fibred and have $\beta_1(M)=3$, and none at all with more than three cusps. This now leaves only 1-cusped 3-manifolds, and v3384 and v3396.

4.3 The Alexander Polynomial

We now turn to the the original suggestion of Dunfield of calculating Alexander polynomials. Once some practice is gained, the process becomes much faster so we might as well apply it to all the remaining unknowns. Let us first assume that M is a 1-cusped 3-manifold with $\beta_1(M)=1$. As mentioned in Section 3, upon taking t as a generator (by symmetry it does not matter which one) for $ab(\pi_1 M)$ we have that the Alexander polynomial of M is an element of the ring $\mathbb{Z}[t,t^{-1}]$, up to units which are $t^{\pm k}$ for $k\in\mathbb{Z}$.

In the process of calculating the polynomial, we found it quickest to make substitutions so that we always have a presentation for $\pi_1 M$ which is in standard form with respect to one of the generators, say x. Then it is seen that $\partial r_i/\partial x = 0$ upon evaluation for each of the relations r_i: first note that $\alpha(g_j) = 1$ for all the other generators g_j of our presentation. Thus whenever we have an x appearing in r_i it contributes a term which is (upon evaluation) t^k, where k is the exponent sum of x in the subword of r_i strictly to the left of this appearance of x, whereas an X contributes $-t^k$ where k is the exponent sum of x in the subword to the left of and including X. The result then follows by pairing off each x and the X with which it cancels when all other g_j are set to the identity. A special case of a presentation in standard form is when each relator has only one appearance of x, which we refer to as simple form with respect to x, so we get

$$r_i = x u_i X v_i \qquad \text{and} \qquad \frac{\partial r_i}{\partial g_j} = k_{ij} t + l_{ij}, \tag{4–1}$$

where u_i, v_i contain no appearance of x and X, with k_{ij} the exponent sum of g_j in u_i and l_{ij} that of g_j in v_i. In particular, if M is fibred over the circle with fibre the surface S, so that $\pi_1 S$ is free of rank n, then we can take a presentation for $\pi_1 M$ of the form $\langle g_1, \ldots, g_n, x | r_1, \ldots, r_n \rangle$, where $r_i = x g_i X v_i$. Thus $\partial r_i/\partial g_j = \delta_{ij} t + l_{ij}$ so that the Alexander polynomial is the characteristic polynomial of the $n \times n$ monodromy matrix $-l_{ij}$ induced by the gluing map, and hence is monic with degree n. Thus we look for nonmonic Alexander polynomials in our calculations and conclude that these 3-manifolds are nonfibred.

In fact, in the case of a 2-generator 1-relator group G with $\beta_1(G) = 1$, there is a straightforward connection between Brown's algorithm and the Alexander polynomial Δ_G: the way to see this is to assume that $G = \langle a, b | r \rangle$ is in standard form with respect to a and then once the relation is drawn out we note that the process given of calculating Δ_G is merely that of counting the appearance of bs (which contribute $+1$) and Bs (-1) in the relation at each level, and these values are the coefficients of Δ_G. In particular, we obtain a very visual insight into how a 2-generator 1-relator knot could have monic Alexander polynomial but not be fibred: the relation must reach its peak more than once and all but one of them must cancel out. Another example is that we can easily recognise 1-punctured torus bundles amongst hyperbolic 3-manifolds with 2-generator 1-relator fundamental groups: if $\pi_1 M = \langle a, b | r \rangle$, with r reduced and cyclically reduced, is the fundamental group of a hyperbolic 3-manifold M then M is a 1-punctured torus bundle if and only if $\beta_1(M) = 1$ and the relation lies on only three levels with a unique maximum and minimum when drawn out in standard form. This is because hyperbolic 1-punctured torus bundles M must have $\beta_1(M) = 1$ and the other condition is exactly what is needed to conclude that M fibres with Alexander polynomial of degree 2, thus the fibre must be a 1-punctured torus or a 3-punctured sphere, but the bundle is not hyperbolic in the latter case. Now 1-punctured torus bundles might need three generators, as seen by looking at their homology, but we cannot conclude in general that a hyperbolic 3-manifold M is a 1-punctured torus bundle if it has a monic quadratic Alexander polynomial. However, if we already know that M is fibred then we can.

Returning to the unknown cusped 3-manifolds M, all our calculations are on 3-generator 2-relator groups so that we put $\pi_1 M = \langle g_1, g_2, x | r_1, r_2 \rangle$ into standard form with respect to x and then we calculate the determinant of the 2×2 matrix $\partial r_i/\partial g_j$. If, furthermore, our two relations are in simple form with respect to x, that is as in Equation (4–1) (which happens often) then we can take a shortcut, since the Alexander polynomial will be (at most) quadratic. We calculate $\det(k_{ij})$ which will be the coefficient of t^2, and then $\det(l_{ij})$ which is the constant. These must be equal which acts as a useful check, given that we are doing these by hand (and are here not interested in the middle term). More generally, we ensure that our result is a Laurent polynomial that is symmetric under $t \mapsto t^{-1}$. The results are listed in Table 1 with only six of these unknown 3-manifolds, written in bold, having a monic Alexander polynomial. We can draw definite conclusions for two of them: recall from Section 4.1 that s704 is a fibred knot, whereas v2530 with Alexander polynomial $t+1$ of degree 1 cannot be fibred because the fibre subgroup would have to be cyclic.

We can see from the table that some properties of the Alexander polynomial of a knot are no longer true in this wider setting: for instance we no longer have $|\Delta_M(1)| = 1$. In fact, we can see from our method of calculation of Δ_M on a presentation in standard form that for $t = 1$ we are just forming the equations of the exponent sums of those generators (all but one) which have finite order in homology, so Δ_M is never zero because $|\Delta_M(1)|$ is always the order of the finite part of the homology. (As this was not known to us when first compiling the table, it provided another useful check.) We can even have a common factor of all the coefficients, as in $\Delta_{s773}(t) = 2(t^3 + t^2 + t + 1)$. Moreover, this example shows that Alexander polynomials are not necessarily of even degree as they are for knots; other examples of hyperbolic 3-manifolds with Alexander polynomials having odd degree are if M is fibred by a surface with an even number of boundary components (whereas knots can only be fibred by a surface with one boundary component).

We also need to consider the unknown 2-cusped 3-manifolds v3384 and v3396. Taking the given presentation for $G = \pi_1(\text{v3384})$ and putting it into standard form with respect to (b, c) via the substitution $a = yB^2$ gives us the two relations

$$y^3B^2Cb^2y^2Bcb, \qquad yB^2Cb^2Yc,$$

so the Alexander matrix is (ordering the generators as (b, c, y) and using the images of b, c in $ab(G)$ as a basis, for which we also write b, c)

$$\begin{pmatrix} b^{-2}c^{-1}(1-c) & b^{-2}c^{-1}(b-1) & c^{-1}(2+3c) \\ b^{-2}c^{-1}(1+b)(1-c) & b^{-2}c^{-1}(b-1)(b+1) & c^{-1}(c-1) \end{pmatrix}$$

giving the three minors (up to units)

$$\begin{aligned} m_1 &= -(b-1)(3bc+2b+2c+3) \\ m_2 &= (c-1)(3bc+2b+2c+3) \\ m_3 &= 0, \end{aligned}$$

thus the Alexander polynomial is $3bc + 2b + 2c + 3$. Similarly the given presentation for $\pi_1(\text{v3396})$ is already in standard form with respect to (b, c) so adopting the same notation we find its Alexander polynomial is $2(b-1)(c-1)$. As mentioned at the end of Section 3, this gives us that v3384 and v3396 are not fibred.

4.4 Fibred after All?

We now have to face up to the four remaining unknowns s594, v2869, v3093, v3541, and should take seriously the possibility that they are fibred. If so then we must have a presentation

$$\pi_1 M = \langle t, a_1, \ldots, a_r | t a_i t^{-1} = w_i \rangle, \qquad (4\text{–}2)$$

where each w_i is a word in $a_1, \ldots, a_r$ equal to $\phi_*(a_i)$, for ϕ_* the induced automorphism of $\pi_1 M$ obtained from the gluing homeomorphism ϕ. These words, as well as $a_1, \ldots, a_r$, generate the fibre subgroup F which will be free of rank r equal to the degree of the Alexander polynomial. Such a presentation will need more than the three generators that we have been given for our 3-manifolds, and it might not be easy to move between the two different presentations. However some points are clear: since $\beta_1(M) = 1$, the elements of F are precisely those in $\pi_1 M$ with finite order in homology, and in looking for a candidate for t, any element generating the infinite part of the homology can be used because we can replace t with kt for any $k \in F$, and w_j with kw_jk^{-1} in the presentation above.

In order to get around the number of generators, we use finite covers. If $\pi_1 M$ is fibred then we will have the cyclic covers $\pi_1 M_n$ of degree n, generated by the $r+1$ elements $t^n, a_1, \ldots, a_r$ and with r relations, that correspond to the gluing homeomorphisms ϕ^n. When we ask Magma for a presentation of an index n subgroup of our 3-generator 2-relator group, it employs the Reidermeister-Schreier process which will obtain a presentation of $2n+1$ generators and $2n$ relators, but some of these might be redundant so the output could be less. Therefore, we start with our unknown $\pi_1 M$, using a presentation in standard form with respect to a generator x. We ask Magma for (the generators of) subgroups of index n (it gives a subgroup in each conjugacy class) and pick the cyclic cover H_n, that is the one with the exponent sum of $x \equiv 0 \bmod n$ (which is easy to spot by checking this condition holds for all of the given generators). We then demand a presentation of H_n, hoping not only that it is a $(d+1)$-generator and d-relator presentation for d the degree of the Alexander polynomial, but also that the presentation $\langle h, x_1, \ldots, x_d | r_1, \ldots, r_d \rangle$ is in simple form with respect to the generator $h = x^n$ of H_n. Then we look at the d subwords from h to h^{-1} in each relation and if this is a basis for the free group on $x_1, \ldots, x_d$ we conclude that conjugation by h sends $\langle x_1, \ldots, x_d \rangle$ into itself. If now the subwords appearing from h^{-1} to h are also a basis then $\langle x_1, \ldots, x_d \rangle$ is normal in H_n, with H_n having a presentation exactly as in Equation (4–2) so by Stallings' condition we have a finite cover of M which is fibred, with fibre subgroup $\langle x_1, \ldots, x_d \rangle$.

In fact, we can halve the work, as we need only check that one of the two sets of subwords is a basis. This follows from Proposition 3.1 in K. S. Brown's paper [Brown 87]. Suppose that G is a finitely generated group and $\chi : G \to \mathbb{Z}$ is a surjective homomorphism. To say that a HNN decomposition of G has χ as associated homomorphism means that we can write G as $\langle B, t | B_1 = tB_2t^{-1} \rangle$, for B a subgroup of G and B_1, B_2 subgroups of B, with $\chi(B) = 0, \chi(t) = 1$. Then we use the result that $\chi \in \Sigma$ if and only if every HNN decomposition of G with χ as associated homomorphism is ascending, namely $B_2 = B$. If this is so, then we can further ask whether $\chi \in -\Sigma$, but $-\chi$ is associated with the decomposition of G where B_1 and B_2 are swapped, thus a second yes answer implies that $B_1 = B_2 = B$. However, if G is a 3-manifold group then $\Sigma = -\Sigma$ by [Bieri et al. 87, Corollary F], meaning that one condition is enough.

To move from the fibred cover back to the original 3-manifold we use [Button 04b, Corollary 2.6] which says that if the fibred 3-manifold N is a finite cover of the compact orientable 3-manifold M, so that $\beta_1(N) \geq \beta_1(M)$, then M is fibred if the natural map given by inclusion between the infinite part of the abelianisations $\overline{\pi_1 N}$ to $\overline{\pi_1 M}$ has kernel coming from the fibre subgroup of N. But if $\pi_1 N$ is equal to H_n as above and $x_1, \ldots, x_d$ are elements of finite order in the homology of $\pi_1 M$ (which just means that when expressed as elements of $\pi_1 M$ they have zero exponent sum in t) then, since h has infinite order in $\overline{\pi_1 M}$, we have that the kernel will be generated by $x_1, \ldots, x_d$ (considered as elements of $\overline{\pi_1 N}$) so it will be contained in the fibre subgroup of N. We shall see directly that this condition always holds so we can conclude that M is fibred as well.

Starting with s594, the Alexander polynomial has degree 3 so, using the presentation $\langle a, c, x\rangle$ in standard form with respect to a as obtained from Table 1, we see that the index 2 subgroup H corresponding to the cyclic cover has abelianisation $\mathbb{Z}_2 + \mathbb{Z}_4 + \mathbb{Z} + \mathbb{Z}$, so it is at least a 4-generator group. After rewriting we see that it is generated by $p = x, q = c, r = axa^{-1}, t = a^2$ with relations

$$RQR\mathbf{t}pqp\mathbf{T}, \quad PQP\mathbf{T}qP\mathbf{t}P, \quad QP\mathbf{T}R\mathbf{t}RqP$$

and taking the subwords between T and t we easily see that these generate the free group on p, q, r so the cover is fibred, as is s594. We can detect the fibre by noting that it must have fundamental group free of rank 3, so it is a 4-punctured sphere or a 2-punctured torus. In fact, it must be the latter because the gluing homeomorphism must permute the boundary components and any one that is fixed must be sent to a conjugate of itself in the fundamental group of the fibre under the induced automorphism (it is not sent to its inverse as the map is orientation preserving), thus adding 1 to the Betti number of the 3-manifold. Thus, if we have a 4-punctured sphere for s594 then as it has Betti number 1, the induced permutation must be without fixed points. But we can check that the cyclic cover of degree 4 has Betti number 3, whereas we would need the Betti number 5.

Moving onto v3093, we have $\pi_1(\text{v3093}) = \langle b, x, y\rangle$ in standard form with respect to b and with degree 4 Alexander polynomial. Looking with Magma at the finite index subgroups, the fundamental groups H_n of the cyclic covers of degree 2 and 3 are given with four generators, whereas for degree 4 and 5 we have six generators. Upon rewriting H_2 and H_3, the number of generators cannot increase, so we try the rewriting process for H_4 and H_5 that do then have the required five generators and four relations, with $t = b^n$ appearing as a generator. Unsurprisingly, t appears too many times in the relations for H_4, but luckily we have H_5 with abelianisation $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}$ in simple form with respect to t. Setting $p = x, q = y, r = b^{-1}xb, s = b^{-1}yb$, and $t = b^5$ we have relations

$$rsQPqrp^2q\mathbf{T}qSrsPRs\mathbf{t},$$
$$sP\mathbf{t}QPqrpqrpqr\mathbf{T}SrsPsP,$$
$$Qpq\mathbf{T}pSQpqSrsPRsPq\mathbf{t}RQPR,$$
$$s\mathbf{T}QpqSrsPRsqSrsPRsqSrsPRs\mathbf{t}r^2,$$

and we get the computer to show that the subwords between t and T are a basis, by setting up a homomorphism from the free group $F_4 = \langle p, q, r, s\rangle$ to itself with these as images and asking if it is a surjection. It is. (We later confirmed this by hand, after obtaining practice with similar calculations in Section 6.)

With the two remaining unknowns, v2869 and v3541, their Alexander polynomials have degree 6 and 10, respectively. For v2869 we need a subgroup of at least index 3 to have a hope of seven generators, but the cyclic covers of degree 3,4,5 all fall short. For v3541 we need index at least 5 for 11 generators, but index 5,6,7 all have eight or less generators after rewriting. When trying to list all subgroups of higher index we run into the problem that there are just too many. Instead, we rely on the fact that we have a good idea what the generators of these particular cyclic covers should look like: if our original fundamental group $G = \langle u, v, s\rangle$ is in standard form with respect to s then H_n has a generating set $s^i u s^{-i}, s^j v s^{-j}, s^n$ for various values of i, j, and after guessing such a generating set we can ask for the index of H_n in G to check whether or not we are correct.

Therefore, as we have $\pi_1(\text{v2869}) = \langle x, y, z\rangle$ in standard form with respect to x, we look at the subgroup H generated by $x^i y x^{-i}, x^j z x^{-j}, x^n$ for $i = 0, \pm 1, \pm 2$, $j = 0, \pm 1, -2$, and $n = 6$. We do indeed find that H has index 6 in G with abelianisation $\mathbb{Z}_{13} + \mathbb{Z}$ and after rewriting we get the magic 7-generator 6-relator presentation, with generators

$$(a, b, c, d, e, f, t) = (y, z, xyX, Xzx, x^2yX^2, x^2zX^2, x^6),$$

that is in simple form with respect to t and with the following free basis to be found between T and t:

$$(F^2eBdBaceBabf, Fef,$$
$$FBabFeBAbEceBabf,$$
$$F^2eBdBaeBadBaCAbDbEf^2,$$
$$F^2eBdBacAbDbEFeBdBabFeBAbEcef,$$
$$F^2eBdBacAbDbf).$$

Finally for $\pi_1(\text{v3541}) = \langle x, y, z \rangle$ in standard form with respect to z, we try the subgroups H_n generated by $z^i x z^{-i}, z^j y z^{-j}, z^{-n}$ for $i = 0, \pm 1, \pm 2$, $j = 0, \pm 1, \pm 2$, and with n running from 8 to 15. All have the correct index: for $n = 8, 9, 10$ we get too few generators, but for the other n we get exactly the required 11 generators and ten relations. For $n = 11$ the presentation is in standard but not simple form with respect to $t = z^n$, for the others the presentation is in standard and simple form but with the relations becoming progressively longer, so we take $n = 12$. The subgroup has abelianisation $\mathbb{Z}_7 + \mathbb{Z}_{35} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z}$ with the other ten generators

$$(a, b, c, d, e, f, g, h, i, j) = \\ (x, y, zxZ, zyZ, Zxz, Zyz, z^2xZ^2, Z^2yz^2, Z^3yz^3, Z^4yz^4).$$

Happily we find a basis between t and T of the form below:

$$(WJ, jwJ, jwfBAweJiCIjaI, jI, jEWaJ, iAJicI, \\ jbDIjEWabFeWJ, iH, hFEfBAweJidBAweJ, \\ hCIhgFEfBAweJidFWJ)$$

where $w = bDCIjaIhGHicH$.

We already mentioned in Section 4.1 [Callahan et al. 99], which lists the knots in S^3 from the m and s part of the census. Recently, we were informed of [Champanerkar et al. 04] which does the same for the v section. Although the table does not tell us which of these knots is fibred (and now does not need to, in light of this section and Dunfield's list), we find in it eight of our unknown 1-cusped 3-manifolds including the last three with which we dealt. [Champanerkar et al. 04] describes v3093 as 16n245346 in Knotscape (if it had been an alternating knot then our work would have been in vain because we would have been able to conclude that it was fibred just from the Alexander polynomial), and v2869 and v3541 are given in terms of a (nonalternating) Dowker-Thistlethwaite code with 18 and 21 crossings respectively. Although these may not be the minimal crossing numbers, they must be pretty close because Knotscape tells us that they are not in its census which goes up to 16 crossings. Also, we now know the topological type of their fibres, because as knots in S^3 their fibres will have one boundary component and genus half the degree of the Alexander polynomial.

In conclusion we have:

Proposition 4.1. *The proportion of fibred 3-manifolds in the (orientable) cusped census is exactly 4199/4815=0.87206645898...*

5. VIRTUALLY FIBRED CUSPED 3-MANIFOLDS

As we now know all fibred 3-manifolds in the cusped census, we turn to how we can find nonfibred virtually fibred examples. The crucial point is this: a nonfibred hyperbolic 3-manifold that is commensurable with a fibred hyperbolic 3-manifold is itself virtually fibred, by considering the common finite cover, so the property of being virtually fibred is constant on commensurability classes. Therefore, we ought, in principle, to be able to use our fibred 3-manifolds to obtain nonfibred commensurable examples M. The first case that comes to mind is when $\pi_1 M$ is arithmetic, which in the cusped case means that it has integral traces and the invariant trace field is an imaginary quadratic number field. Here, two arithmetic fundamental groups will be commensurable if they have the same invariant trace field, so upon finding a fibred example we have that all arithmetic hyperbolic cusped 3-manifolds with this imaginary quadratic number field will be virtually fibred.

The paper [Goodman et al. 03] gives an algorithm that determines the commensurator of any nonarithmetic cusped hyperbolic 3-manifold and it is then applied to find commensurability classes for the 3-manifolds in the cusped census, as well as for hyperbolic knots and links for up to twelve crossings. Therefore, it is worth looking at the 616 nonfibred census 3-manifolds to see if any are in the same commensurability class as a fibred 3-manifold, given that we now can recognise all fibred 3-manifolds in the cusped census. Doing this gives us 86 nonfibred virtually fibred cusped hyperbolic 3-manifolds as listed in Table 2 (a few of which would have been known before, see for instance [Calegari and Dunfield 02, Hodgson et al. 92, Leininger 02]). Most of the fibred 3-manifolds certifying that these examples are virtually fibred have more than one cusp; moreover the four nonfibred 3-manifolds with 2 cusps (v2943, v3379, v3384, v3396) all appear thus we can say that any hyperbolic 3-manifold in the census with more than one cusp is virtually fibred.

We can further add to this table because the data we are using includes commensurability classes of knots and links in S^3. However, rather than just looking for fibred knots and links, we use the recent result [Walsh 04] that all 2-bridge knots and links are virtually fibred. We can identify 2-bridge knots and links in the tables by their Conway notation. This gives us another 51 examples to add to our table. Most of these are themselves nonfibred 2-bridge knots or next to one in the census, although a few are shown virtually fibred by being commensurable with a 2-bridge knot that is not in the cusped census. We have also two links not from the census that make an appearance: there is the fibred 2-bridge link 8a31 (or 8^2_4 in the tables) with Conway notation 323 and the nonfibred 2-bridge link 10a171 with Conway notation 262 (in fact the 2-cusped 3-manifolds v2943 and v3379 mentioned above are also 2-bridge links identifiable as 7a11 or 7^2_3 or 232 and 8a24 or 8^2_6 or 242 respectively).

One amusing consequence of the ubiquity of 2-bridge knots amongst those with low crossing number is that just by striking out from the tables of knots with nine crossings or less the 2-bridge knots and the knots with monic Alexander polynomial (which for these crossing numbers will be fibred), we see that all but ten knots are confirmed virtually fibred. In fact [Walsh 04] also shows that spherical Montesinos knots are virtually fibred, which leaves only five unknowns up to nine crossings. There may be a few more cusped 3-manifolds in the census that could be added to this table by having full knowledge of which knots and links up to twelve crossings are fibred, but certainly some nonfibred 3-manifolds are listed alone in their commensurability class so this process would not finish the job off. However we have pushed the number of virtually fibred 3-manifolds in the cusped census up to 4,336 which is a fraction over 90%.

6. CLOSED FIBRED HYPERBOLIC 3-MANIFOLDS

In the Hodgson-Weeks census [Hodgson and Weeks 01] of closed hyperbolic 3-manifolds, consisting of just under 11,000 examples (the number given is 11,031 but there are a few duplications), nearly all have finite first homology: only 127 have first Betti number 1 and above that there is only one 3-manifold with first Betti number 2. Thus only these few special closed 3-manifolds have a chance of being fibred, but in fact there is a reason why it is likely to be a good chance. All 3-manifolds in the closed census are obtained by Dehn surgery on 1-cusped 3-manifolds from the cusped census and this process either preserves the first Betti number or reduces it by one. Therefore the closed 3-manifolds M with $\beta_1(M) = 1$ come from 1-cusped 3-manifolds M' with $\beta_1(M') = 1$ or 2. But there are only three examples of the latter and moreover we now know that the vast majority of 3-manifolds M' in the cusped census are fibred. If M' is fibred and if $\beta_1(M') = 1$ then we have mentioned in Section 2 that M must also be fibred.

In addition, the one closed 3-manifold M with $\beta_1(M) = 2$ happens to be v1539(5,1), so it is irreducible and therefore Section 2 tells us it is fibred, as well as v1539(-5,1) which also appears in the census. Otherwise, we work through the closed 3-manifolds M with $\beta_1(M) = 1$, seeing if they result from surgery on a 1-cusped 3-manifold M' that is listed as fibred but which is not one of the three special cases with $\beta_1(M') = 2$. In this way, we find 80 additional closed fibred 3-manifolds in the census, which is a big proportion of those with positive first Betti number. The results are listed in Table 3.

As for the remaining 46 closed 3-manifolds M with $\beta_1(M) > 0$ in the census, we calculate the Alexander polynomial of the given fundamental group presentation which proves that all but five are not fibred. As we have $\beta_1(M) = 1$, we can do this in exactly the same way as we did for 1-cusped 3-manifolds, and indeed it is still invariant under $t \mapsto t^{-1}$. Moreover, it is again the case that if M is fibred over the circle then Δ_M must be monic, and here the degree of Δ_M must be twice the genus of the fibre: we can see this from Equation (4–2) by noting that we need to add a relation for the closed surface, but this results in an extra row of zeros upon application of the free differential calculus.

Our fundamental groups are usually 2-generator 2-relator with a few 3-generator 3-relator examples but we can use short cuts that might avoid calculating the whole Alexander polynomial. If we have $\pi_1 M = \langle g, x | r_1, r_2 \rangle$, which we always assume is in standard form with respect to x, then $\partial r_i / \partial x = 0$, thus the Alexander polynomial is the highest common factor of the two polynomials $\partial r_i / \partial g$. But since we know M is hyperbolic, if it is fibred then this must be by a surface of genus at least two, so the Alexander polynomial must be monic of even degree at least four. We thus calculate only one polynomial corresponding to the nicest looking relation and if this does not have such a factor then we are done. It turns out, as seen in Table 4, that in all but three of the cases the polynomial obtained was quartic, nonmonic, and not a scalar multiple of a monic quartic polynomial, so these 3-manifolds are not fibred. The three exceptions were that with v2018(-4,1) a quintic was obtained which factors as $(t+1)(t^2+1)(2t^2-3t+2)$ so this is nonfibred, indeed the other relation gives $(t^2+t+1)(t^2+1)(2t^2-3t+2)$ so the last two factors are the Alexander polynomial. This 3-manifold will feature again in Section 7 where we will find that it is virtually fibred. The next exception that needs to be checked is v2238(-5,1), but here a quintic is obtained that factors into irreducibles as $(t+1)(2t^4 - t^3 - t + 2)$ so this is fine. The only other problem is v3183(-3,2) which yields $2(t^4+1)$ so we worry that $t^4 + 1$ might be the Alexander polynomial, but looking at the other relation we see this cannot be the case.

For the three 3-generator cases, we similarly take two relations and calculate the relevant 2×2 determinant; these are all quartic and present no problems.

We treat the closed 3-manifolds which come from the three special 1-cusped 3-manifolds s789, v1539, v3209 separately. For the 2-generator group $\pi_1(\text{v1539})$ we have already stated in Section 2.1 that $(Ab, B^3a^5B^2)$ is a basis for the cusp, so taking the relation $(Ab)^p(B^3a^5B^2)^q$ from v1539(p, q) and substituting $a = bx$ so that it is in standard form with respect to b gives us the polynomial

$$qt^4 + qt^3 + (q - p)t^2 + qt + q,$$

whereas the original relation gives 0, so this is the Alexander polynomial (except for $(p, q) = (5, 1)$ where $\beta_1(M) = 2$) and $q \neq 0, 1$ implies that the 3-manifold is not fibred. We now have built up the complete picture for these hyperbolic 3-manifolds: we saw in Section 2 that v1539$(p, 1)$ is fibred (it is clear that v1539$(1, 0)$ has cyclic fundamental group so is not hyperbolic) and, in particular, v1539(5,2) that appears in Table 4 is nonfibred. Similarly for s789 we have (abc^2, a^3cbcA^3C) as a basis for the cusp and we take this Dehn filling relation for s789(p, q) along with either one of the two original relations (they result in the same polynomials). We put $c = Ax$ and $b = ya$ to get two relations in standard form with respect to a and this yields the Alexander polynomial

$$qt^4 - qt^3 + (p + q)t^2 - qt + q,$$

so once again it is not fibred if $q \neq 0, 1$ (with π_1s789$(1, 0) = \mathbb{Z}$ again), sorting out s789(-5,2). Finally we do this for v3209, with basis $(aCbc^2, aCacAcAC)$ and either one of the original relations, setting $a = Cx$ so that we are in standard form with respect to c. For v3209(p, q) we have π_1v3209$(1, 0) = \mathbb{Z}$ and Alexander polynomial

$$qt^4 - 2qt^3 + (p + 2q)t^2 - 2qt + q,$$

which reveals nine closed 3-manifolds in Table 4 as not fibred when $q > 1$.

We guess that s789$(p, 1)$ and v3209$(p, 1)$ are all fibred; not only would this fit into the same pattern as v1539 but we have already seen in Table 3 that s789$(p, 1)$ for $p = \pm 5$ and v3209$(p, 1)$ for $p = \pm 3$ are fibred, as they have alternative descriptions as Dehn fillings on 3-manifolds M' with $\beta_1(M') = 1$. We can say that if so, they must have fibres of genus two.

However, this still leaves in the census five 3-manifolds v3209$(p, 1)$ for $p = \pm 4, \pm 5, 6$ whose status is unknown. In the hope of finishing this off, it is worth looking for cyclic covers which we can show are fibred, just as we did with the remaining 1-cusped 3-manifolds in Section 4. Happily this works for all five, thus the fibred status of every 3-manifold in the closed census is known: 87 are fibred, 41 are nonfibred with $\beta_1(M) = 1$ and the rest are nonfibred with $\beta_1(M) = 0$.

We summarise the details of the last five unknowns so as to allow the claims to be checked. All five cases are very similar. We put $a = xC$ in our presentation and then we have the fundamental group $\langle a, c, x\rangle$ in standard form with respect to c. We know the fibre would be a genus 2 surface so we are after a 5-generator presentation. In each case, the cyclic covers of degree 2 and 3 have too few generators (at least after rewriting) but Magma tells us that the cyclic cover of degree 4 yields a 5-generator presentation of the form $\langle g_1, g_2, g_3, g_4, t\rangle$ for

$$(g_1, g_2, g_3, g_4, t) = \begin{cases} (x, cxC, Cbc, c^2xC^2, c^4), & p = 4, \pm 5, \\ (cbC, cxC, Cbc, c^2xC^2, c^4), & p = -4, \\ (x, cxC, Cbc, Cxc, c^4), & p = 6. \end{cases}$$

Since $t = c^4$ has infinite order but all g_i have finite order in the homology of M, we know the presentation obtained in each case will be in standard form with respect to t. What is most promising is that we always find the first relation given has no appearance of t (but t does appear in the others). Indeed, in all, but $p = -4$, this relation is of length 8 with each $g_i^{\pm 1}$ appearing once, and is a relation defining the closed surface of genus 2. For $p = -4$ it is of length 12, but as a consequence of showing the 3-manifold is fibred, this relation also has to define the genus 2 closed surface group.

We then proceed, just as in Section 4, by looking at the subwords from t to T, or from T to t (we did in fact do both). In all, but $p = -4$, we are given more than five relations, so we are looking for generating sets for the free group on g_1, g_2, g_3, g_4 rather than a free basis, but we always proceed by taking our n subwords (where n can be 4, 5, or 6) and using the shorter subwords to knock letters off the longer subwords until we have each generator g_i.

We do this by hand. For $p = \pm 4$, the relations are in simple form with respect to t. For $p = 5$, the fourth and sixth of the seven relations have two appearances of t (whereas the first relation has none and the rest have one). They are of the form $tw_1Tw_2tu_1Tu_2$ and $v_1tv_2TW_2tW_1T$ for u_j, v_j, w_j words in the g_i, so we can concatenate them to obtain a relation in simple form which we now use. For $p = -5$ we have six relations with the third, fifth, and sixth in this double form but each pair of these three can be concatenated as above to obtain five relations in simple form. Then for $p = 6$ we are given seven relations with the last three simple. We put together the second and fifth to obtain tsT, where $s = Cxc$, which we can now insert into the three relations in double form, resulting in enough relations in simple form to obtain all the generators.

Finally to show the original 3-manifolds are fibred, we look at the homology of the degree 4 covers. These are listed below and all have first Betti number 1, so we are done.

3-manifold	Homology of cover
v3209(4,1)	$\mathbb{Z}_2 + \mathbb{Z}_4 + \mathbb{Z}_4 + \mathbb{Z}_{24} + \mathbb{Z}$
v3209(-4,1)	$\mathbb{Z}_2 + \mathbb{Z}_4 + \mathbb{Z}_4 + \mathbb{Z}_8 + \mathbb{Z}$
v3209(5,1)	$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_{65} + \mathbb{Z}$
v3209(-5,1)	$\mathbb{Z}_5 + \mathbb{Z}_5 + \mathbb{Z}_{15} + \mathbb{Z}$
v3209(6,1)	$\mathbb{Z}_2 + \mathbb{Z}_6 + \mathbb{Z}_6 + \mathbb{Z}_{42} + \mathbb{Z}$

Thus we now know all the fibred 3-manifolds in the closed census. We have seen that if M' is a 1-cusped fibred 3-manifold with $\beta_1(M') = 1$ and we Dehn fill along its longitude to create M then M is fibred. We might expect that if instead M' is nonfibred then M is not, but this is unlikely to be true in full generality. For instance, let us take the 1-cusped 3-manifold m137 (an interesting example as it has a quadratic imaginary invariant trace field but is the first in the cusped census not to have integral traces). It is not fibred (indeed is not known to be virtually fibred) and is a knot in an integral homology sphere. We find from SnapPea a fundamental group presentation and basis for the cusp, whereupon it is easily seen that the group $\mathbb{Z}$ is obtained upon Dehn filling of the longitude, thus (assuming Poincaré) $M = S^2 \times S^1$ and so is fibred. (Another 3-manifold M' in the census with $\beta_1(M') = 1$ where $\mathbb{Z}$ is obtained upon Dehn filling is the nonfibred s783. The similar fact is true for the three 1-cusped examples with $\beta_1(M') = 2$.)

However, if M' is the exterior of a nontrivial knot in S^3 then Gabai shows in [Gabai 87] that $\pi_1 M \neq \mathbb{Z}$. He goes on to prove that for knots, M' is fibred if and only if M is, in which case the fibres have the same genus. Although this seems useful, and certainly we have included in Table 3 the genus of the fibre of those closed 3-manifolds M where the given M' is a knot exterior in S^3, there was only one case where this would have proved M is nonfibred: s862 is the nonfibred knot 8_4 so s862(7,1) in Table 4 is not fibred. In trying to generalise Gabai's result, a conjecture of Boileau (Problem 1.80 (C) in the Kirby problem list [Kirby 93]) states that if K is a null-homotopic knot in a closed orientable irreducible 3-manifold M then a nontrivial Dehn surgery on $M - K$ produces a fibred 3-manifold if and only if $M - K$ is fibred and it is the longitudinal surgery. Here the trivial surgery is just filling in K to obtain M thus destroying the meridian, and a null-homotopic knot can be detected because the longitude then becomes trivial. A fair variant on this question might be: if M' is a 1-cusped hyperbolic 3-manifold with $\beta_1(M') = 1$ where the longitudinal surgery produces a closed fibred 3-manifold M that is hyperbolic then is M' fibred? This is true for all examples we have considered.

7. VIRTUALLY FIBRED CLOSED 3-MANIFOLDS

We will now use our data to find nonfibred virtually fibred closed hyperbolic 3-manifolds. There seem to be even less examples of these than in the cusped case: until this point the only known ones in the literature consisted of the union of two twisted I-bundles over a nonorientable surface (originally due to Thurston), which have a fibred double cover, and the pair of non-Haken examples in [Reid 95] (one of which is the unique double cover of the other). However, just as in the cusped case, we merely need to find nonfibred hyperbolic 3-manifolds that are commensurable with fibred hyperbolic 3-manifolds. In particular, any 3-manifold M in the closed census which is commensurable with something in Table 3, but which is not in Table 3 itself, is a nonfibred virtually fibred example. We certainly do not have a full enumeration of the commensurability classes as in the cusped case, so we turn to the theory of arithmetic Kleinian groups: that is, if we have arithmetic hyperbolic 3-manifolds M_1, M_2 then they are commensurable if and only if their invariant trace fields and invariant quaternion algebras are isomorphic. In the closed arithmetic case we are guaranteed more invariant trace fields than just the imaginary quadratic ones: in fact, the fields that occur are precisely those with exactly one conjugate pair of complex embeddings. In order to determine this we utilise the program Snap [Snap 03] (see [Coulson et al. 00] for a description) and look for the file snap_data/closed.fields which lists (in order of volume) all closed 3-manifolds in the closed census for which the invariant trace field and invariant quaternion algebra could be found. It is known that $M = \mathbb{H}^3/\Gamma$ is arithmetic if and only if the invariant trace field $k\Gamma$ has exactly one complex place, the invariant quaternion algebra $A\Gamma$ is ramified at every real place, and Γ has integer traces. Thus, if M is a fibred 3-manifold from Table 3 appearing in this list we next look at the file snap_data/closed_census_algebras which gives (listed in order of trace field) 3-manifolds grouped together by invariant trace field, quaternion algebra, and whether or not they are arithmetic. Hence, if M is arithmetic then all 3-manifolds appearing together in the same grouping as M are commensurable with M, and so are virtually fibred.

The results are listed in Table 5. In particular, we find that the Weeks 3-manifold m003(-3,1), conjectured to be the smallest volume closed hyperbolic 3-manifold and known [Chinburg et al. 01] to be the smallest volume arithmetic 3-manifold, is virtually fibred as it is commensurable with m289(7,1). The third entry m007(3,1) in the closed census is one of the two non-Haken virtually fibred closed 3-manifolds in [Reid 95] and is called Vol(3) as it is the conjectured third smallest closed hyperbolic 3-manifold. This is known to be arithmetic (see [Jones and Reid 01]) so we can add it and the other 3-manifolds that Snap lists in its commensurability class to Table 5. Work of Dunfield [Dunfield 99] determines that out of the 246 3-manifolds in the closed census with volume less than 3, exactly 15 are Haken. Only one from that list appears here (this is m140(4,1) with volume 2.6667) so all other 3-manifolds in Table 5 with volume less than 3 are non-Haken virtually fibred hyperbolic examples. For other specific examples of Haken nonfibred virtually fibred closed hyperbolic 3-manifolds, one can use Theorem 2 in [Reid 95] which shows that the $3k$-fold cyclic branched cover M_{3k} of the figure eight knot is a double twisted I-bundle with $\beta_1(M_{3k}) = 0$. However, we also have, as promised, a closed nonfibred virtually fibred 3-manifold in the form of v2018(-4,1) with positive Betti number. Incidentally, it can be checked that this 3-manifold is genuinely a new example and not a union of two twisted I-bundles because if so it would have a fibred double cover, but all of its three index 2 subgroups have first Betti number 1. We claim that this is the first known example of its kind: for instance in [Boileau and Wang 96] it is shown that for every $n > 0$ there exist nonfibred closed hyperbolic 3-manifolds M_n with $\beta_1(M_n) = n$ but it is not known if they are virtually fibred.

We end up with 129 nonfibred virtually fibred 3-manifolds from the closed census. One might say that this is only a small proportion of the whole census, but of course our method only gives rise to arithmetic examples because $(k\Gamma, A\Gamma)$ is not a complete commensurability invariant in the nonarithmetic case. Another point is that all the examples of virtually fibred 3-manifolds we have given are commensurable with fibred 3-manifolds that necessarily must appear in the census, whereas as the volume grows and we have more and more 3-manifolds one would expect to have to look further for commensurable fibred 3-manifolds. This could explain why we do better with the 3-manifolds of smallest volume: of the first 51 census 3-manifolds (which goes up to volume twice that of the regular ideal tetrahedron), 34 are arithmetic, with 15 of these now known to be virtually fibred.

8. CORANK OF THE CENSUS 3-MANIFOLDS

The corank $c(G)$ of a finitely generated group G is the maximum n for which there is a homomorphism from G onto the free group F_n of rank n. Clearly $\beta_1(G) \geq c(G)$ and $\beta_1(G) \geq 1$ implies $c(G) \geq 1$. This quantity is of algebraic interest and we can think of the property $c(G) > 1$ as giving rise to one of the several notions of "largeness" of a group; see for instance [Button 04a]. But if $G = \pi_1 M$ for M a compact orientable 3-manifold (for which we write $c(M)$) then we have a geometric interpretation which allows us to think of it as a measure of "largeness" of a 3-manifold: this is because $c(M)$ is the maximal number of disjointly and properly embedded orientable connected surfaces S_i for which $M \setminus \cup S_i$ is connected (and in this context is also called the cut number of M). We can ask about the corank of 3-manifolds in the closed or cusped censuses: this can quickly be determined for every single one, and it turns out that we do not have any examples of "large" 3-manifolds here. As pointed out in [Holt and Rees 96], there is a (computationally very inefficient) procedure to determine if a finitely presented group surjects onto F_n, but it will not prove the nonexistence of such a surjection. However, in this setting we have available properties of 3-manifold groups to help us.

Theorem 8.1. *If M is a 3-manifold appearing in the closed census then $c(M) = 0$ if $\beta_1(M) = 0$ and otherwise $c(M) = 1$. If M is a 3-manifold appearing in the cusped census then $c(M) = 1$.*

Proof: We only need to do anything when $\beta_1(M) > 1$. However, if so and if M is fibred then $\beta_1(M) > c(M)$. This is Theorem 4.2 in [Button 04b], but here is a variation on that proof. If $\beta_1(M) = c(M) = n$ with $\theta : \pi_1 M \to F_n$ a surjective homomorphism then any homomorphism from $\pi_1 M$ to $\mathbb{Z}$ factors through θ. If M is fibred then we have our finitely generated kernel K of our relevant surjective homomorphism in $\pi_1 M$ which is normal and of infinite index, so $\theta(K)$ must be the same in F_n. But nonabelian free groups do not have finitely generated normal subgroups of infinite index except for the trivial group.

Thus this sorts out v1539(5,1), the only closed 3-manifold with Betti number 2. It also sorts out all cusped 3-manifolds M (which must have $\beta_1(M) \geq 1$) except for the four nonfibred examples in Section 4.2 with $\beta_1(M) = 2$ and the three fibred examples in the census with $\beta_1(M) = 3$. For these seven, we have to eliminate the possibility that $c(M) = 2$.

Firstly, v2943 and v3379 are 2 generator, so we cannot have $\pi_1 M$ surjecting onto F_2 unless $\pi_1 M = F_2$ which is not true. The given presentation for π_1(v3384) is

$$\langle a, b, c | ab^2ab^2aCb^2ab^2abcb,\ aCAc \rangle.$$

The second relation means that our surjection θ to F_2 would have to send a and c to powers of the same element $v \in F_2$ because that is the only way elements can commute in a nonabelian free group. So $u = \theta(b)$ and v must generate F_2, hence be a free basis, but this is not possible by looking at the image of the first relation which would always give a nontrivial relation between u and v.

This argument also works for the three 3-manifolds s776, v3227, v3383 with $\beta_1(M) = 3$: we know $c(M) = 3$ is not possible and to eliminate $c(M) = 2$ we use the second relations given in each case. Respectively they are $aCAc$, $bCBc$, both of which work in exactly the same way above, and aCb^2AcB^2, which by first setting $a = cx$ and then $c = b^2Y$ becomes b^2YxyXB^2, so we now just use the pair of generators x, y.

This leaves only

$$\pi_1(\text{v3396}) = \langle a, b, c | aBca^2bC,\ a^2cba^2CAB \rangle$$

with abelianisation $\mathbb{Z}_3 + \mathbb{Z} + \mathbb{Z}$. We suppose $\theta : \pi_1(\text{v3396}) \to F_2$ is onto and to finish we derive three quick contradictions. Both groups have three subgroups of index 2, which in the case of F_2 are all copies H_i of F_3. As each $\theta^{-1}(H_i)$ is distinct and has index 2, these must be the three index 2 subgroups K_i of π_1(v3396) so $c(K_i) \geq 3$, which implies that $\beta_1(K_i) \geq 3$ and K_i will need at least four generators. Two subgroups pass those tests but the third is $\langle a, cb^{-1}, b^2 \rangle$ and has abelianisation $\mathbb{Z}_{24} + \mathbb{Z} + \mathbb{Z}$ so it fails on both counts. Or we could try the lazy approach: by considering $\theta^{-1}(H)$ for H finite index in F_2, as before we have that π_1(v3396) must have as many subgroups of index n as F_2 does, so we ask the computer. The numbers we get from index 2 onwards are 3, 15, 32, 64 for π_1(v3396) whereas for F_2 they are 3, 7, 26, 97 so we have already been overtaken at index 5. In fact this is actually the number of subgroups up to conjugacy but our point still holds. □

9. GUIDE TO TABLES

9.1 Table 1: Alexander Polynomials of Unknown Cusped Census 3-Manifolds

This lists in the column "Name" the 165 cusped 3-manifolds M with $\beta_1(M) = 1$ which are unknown in Dunfield's list (available at http://www.its.caltech.edu/~dunfield/snappea/tables/mflds_which_fiber) of fibred and nonfibred cusped 3-manifolds. For each one, we take the presentation for its fundamental group (as given in http://www.its.caltech.edu/~dunfield/virtual_haken/virtual_haken_data/manifolds/cusped.gap) which is always (with the exception of v3036 which is marked by *2 gen*) generated by a, b, c and with two relations. The "Standard column" indicates the substitutions we must make, in order, to put the presentation into standard form with respect to a generator (meaning that the generator has zero exponent sum in both relations); this generator is then given at the end. Then the column "Poly" gives the Alexander polynomial which is written in a compact form. If a single number n is given without brackets then the presentation obtained was in simple form, as described in Section 4.3, so that the Alexander polynomial must be of the form $nt + m + nt^{-1}$. Here n can be obtained quickly and we do not need to calculate m, unless n is zero in which case we do and we write $0 = [m]$. The brackets notation that we use in general is because the Alexander polynomial is equal, up to units, when t is substituted for t^{-1} and it is nonzero when evaluated at 1. Thus it is either of even degree and in the form

$$a_k t^k + \ldots + a_0 + \ldots + a_k^{-k}, \text{ written } [a_k, \ldots, a_0],$$

or of odd degree in the form

$$a_k t^k + \ldots + a_1 t + a_1 + \ldots + a_k^{-(k-1)}, \text{ written } (a_k, \ldots, a_1).$$

The six 3-manifolds that have monic Alexander polynomial are printed in bold, as is the leading coefficient. They are all fibred except v2530.

9.2 Table 2: Cusped Virtually Fibred Nonfibred Census 3-Manifolds

Here we list under "Name" the nonfibred virtually fibred cusped census 3-manifolds that we found (we know they

Name	Standard	Poly	Name	Standard	Poly
m306	$c = xA; a$	-3	m307	$b = Ax; a$	[3,2]
m372	a	-2	m373	a	-2
m410	a	[-2]			
s386	a	-2	s387	c	-2
s426	$b = Ax; a$	[-4,2]	s427	$b = Ax, c = Ay; a$	[-4,-2]
s435	a	-3	s436	$c = Bx; b$	3
s486	$b = xA; a$	[-5,4]	s487	$c = xA; a$	[-5,-4]
s491	$c = yA, b = Ax; a$	[-5,-6]	s492	$b = xA; a$	[5,-6]
s594	$b = Ax; a$	(**1**,3)	s626	$b = YX^3, c = x^2y; x$	[3,-3,2]
s673	$a = x^2y, b = YX^3; x$	[3,-3,4]	**s704**	$c = Z^3y, a = Yz^2; z$	[**1**,-2,3]
s707	b	-3	s708	$b = Ax; a$	[3,0]
s732	c	[-2,-11]	s733	a	2
s773	$c = Ax; a$	(2,2)	s779	$c = Ax; a$	(2,0)
s784	b	3	s788	c	-3
s818	a	-2	s819	a	-2
s837	c	-3	s838	a	3
s878	b	2	s879	$c = xA; a$	[-2,5]
s899	b	4	s900	$b = Cx; c$	[4,-1]
s938	c	0=[-3]	s939	c	0=[-3]
v0895	b	[2,-14]	v0896	a	2
v0948	$b = Ax, c = Ay; a$	[-5,2]	v0949	$b = Cx, a = yC^2; c$	[5,2]
v0950	b	-3	v0951	c	3
v1000	b	3	v1001	$b = Ax; a$	-3
v1016	$b = xa; a$	4	v1017	c	-4
v1066	$b = xc; c$	5	v1067	a	-5
v1083	a	-5	v1084	$a = xb; b$	-5
v1095	b	[7,-4]	v1096	$b = ax; a$	[-7,-4]
v1097	$b = xA; a$	[7,6]	v1098	$c = Ax; a$	[7,-6]
v1104	$b = Cy, c = az; a$	[-7,8]	v1105	$c = xA; a$	[-7,-8]
v1110	$c = xA; a$	7	v1111	$b = xA; a$	-7
v1123	$c = xa; a$	[-8,-6]	v1124	$c = Ax; a$	[8,-6]
v1128	$c = A^2x; a$	[8,-10]	v1129	$c = Ax; a$	[-8,-10]
v1491	c	-2	v1492	b	-2
v1684	$c = Ax, a = yb^2; b$	[2,-2,-3]	v1737	$a = yz^2yz^3, b = Z^2Y; z$	[-4,4,-3]
v1781	a	-3	v1782	c	3
v1793	$b = Ax, a = yz^2,$ $c = Z^3Y; z$	[4,-4,5]	v1858	$a = xC^2; c$	(3,-1)
v1863	$a = C^2x; c$	[2,-2,7]	v1893	$a = B^2x; b$	[-3,3,2]
v1897	c	-2	v1898	c	2
v1901	$a = xB; b$	[-4,1]	v1902	$a = xB; b$	[4,1]
v2001	$b = Cx; c$	[4,-7]	v2002	$b = xC; c$	[4,7]
v2022	b	[-3,-15]	v2023	c	-3
v2037	$c = A^3x; a$	(3,1)	v2066	$a = xy^3, b = Y^2X; y$	[3,-3,8]
v2103	$b = C^2x, c = X^2y; x$	[-5,5,-3]	v2130	$c = az, a = xy^2,$ $b = Y^3X; y$	[-5,5,-2]

TABLE 1. Alexander polynomials of unknown cusped census 3-manifolds.

Name	Standard	Poly	Name	Standard	Poly
v2134	b	3	v2135	c	-3
v2146	c	4	v2147	a	4
v2151	$a = xy^3, c = Y^2X; y$	[-5,5,-7]	v2174	$b = Ax; a$	[-5,-1]
v2175	c	-5	v2182	b	5
v2183	b	-5	v2205	$a = xy^2, c = Y^3X; y$	[-5,5,-8]
v2257	$c = Bx; b$	[-5,11]	v2258	a	-5
v2304	$c = Bx; b$	[5,-14]	v2305	a	5
v2308	$c = xa, b = yX^2,$ $a = zx^3; x$	[-3,3,-3,2]	v2346	c	3
v2347	a	3	v2365	$a = xy^3, c = Y^2X; y$	[-3,3,-3,4]
v2388	b	-2	v2389	a	2
v2438	$b = A^3x; a$	[2,-2,0,3]	v2467	c	-2
v2468	b	2	**v2530**	$a = bx, c = yX^2,$ $b = xz; x$	(**1**)
v2575	c	3	v2576	a	[-3,10]
v2605	$c = xA^2; a$	[-3,-1,0]	v2706	$c = Ax; a$	[-3,-14]
v2707	a	3	v2708	$c = Ax; a$	[-3,1,0]
v2743	a	4	v2744	a	[-4,-7]
v2787	a	(-2,0)	v2807	a	6
v2808	a	[-6,6]	v2861	a	-2
v2862	b	2	**v2869**	$c = Ax, b = Ay,$ $a = zx^2; x$	[-**1**,2,-2,3]
v2874	b	(-2,3)	v2926	$c = Ax; a$	[6,0]
v2927	$c = Ax; a$	[-6,0]	v2997	a	2
v2998	b	-2	v3003	$b = Ax; a$	[-6,-3]
v3004	a	-6	v3036	*2 gen* a	[3,4,5]
v3092	$c = xa^3, b = Ay; a$	[2,-2,0,2,-1]	**v3093**	$a = xB^2, c = yB; b$	[-**1**,1,1]
v3102	b	4	v3103	b	-4
v3145	$c = By, a = bx; b$	[2,1,2]	v3168	a	-3
v3169	b	-3	v3188	c	-2
v3189	a	2	v3210	c	2
v3219	a	0=[4]	v3221	c	0=[4]
v3226	b	-2	v3228	c	-4
v3243	$a = B^2x, c = b^2y; b$	[-2,-1,-2]	v3244	$b = A^2x; a$	[-2,1,-2]
v3245	$a = C^3x, b = c^4y; c$	[2,-1,-1,3]	v3272	c	[3,-10]
v3273	a	3	v3293	$b = xA; a$	(-2)
v3329	$a = b^3x, c = By; b$	[2,-1,-1,2,-1]	v3337	b	[-4,-10]
v3338	b	4	v3377	$b = xA; a$	(-2,-1)
v3382	c	-5	v3394	b	[-3]
v3395	$b = Ax; a$	[3]	v3452	c	[4,-2]
v3453	a	[-4,-2]	v3492	c	-2
v3493	a	2	v3498	$a = C^3x, b = cy; c$	[2,0,-2,3]
v3526	$b = Ax, c = a^2y; a$	(-2)	**v3541**	$b = ax, a = yz^4,$ $c = Z^3Y; z$	[**1**,-2,1,0,2,-3]

TABLE 1. (continued) Alexander polynomials of unknown cusped census 3-manifolds.

Name	Ratio	Name of fibred	Name	Ratio	Name of fibred
m006,m007	1/2	v1241	m015,m017	1/1	m015=5a1($5_2 = 32$)
m029,m030	1/2	v3140^2	m032,m033	1/1	m032=6a3($6_1 = 42$)
m035,m037	1/1	m039,m040	m045,m046	1/2	v3383^3
"	1/2	v3218^2,v3220^2,v3222^2, v3225^2,v3227^3			
m053,m054	1/1	m053=7a4($7_2 = 52$)	m073,m074	1/1	m074=8a11($8_1 = 62$)
m079,m080	1/2	10a171^2(262)	m093,m094	1/1	m094=9a27($9_2 = 72$)
m139		$\mathcal{A}1$	m148,m149	1/2	8a31^2($8_4^2 = 323$)
m208		$\mathcal{A}3$	m287,m288	1/2	9a39($9_{10} = 333$)
m306,m307	1/1	s298,s299	m340	1/1	m340=7a5($7_3 = 43$)
m410		$\mathcal{A}3$			
s016,s017	1/1	s016=10a75($10_1 = 82$)	s022,s023	1/1	s023=11a247(92)
s119		$\mathcal{A}3$	s348	1/1	m329^2
s349,s350	1/1	m328^2	s423,s424	1/1	m359^2
s437	1/1	m367^2	s477	1/1	m391^2
s478,s480	1/1	s479,v0953	s558	1/1	s558=9a38($9_3 = 63$)
s643,s644	1/2	11a333(41114)	s648,s649	1/1	v1241, s648=7a6(7_4=313)
s673,s674	1/1	v1276,v1277	s725,s726	1/1	s726=8a18($8_3 = 44$)
s763,s764	1/2	9a16($9_{23} = 22122$)	s772,s773, s779,s784		$\mathcal{A}7$
s788	1/1	s789,v1539,v1540	s818,s819	1/1	s817,v1638
s862	1/1	s862=8a17($8_4 = 413$)	s870	1/1	s870=9a35($9_4 = 54$)
s899,s900	2/1	m015=5a1($5_2 = 32$)			
v0016,v0017	1/1	v0016=12a803 ([10]2)	v0024,v0025	1/1	v0025=13a3143 ([11]2)
v0571	1/1	m340=7a5($7_3 = 43$)	v0785	1/1	m357^2
v0819	1/1	m366^2,v0820	v0954	1/1	m388^2
v1010,v1012	1/1	s506^2	v1011	1/1	s503^2
v1035,v1036	1/2	11a365(353)	v1112,v1113	1/1	s549^2
v1152	1/1	s568^2	v1168	1/1	s577^2
v1172	1/1	s578^2	v1179	1/1	v1178^2
v1194	1/1	s602^2	v1205	1/1	v1204^2
v1210	1/1	s621^2	v1229	1/1	s638^2
v1243	1/1	v1243=11a364(83)	v1256	1/1	s661^2
v1676	1/1	s831^2	v1858		$\mathcal{A}1$
v2018	1/1	s876^2	v2037	1/1	s880^2
v2078	1/1	s887^2	v2158	1/1	s895^2
v2203	1/1	s898^2,v2202	v2238	1/1	s906^2
v2284,v2285	1/1	v2284=9a36($9_5 = 513$)	v2297,v2298	1/1	v2296
v2339	1/1	s914^2	v2346,v2347	1/1	v2345
v2361,v2362	1/1	v2362=10a117 ($10_3 = 64$)	v2467,v2468	1/1	v2469
v2488	1/1	v2488=10a113 ($10_4 = 613$)	v2520	1/1	v2520=11a342 (74)
v2575,v2576	1/1	v2574	v2706,v2707	1/1	v2705
v2787		$\mathcal{A}2$	v2796,v2797	1/2	11a119(23132)
v2858	1/1	v2858=10a114 ($10_8 = 514$)	v2874		$\mathcal{A}3$

TABLE 2. Cusped virtually fibred nonfibred census 3-manifolds.

Name	Ratio	Name of fibred	Name	Ratio	Name of fibred
v2894	1/1	v2894=11a358 (65)	v2943**2**,v2944	1/1	v2942**2**
v3128	1/1	v3126,v3127**2**	v3210	1/1	v3207,v3208, v3209
v3243,v3244	1/1	v3246,v3247	v3310	1/1	v3310=7a3 ($7_5 = 322$)
v3377	1/1	v3376**2**,v3378	v3379**2**,v3384**2**	1/1	v3383**3**
v3396**2**	1/1	v3393**2**	v3427	1/1	v3426**2**
v3457	1/1	v3456**2**	v3492,v3493	1/1	v3490,v3491

TABLE 2. (continued) Cusped virtually fibred nonfibred census 3-manifolds.

are nonfibred by Dunfield's list and the results of Section 4) using the file of cusped commensurability classes that makes up the data resulting from [Goodman et al. 03] (supplied to us by the authors, for which we thank them). In the column "Name of fibred" we list the fibred 3-manifolds with which the listed 3-manifolds are commensurable, thus showing that they are virtually fibred. The column before this is headed "Ratio" and is the ratio of the volume of the virtually fibred 3-manifold(s) to that of the corresponding group of fibred 3-manifolds. The 3-manifolds with **2** or **3** as a superscript have that number of cusps whereas the rest all have one cusp. As mentioned in Section 5, we also use 2-bridge knots and links. Here several notations are in use, so we give its name as a census 3-manifold (if it is one) as obtained from [Callahan et al. 99] and [Champanerkar et al. 04], then the Knotscape name (crossing number, a (or n) for (non)alternating and the reference number), then the ordering in the knot tables started by Alexander and Briggs and extended by Rolfsen and Bailey using work of Conway (this only applies for knots with ten or less crossings and links of nine or less). Then we give the Conway notation, needed to confirm it is 2-bridge, in which case this is just a string of integers (written together, with two digit numbers denoted [10] etc).

In order to move between these different notations, the file has commensurability classes of knots and links up to twelve crossings given under the Knotscape name, which it lists as equal to the relevant cusped census 3-manifold if appropriate. For knots of 10 crossings or less we can use the file in Knotscape that converts between its notation and the Rolfsen-Bailey tables, then look up the Conway notation in [Rolfsen 76]. For 11 crossing alternating knots, the original enumeration is due to Little but it was then taken up by Conway. We found http://www.indiana.edu/~knotinfo/ which converts from Knotscape to Conway notation. To check this, we then used http://www.scoriton.demon.co.uk/knots.html which allows us to go from Conway notation to braid notation (this table is in order of Little's notation so we confirm it with Conway in [Conway 70]) which we can then enter into Knotscape and ask it to identify the knot, thus taking us back.

There was one census knot each for 12 and 13 crossings that were featured; by getting Knotscape to draw them it was immediately seen that they were both twist knots. For the two links mentioned in Section 5, we used [Adams et al. 91] to go between Thistlethwaite's notation as given in the file and the Rolfsen-Bailey tables by recognising volumes in one case, whereas for the ten crossing link we recognised it as a 2-bridge link from the picture in http://www.math.toronto.edu/~drorbn/KAtlas/Links/

Finally, nonfibred arithmetic 3-manifolds are confirmed virtually fibred by the symbol $\mathcal{A}n$ in the "Name of fibred" column, where n can be 1, 2, 3, or 7 which refers to the imaginary quadratic number field which is its invariant trace field. As we know of arithmetic fibred cusped 3-manifolds with each of these invariant trace fields, they will be commensurable with those listed under "Name".

9.3 Table 3: Closed Fibred Census 3-Manifolds

This lists all closed 3-manifolds in the census which are fibred, as shown in Section 6. There are 87 entries listed in order of volume, which is given in the first column since it can be time consuming to find a 3-manifold by hand on name alone. To aid this, the volume is given to 4 decimal places, which should be enough to find the right part of the census, and is always rounded down to avoid having to look back. The " symbol indicates a volume which is the same as the preceding volume to the accuracy given in the census. Next we give the name of the 3-manifold as listed in the census, which we take to be ftp://www.geometrygames.org/priv/weeks/SnapPea/SnapPeaCensus/ClosedCensus/ClosedCensusInvariants.txt.

Volume	Name	ℤ	neg	Alternative
3.1663	m160(3,1)			
"	m159(4,1)			
3.1772	m199(-4,1)	2		
"	m122(-4,1)			
3.6638	s942(-2,1)		-	s957(-1,2)
"	m336(-1,3)			
3.7028	m345(1,2)	•		
3.7708	m289(7,1)	2		
"	m280(1,4)			
3.8534	m304(-5,1)			
"	m305(-1,3)			
3.9466	s385(5,1)	3		
3.9702	s296(-1,3)			
"	s297(5,1)			
4.0597	s912(0,1)	2		
"	m401(-2,3)			
"	m371(-1,3)			
"	m368(-4,1)			
4.4081	s580(-5,1)	2		
"	s581(-1,3)			
4.4153	s869(-1,2)	•		
"	s861(3,1)			
4.4191	v1191(-5,1)			
"	v1076(-5,1)			
4.4646	s924(3,1)			
"	v1408(4,1)			
4.5169	s677(1,3)			
"	s676(5,1)			
4.5559	v2641(-4,1)	•		
"	s745(3,2)			
4.6307	s646(5,2)			
4.7135	v1539(5,1)	$\beta 2$		
"	s789(-5,1)			v1540(1,3)
4.7252	s719(7,1)			
"	v1373(-2,3)			
4.7517	v3209(3,1)			v3514(-2,1)
"	v2420(-3,1)			
4.7659	v2099(-4,1)			
"	v2101(3,1)			
4.7740	s789(5,1)			v1670(-1,3)
"	v1539(-5,1)			
4.7874	v1721(1,4)	•		
4.9068	v2771(-4,1)			
4.9069	s836(-6,1)	4		

Volume	Name	ℤ	neg	Alternative
4.9094	v2986(1,2)	•		
4.9717	v2209(2,3)	•		
5.1171	v2054(-7,1)			
5.1379	v3066(-1,2)	•		
"	v2563(5,1)			
"	v2345(5,1)			
5.1706	v3209(-3,1)			v3486(3,1)
5.1984	v3077(5,1)		-	
"	v2959(-3,1)		-	
5.2007	v2671(-2,3)	•		
5.2983	s928(2,3)	•		
5.3334	v3390(3,1)			
"	v3209(4,1)			
"	v2913(-3,2)			
"	v3505(-3,1)	2		
"	v3261(4,1)			
"	v3262(3,1)			
5.3488	v2678(-5,1)			
5.4633	v3107(3,2)	•		
5.4957	v3216(4,1)			
"	v3217(-1,3)			
5.4962	v3320(4,1)	3		
5.5410	v3091(-2,3)			
5.5636	v3214(1,3)			
"	v3215(-4,1)			
5.5736	v3209(-4,1)			
5.6510	v2984(-1,3)	•		
5.6664	v3209(5,1)			
5.6743	v3019(5,2)	•		
5.7024	v3212(1,3)	•		
5.8111	v3209(-5,1)			
5.8524	v3425(-3,2)			
5.8664	v3209(6,1)			
5.8760	v3318(4,1)	•		
5.9780	v3352(1,4)	•		
6.0075	v3398(2,3)	•		
6.0502	v3378(-1,4)	•		
6.1102	v3408(1,3)	•		
6.1203	v3467(-2,3)	•		
6.1254	v3445(6,1)	•		
6.2391	v3509(4,3)			
"	v3508(4,1)			
6.2428	v3504(-2,3)			

TABLE 3. Closed fibred census 3-manifolds

The column "ℤ" refers to those 3-manifolds whose homology is ℤ and a dot indicates this. If the associated cusped 3-manifold is a knot in S^3 (as given by [Callahan et al. 99] and [Champanerkar et al. 04]) then the corresponding closed 3-manifold is then formed by a surgery along a longitude so its fibre will have the same genus as the knot, in which case we put this number in the column instead. As shown in Section 6, the genus of the fibre of any of the 3-manifolds in this table can be calculated from the fundamental group presentation if required. The $\beta 2$

indicates the one 3-manifold with homology $\mathbb{Z}+\mathbb{Z}$. The "neg" column marks with "-" those 3-manifolds that are listed in the census as having negatively oriented tetrahedra present. The program SnapPea has alternative descriptions for some 3-manifolds which might not involve negative orientations. In the "Alternative" column we have included such a description in one case, as well as alternative descriptions known to us for 3-manifolds obtained by $(p,1)$ surgery on the 3-manifolds s789 and v3209 as this is required to prove they are fibred.

9.4 Table 4: Closed Nonfibred Census 3-Manifolds with Infinite Homology

This lists the remaining 41 closed 3-manifolds in the census with infinite homology, along with evidence to show that they are nonfibred. They are given by volume and name, then in the "Standard" column we give the substitutions we used to put their fundamental groups in standard form, followed by the relevant generator, starting from the presentations given in `virtual_haken_data/manifolds/final.gap` at http://www.its.caltech.edu/~dunfield/virtual_haken/. We note in this column that s862 is the (nonfibred) knot 8_4. In the "Poly" column we give the polynomial obtained from the first relation, using the same notation for polynomials as in Table 1 (so the Alexander polynomial is a factor of this but we have not confirmed that they are equal). From Section 6 this polynomial immediately tells us that the 3-manifold is nonfibred except for the three indicated in bold for which we refer back to that section.

For a 3-manifold that is (p,q) surgery on s789, v1539, or v3209, we show in Section 6 that $q \neq 1$ implies it is nonfibred and so we mark these with x.

9.5 Table 5: Closed Virtually Fibred Nonfibred Census 3-Manifolds

This lists the closed virtually fibred 3-manifolds found in Section 7; they are all arithmetic. Also they all have finite homology (hence are nonfibred) with one exception, marked by the suffix $\beta 1$ and printed in bold. Again we list volume, name (at 2.5689 we list m130(-3,1) with "?" because it is given as m130(1,3) in the original census but the former in all other sources) and the column "neg" marks those 3-manifolds with negatively oriented tetrahedra (at this point we did not have access to possible alternative descriptions).

As in Table 2 for the cusped case, in the column "Name of fibred" we give the fibred 3-manifolds from Table 3 with which the listed 3-manifolds (put together in a group if they are commensurable and have the same volume) are commensurable, thus showing that they are virtually fibred. There is one commensurability class that is proved virtually fibred by using Vol(3) in [Reid 95] which is in the census as m007(3,1). We put a zero superscript

Volume	Name	Standard	Poly
4.4559	s528(-1,3)	$a = xB; b$	[-2,2,1]
"	s527(-5,1)	$a = xB; b$	[2,2,-1]
4.5760	s644(-4,3)	$a = xB^2; b$	[2,-2,5]
"	s643(-5,1)	$a = xB; b$	[2,2,5]
4.7494	**v2018(-4,1)**	$a = xB; b$	(2,-1,1)
4.7809	v1436(-5,1)	b	[3,-1,3]
4.7904	s750(4,3)	$a = xy^3$, $b = Y^2X; y$	[-3,3,5]
"	s749(5,1)	$a = xB; b$	[3,3,5]
4.8461	s789(-5,2)		x
"	v1539(5,2)		x
4.8511	**v2238(-5,1)**	$a = xB; b$	(2,1,-1)
"	v3209(1,2)		x
"	s828(-4,3)	$a = xB; b$	[2,4,5]
4.8810	v1695(5,1)	a	[3,-2,3]
5.0362	s862(7,1)	(The knot 8_4) b	[-2,5,-5]
"	v2190(4,1)	$a = xB; b$	[2,5,5]
5.2283	v3209(-1,2)		x
"	v2593(4,1)	$a = xB; b$	[2,4,3]
5.3811	v3209(3,2)		x
"	v3027(-3,1)	$a = xB; b$	[2,4,7]
5.4334	v2896(-6,1)	$a = xB; b$	[-2,3,0]
"	v2683(-6,1)	$a = xB; b$	[2,3,0]
5.4561	v2796(4,1)	a	[2,-1,5]
"	v2797(-3,4)	a	[2,1,5]
5.5573	v2948(-6,1)	$a = xB; b$	[3,-2,0]
"	v2794(-6,1)	$a = xB; b$	[3,2,0]
5.5736	**v3183(-3,2)**	$a = xy^3$, $b = Y^2X; y$	[-2,0,0]
5.6562	v3145(3,2)	b	[-2,-1,-2]
"	v3181(-3,2)	$a = xB; b$	[2,5,8]
5.6872	v3036(3,2)	b	[3,4,5]
5.7024	v3209(1,3)		x
"	v3269(4,1)	a	[3,6,7]
5.7057	v3209(-3,2)		x
5.7243	v3209(2,3)		x
"	v3313(3,1)	b	[3,6,8]
5.8041	v3239(3,2)	$a = xB; b$	[3,5,7]
5.8060	v3209(5,2)		x
5.8073	v3209(-1,3)		x
5.8759	v3209(4,3)		x
5.8882	v3244(4,3)	$C = ax, a = yz^2$, $b = Z^3Y; z$	[2,-1,2]
"	v3243(-4,1)	$a = xc, b = yx$, $c = Z^2Y; z$	[2,1,2]

TABLE 4. Closed nonfibred census 3-manifolds with infinite homology.

Volume	Name	neg	Ratio	Name of fibred
0.9427	m003(-3,1)		1/4	m289(7,1)
				m280(1,4)
1.0149	m007(3,1)	-	1/1	$m007(3,1)^0$
1.4140	m009(4,1)		3/8	m289(7,1)
				m280(1,4)
1.5831	m007(4,1)		1/2	m160(3,1)
				m159(4,1)
1.5886	m006(3,1)	-	1/3	v2099(-4,1)
	m003(-5,4)			v2101(3,1)
1.8319	m009(5,1)		1/3	v3217(-1,3)
	m010(-2,3)			
	m009(-5,1)			
	m006(1,3)			
1.8854	m007(5,1)		1/2	m289(7,1)
	m006(-1,3)			m280(1,4)
2.0298	m036(-3,2)	-	2/1	$m007(3,1)^0$
	m010(-4,3)			
"	m010(4,1)		1/2	m371(-1,3)
				m368(-4,1)
2.5689	m039(6,1)		2/3	m304(-5,1)
	m035(-6,1)			m305(-1,3)
	m037(2,3)			
	m130(-3,1)?			
	m120(-4,1)	-		
	m223(3,1)			
	m038(-6,1)			
	m036(-2,3)	-		
2.6667	m135(-1,3)		1/2	v3505(-3,1)
	m135(1,3)	-		v3261(4,1)
	m168(3,2)			v3262(3,1)
	m140(4,1)	-		
2.8281	m221(3,1)		3/4	m289(7,1)
	m070(1,4)			m280(1,4)
	m139(2,3)			
3.0448	m247(-1,3)	-	3/1	$m007(3,1)^0$
3.1772	m303(-3,1)		1/1	m199(-4,1)
				m122(-4,1)
"	m141(4,1)		2/3	v2099(-4,1)
	m249(1,2)			v2101(3,1)
	s254(-3,1)			
	s479(-3,1)	-		
	m146(-2,3)			
	m188(4,1)			
	m148(6,1)			
	m149(-2,3)			
	m206(3,2)			
	m159(-2,3)			
3.6638	s960(-1,2)		1/1	s942(-2,1)
	m304(5,1)			m336(-1,3)
"	s572(1,2)	-	2/3	v3216(4,1)
	m293(4,1)			
	s645(-1,2)			
	s297(-1,3)			
	s778(-3,1)			
	s775(-1,2)			
	s682(-3,1)			
"	s296(5,1)		2/3	v3217(-1,3)
	s779(2,1)	-		
	m312(-1,3)			
	s595(3,1)			
	s775(-3,1)			
	s350(-4,1)			
	m294(4,1)			
	s495(1,2)	-		
3.7708	m369(-3,2)		1/1	m289(7,1)
	m371(3,2)			m280(1,4)
	s478(-1,2)			
	s479(1,2)	-		
3.9702	s784(1,2)	-	1/1	s296(-1,3)
	m303(5,1)			s297(5,1)
	m376(3,2)			

TABLE 5. Closed virtually fibred nonfibred census 3-manifolds

Volume	Name	neg	Ratio	Name of fibred
4.0597	v0825(4,1)		4/1	$m007(3,1)^0$
	m358(1,3)			
	s775(1,2)			
	s778(-3,2)			
	s779(1,2)	-		
	m395(-2,3)			
	s787(1,2)			
	s440(-1,3)			
"	s705(-3,1)		1/1	m371(-1,3)
	s779(-3,2)			m368(-4,1)
	s772(-3,2)			
4.2421	v2101(1,2)		5/4	m289(7,1)
				m280(1,4)
4.4153	v2101(-1,3)		1/1	s869(-1,2)
	s779(-4,1)			s861(3,1)
	s775(-4,1)			
	s778(3,1)			
	s772(-4,1)			
	s773(3,1)			
	s786(3,1)			
	s781(-4,1)			
4.4646	s781(-2,3)		1/1	s924(3,1)
	s786(-1,3)			v1408(4,1)
	s773(-1,3)	-		
	s777(-5,1)			
	v2787(1,2)	-		
4.6307	s645(5,2)		1/1	s646(5,2)
4.7135	s889(3,2)		5/4	m289(7,1)
	v2739(1,2)			m280(1,4)
"	v2797(2,1)		1/1	v1539(5,1)
	v2573(-3,2)			s789(-5,1)
	s788(-1,3)			
4.7494	**v2018(-4,1)** $\beta 1$		3/2	m160(3,1)
				m159(4,1)

Volume	Name	neg	Ratio	Name of fibred
4.7659	v2787(-3,1)		3/2	m199(-4,1)
	v1644(-2,3)			m122(-4,1)
	v2100(-3,1)			
"	s916(-3,2)		1/1	v2099(-4,1)
	s957(1,2)			v2101(3,1)
	s821(2,3)			
	s960(1,2)			
4.9068	v2018(2,3)	-	1/1	v2771(-4,1)
5.0747	v3216(-4,1)		5/1	$m007(3,1)^0$
	v3210(3,1)			
	v2636(2,3)			
	v2417(-1,3)			
"	v3213(-3,1)		5/4	m371(-1,3)
				m368(-4,1)
5.1379	v3100(-3,1)		4/3	m304(-5,1)
	v2346(-1,3)			m305(-1,3)
	v2345(-1,3)			
	v3469(3,1)			
	v3106(1,3)			
	s916(5,1)			
	v3214(-3,1)			
"	v2346(5,1)		1/1	v2563(5,1)
				v2345(5,1)
5.3334	v3210(-3,1)			v3209(4,1)
	v3207(-3,1)			v3505(-3,1)
	v3208(4,1)			v3261(4,1)
	v3106(-3,1)			v3262(3,1)
	v3107(-4,1)			
	v3331(-2,3)			
5.4957	v3213(-1,3)		1/1	v3216(4,1)
"	v3412(5,1)		1/1	v3217(-1,3)
5.6562	v3387(3,2)		3/2	m289(7,1)
	v3136(-1,3)			m280(1,4)

TABLE 5. (continued) Closed virtually fibred nonfibred census 3-manifolds.

on this to remind ourselves it has zero first Betti number. We then have in "Ratio" the ratio of the volume of the virtually fibred 3-manifolds in each group to that of the corresponding group of fibred 3-manifolds (it happens that the latter always have the same volume within a group). They are given as fractions with small coefficients; although this is likely to be correct, it could be argued that, unlike in the cusped case where we are able to use the index of the 3-manifold in its commensurator, we have only confirmed it to the number of decimal places available. This does not concern us here because the aim is to allow quick access to the volumes of those 3-manifolds in the right hand column for ease of reference.

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