

# On the Vanishing of Twisted $L$ -Functions of Elliptic Curves

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Let  $E$  be an elliptic curve over  $\mathbb{Q}$  with  $L$ -function  $L_E(s)$ . We use the random matrix model of Katz and Sarnak to develop a heuristic for the frequency of vanishing of the twisted  $L$ -functions  $L_E(1, \chi)$ , as  $\chi$  runs over the Dirichlet characters of order 3 (cubic twists). We also compute explicitly the conjecture of Keating and Snaith about the moments of the special values  $L_E(1, \chi)$  in the family of cubic twists. Finally, we present experimental data which is consistent with the conjectures for the moments and for the vanishing in the family of cubic twists of  $L_E(s)$ .

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## 1. INTRODUCTION

Let  $E$  be an elliptic curve defined over  $\mathbb{Q}$  with conductor  $N_E$ , and let

$$\begin{aligned} L_E(s) &= \prod_{p \nmid N_E} \left(1 - \frac{a_p}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \prod_{p \mid N_E} \left(1 - \frac{a_p}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \end{aligned} \tag{1-1}$$

be the  $L$ -function of  $E$ . Then, from the work of Wiles and Taylor [Wiles 95, Taylor and Wiles 95] and Breuil, Conrad, Diamond, and Taylor [Breuil et al. 01],  $L_E(s)$  has analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda_E(s) = \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L_E(s) = \omega_E \Lambda_E(2-s),$$

where  $-\omega_E = \pm 1$  is the eigenvalue of the Fricke involution. Let  $\chi$  be a primitive character of conductor  $\mathfrak{f}$  coprime to  $N_E$ . We can then form the twisted  $L$ -function

$$L_E(s, \chi) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s},$$

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which also has analytic continuation to the whole complex plane, and satisfies the functional equation,

$$\begin{aligned} \Lambda_E(s, \chi) &= \left( \frac{f\sqrt{N_E}}{2\pi} \right)^s \Gamma(s)L_E(s, \chi) \\ &= \frac{\omega_E\chi(N_E)\tau(\chi)^2}{f} \Lambda_E(2-s, \bar{\chi}), \end{aligned} \tag{1-2}$$

where  $\tau(\chi)$  is the Gauss sum [Shimura 71, Theorem 3.66].

In the particular case where  $\chi_d$  is a quadratic character of discriminant  $d$ , the functional equation is

$$\begin{aligned} \Lambda_E(s, \chi_d) &= \left( \frac{|d|\sqrt{N_E}}{2\pi} \right)^s \Gamma(s)L_E(s, \chi_d) \\ &= \omega_E\chi_d(-N_E)\Lambda_E(2-s, \chi_d). \end{aligned} \tag{1-3}$$

Then, for about half of the discriminants  $d$ ,  $\omega_E\chi_d(-N_E) = -1$  and  $L_E(s, \chi_d)$  vanishes at  $s = 1$ . For each quadratic character  $\chi_d$ , let  $r_d$  be the order of vanishing of  $L_E(s, \chi_d)$  at  $s = 1$ . Goldfeld conjectured that [Goldfeld 79]

$$\sum_{|d|\leq X} r_d \sim \frac{1}{2} \sum_{|d|\leq X} 1 \quad \text{as } X \rightarrow \infty,$$

where both sums run over quadratic characters of discriminant  $|d| \leq X$ . In particular, Goldfeld’s conjecture implies that

$$N_{\geq 2}(X) = \#\{|d| \leq X \text{ such that } r_d \geq 2\} = o(X).$$

There are lower bounds for  $N_{\geq 2}(X)$ , first obtained by Gouvêa and Mazur [Gouvêa and Mazur 91], and improved by Stewart and Top [Stewart and Top 95]. More precisely,  $N_{\geq 2}(X) \gg X^{1/2}$  under the Parity Conjecture [Gouvêa and Mazur 91, Stewart and Top 95]. See the review article [Rubin and Silverberg 02] for a more complete account of these results, and for other similar results [Rubin and Silverberg 01].

In the recent years, a new approach to the understanding of zeroes of  $L$ -functions in families emerged from the work of Katz and Sarnak on zeroes of  $L$ -functions and random matrix theory [Katz and Sarnak 99a, Katz and Sarnak 99b]. For example, Goldfeld’s conjecture is a particular case of their Density Conjecture, inspired by their work over function fields. Using similar ideas, Conrey, Keating, Rubinstein, and Snaith [Conrey et al. 02] predicted a precise asymptotic for  $N_{\geq 2}(X)$ . Their work is described in more detail in Section 3.

In this paper, we study vanishing of the twisted  $L$ -functions  $L_E(s, \chi)$  by Dirichlet characters of order 3 (cubic characters). In all the following,  $\chi$  will be a cubic

character of conductor  $f$ . Let  $E$  be an elliptic curve over  $\mathbb{Q}$ , and let

$$N(X) = \#\{\text{cubic characters } \chi \text{ of conductor } f \leq X\} \tag{1-4}$$

$$\mathcal{F}_E = \{L_E(s, \chi) : \chi \text{ is a cubic character}\} \tag{1-5}$$

$$N_E(X) = \#\{L_E(s, \chi) \in \mathcal{F}_E : L_E(1, \chi) = 0 \text{ and } f \leq X\}. \tag{1-6}$$

What can we say about the asymptotic behavior  $N_E(X)$ ? The situation is different from the case of quadratic twists, as the functional equation (1-2) now relates  $L_E(s, \chi)$  and  $L_E(s, \bar{\chi})$  and does not force vanishing of  $L_E(1, \chi)$  when the sign of the functional equation is not 1. There is then no reason to predict that the set of cubic characters for which  $L_E(1, \chi)$  vanishes has positive density. We also note that in the case of cubic twists, the twisted  $L$ -function  $L_E(s, \chi)$  is conjecturally related to the points that  $E$  acquires over cyclic cubic fields. More precisely, let  $K$  be a cyclic cubic field, and let  $\hat{G}$  be the character group of  $\text{Gal}(K/\mathbb{Q})$ . Let  $L(E/K, s)$  denote the  $L$ -function of  $E$  seen as an elliptic curve over the field  $K$ . Then,

$$L(E/K, s) = \prod_{\chi \in \hat{G}} L_E(s, \chi),$$

i.e., the vanishing of  $L_E(s, \chi)$  is related (via the Birch and Swinnerton-Dyer conjecture) to the existence of rational points on  $E(K)$ .

Kuwata [Kuwata 99] and Fearnley and Kisilevsky [Fearnley and Kisilevsky 00] have shown that if there is one cubic twist  $\chi$  such that  $L_E(1, \chi)$  vanishes, then there are infinitely many. When  $E$  is a curve with rational 3-torsion with some additional conditions, Fearnley and Kisilevsky have shown that  $N_E(X) \gg X^{1/2}$ .

We give in this paper a heuristic, based on the connection between zeroes of  $L$ -functions in families and random matrix theory introduced by Katz and Sarnak, to predict the asymptotic behavior of  $N_E(X)$ . As in [Conrey et al. 02], we use the ideas of Keating and Snaith [Keating and Snaith 00a, Keating and Snaith 00b] to predict the value distribution at the central critical point of the  $L$ -functions in our families. Similar heuristics were also developed for families of higher-order twists in [David et al. 04].

We would like to emphasize that the cubic twists we discuss in this paper refer to the  $L$ -functions of elliptic curves over  $\mathbb{Q}$  twisted by cubic Dirichlet characters. These are different from the  $L$ -functions arising from the family of (complex multiplication) elliptic curves

$x^3 + y^3 = m$ . Those curves are isomorphic to the elliptic curve  $x^3 + y^3 = 1$  by an isomorphism of order three, and are also called cubic twists. That family was studied by Zagier and Kramarz [Zagier and Kramarz 88] who obtained some numerical data suggesting that a positive proportion of those curves have rank two or more. The numerical data for this family was extended recently by Watkins [Watkins 04], suggesting that it is more likely that the proportion goes to zero. Watkins also shows that random matrix theory predicts that the number of curves in the family  $x^3 + y^3 = m$  with even nonzero rank has density zero, following the ideas of [Conrey et al. 02] and the present paper.

The structure of the paper is as follows. The second section presents a discretisation of the special values  $L_E(1, \chi)$ . The third section reviews the work of Keating and Snaith, which suggests that the value distribution of the  $L$ -functions at the critical point is related to the value distribution of characteristic polynomials of random matrices. This leads to a random matrix conjecture for the asymptotic behavior of  $N_E(X)$ . In the fourth section, we write a precise conjecture for the integral moments of  $L_E(1, \chi)$  in our family, following from the work of Keating and Snaith. We explicitly compute the arithmetic constant for the family. The conjecture can then be tested numerically, providing support for the random matrix models of the  $L$ -functions  $L_E(1, \chi)$  in the family of cubic twists. The fifth section contains asymptotics for  $N(X)$  and related sums which are needed in the rest of the paper. Finally, the last section presents some experimental results.

## 2. DISCRETISATION OF THE SPECIAL VALUES

Following Mazur, Tate, and Teitelbaum [Mazur et al. 86], we define the algebraic part of  $L_E(1, \chi)$  to be

$$\begin{aligned} L_E^{alg}(1, \chi) &= \frac{2 \mathfrak{f} L_E(1, \chi)}{\Omega \tau(\chi)} \\ &= \sum_{a \pmod{\mathfrak{f}}} \bar{\chi}(a) \Lambda(a, \mathfrak{f}), \end{aligned} \quad (2-1)$$

where  $\Lambda(a, \mathfrak{f}) \in \mathbb{Z}$  and  $\Omega$  is a nonzero rational multiple of the real period  $\Omega_E$ . Then,  $L_E^{alg}(1, \chi)$  is an algebraic integer in  $\mathbb{Z}[\rho]$  where  $\rho$  is a third root of unity. In fact, we have

**Theorem 2.1.**

$$|L_E^{alg}(1, \chi)| = \begin{cases} n_\chi & \text{if } \omega_E = 1; \\ \sqrt{3} n_\chi & \text{if } \omega_E = -1; \end{cases}$$

for some nonnegative integer  $n_\chi$ .

*Proof:* As  $E$  is defined over  $\mathbb{Q}$ , we have that  $\overline{L_E(1, \chi)} = L_E(1, \bar{\chi})$ . Also, as  $\chi$  is a cubic character,  $\chi(-1) = 1$  and  $\overline{\tau(\chi)} = \chi(-1)\tau(\bar{\chi}) = \tau(\bar{\chi})$ . From (2-1), this gives  $\overline{L_E^{alg}(1, \chi)} = L_E^{alg}(1, \bar{\chi})$ . Now, using the functional equation

$$\begin{aligned} L_E^{alg}(1, \chi) &= \frac{2 \mathfrak{f} L_E(1, \chi)}{\Omega \tau(\chi)} \\ &= \frac{2 \omega_E \chi(N_E) \tau(\chi)}{\Omega} L_E(1, \bar{\chi}) \\ &= \omega_E \chi(N_E) \overline{L_E^{alg}(1, \chi)} \\ &= \zeta_\chi \overline{L_E^{alg}(1, \chi)} \quad \text{with } \zeta_\chi = \omega_E \chi(N_E). \end{aligned}$$

Then,  $L_E^{alg}(1, \chi)$  satisfies an equation

$$\lambda = \zeta_\chi \bar{\lambda} \quad (2-2)$$

for  $\zeta_\chi \in \mathbb{C}^*$ . It is easy to see that any two solutions  $\lambda_1, \lambda_2$  of such an equation satisfy  $\lambda_1 = \alpha \lambda_2$  with  $\alpha$  real. Suppose that  $\omega_E = 1$ , which implies that  $\zeta_\chi$  is a third root of unity. If  $\zeta_\chi = 1$ , then  $L_E^{alg}(1, \chi)$  is real, and as  $L_E^{alg}(1, \chi) \in \mathbb{Z}[\rho]$ , we must have  $L_E^{alg}(1, \chi) \in \mathbb{Z}$ . If  $\zeta_\chi$  is a primitive third root of unity, then  $\lambda = \zeta_\chi^2$  satisfies (2-2) and we have  $L_E^{alg}(1, \chi) = \alpha \zeta_\chi^2$  with  $\alpha$  real. As  $L_E^{alg}(1, \chi) \in \mathbb{Z}[\rho]$ , we must have  $\alpha \in \mathbb{Z}$ . Suppose that  $\omega_E = -1$ . If  $\zeta_\chi = -1$ , then  $\lambda = \sqrt{-3}$  satisfies (2-2) and we have  $L_E^{alg}(1, \chi) = \alpha \sqrt{-3}$  with  $\alpha$  real. As  $L_E^{alg}(1, \chi) \in \mathbb{Z}[\rho]$ , we must have  $\alpha \in \mathbb{Z}$ . If  $\zeta_\chi$  is a primitive sixth root of unity, then  $\lambda = (\zeta_\chi - \bar{\zeta}_\chi) \zeta_\chi^2$  satisfies (2-2) and we have  $L_E^{alg}(1, \chi) = \alpha (\zeta_\chi - \bar{\zeta}_\chi) \zeta_\chi^2$  with  $\alpha$  real. As  $L_E^{alg}(1, \chi) \in \mathbb{Z}[\rho]$ , we must have  $\alpha \in \mathbb{Z}$ .  $\square$

As  $L_E(1, \chi)$  vanishes if and only if the integer  $n_\chi$  vanishes, this gives a discretisation on the special values  $L_E(1, \chi)$ . One should mention that the distribution of the integers  $n_\chi$  is very interesting. For example, the experimental data suggests that there are infinitely many cubic characters  $\chi$  for which  $n_\chi = 1$  (see Figure 6). This seems to be very difficult to prove. We also submit the following conjecture, obtained in part by observation of the experimental data, and in part by analogy with the genus theory of number fields.

**Conjecture 2.2.** *Suppose that  $E$  is isogenous to a curve with a rational 3-torsion point. For any positive integer  $n$ , let  $\nu(n)$  be the number of distinct prime divisors of  $n$ . Let  $\chi$  be a cubic character of conductor  $\mathfrak{f}$ , and let  $n_\chi$  be the integer defined by Theorem 2.1. Then*

$$3^{\nu(\mathfrak{f})-1} \mid n_\chi.$$

In order to obtain a heuristic for the vanishing in the family  $\mathcal{F}_E$ , we have to make some assumptions on the distribution of the integers  $n_\chi$ . From the above conjecture, it seems that we should distinguish between the cases where  $E$  has rational 3-torsion or not. This distinction is also suggested by the work of Fearnley and Kisilevsky discussed in the introduction, and fits the experimental data as we will see in Section 6.

### 3. RANDOM MATRIX THEORY

Let  $G(N)$  be one of the classical compact irreducible symmetric spaces. For each  $A \in G(N)$ , let  $\lambda_1 = e^{i\theta_1}, \dots, \lambda_N = e^{i\theta_N}$  be the eigenvalues of  $A$  which are ordered by the eigenangles  $\theta_1, \dots, \theta_N$  such that

$$0 \leq \theta_1 \leq \dots \leq \theta_N < 2\pi.$$

Let  $\mathcal{F} = \{L_f(s)\}$  be a family of  $L$ -functions with symmetry type  $G(N)$ . It is conjectured by Katz and Sarnak that the statistics of the low-lying zeroes of  $\mathcal{F}$  should fit those of the eigenangles of random matrices in  $G(N)$  [Katz and Sarnak 99a, Katz and Sarnak 99b].

Let  $P_A(\lambda) = \det(A - \lambda I)$  be the characteristic polynomial of  $A$ , and let  $\{L_f(1/2)\}_{f \in \mathcal{F}}$  be the central critical values of the  $L$ -functions in  $\mathcal{F}$ . Keating and Snaith [Keating and Snaith 00a, Keating and Snaith 00b] suggest that the value distribution of the  $L$ -functions at the critical point is related to the value distribution of the characteristic polynomials  $|P_A(1)|$  with respect to the Haar measure of  $G(N)$ .

Using this model, vanishing in the family of quadratic twists was studied in [Conrey et al. 02]. More precisely, let

$$\begin{aligned} \mathcal{F}_{E^+} &= \{L_E(s, \chi_d) : \chi_d \text{ quadratic} \\ &\quad \text{with } \omega_E \chi_d(-N_E) = 1\} \\ N_{E^+}(X) &= \#\{L_E(s, \chi_d) \in \mathcal{F}_{E^+} : L_E(s, \chi_d) = 0 \\ &\quad \text{and } |d| \leq X\}, \end{aligned}$$

i.e.,  $\mathcal{F}_{E^+}$  is the family of quadratic twists for which the sign of the functional equation is 1. Then, either  $L_E(1, \chi_d) \neq 0$ , or it vanishes with even order at least 2.

**Conjecture 3.1.** [Conrey et al. 02] *There are constants  $b_E \neq 0$  and  $e_E$  such that*

$$N_{E^+}(X) \sim b_E X^{3/4} \log^{e_E} X$$

when  $X \rightarrow \infty$ .

In this section, we make a similar analysis for the family  $\mathcal{F}_E$  of cubic twists. As the symmetry type of our family is the unitary group  $U(N)$ , we now review the work of Keating and Snaith for this symmetry group. All the results cited below are from [Keating and Snaith 00a]. Let

$$M_U(s, N) = \int_{U(N)} |P_A(1)|^s d\text{Haar}$$

be the moments for the distribution of  $|P_A(1)|$  in  $U(N)$  with respect to the Haar measure. Keating and Snaith prove that

$$M_U(s, N) = \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+s)}{\Gamma^2(j+s/2)}, \tag{3-1}$$

and then  $M_U(s, N)$  is analytic for  $\text{Re}(s) > -1$ , and has meromorphic continuation to the whole complex plane. The probability density function is the Mellin transform

$$P_U(x, N) = \frac{1}{2\pi i} \int_{(c)} M_U(s, N) x^{-s-1} ds$$

for some  $c > -1$ . For  $x$  small, the value of  $P_U(x, N)$  is determined by the first pole of  $M_U(s, N)$  at  $s = -1$ , and this gives

$$P_U(x, N) \sim \frac{1}{\Gamma(N)} \prod_{j=1}^N \frac{\Gamma(j)^2}{\Gamma^2(j-1/2)} = R(N) \quad \text{as } x \rightarrow 0.$$

We have

$$R(N) \sim N^{1/4} G^2(1/2) \quad \text{as } N \rightarrow \infty,$$

where  $G$  is the Barnes  $G$ -function defined by

$$\begin{aligned} G(1+z) &= (2\pi)^{z/2} e^{-((1+\gamma)z^2+z)/2} \\ &\quad \times \prod_{n=1}^{\infty} \left( (1+z/n)^n e^{-z+z^2/2n} \right). \end{aligned}$$

Let  $M_E(s, X)$  be the moments

$$M_E(s, X) = \frac{1}{N(X)} \sum_{\chi \leq X} |L_E(1, \chi)|^s, \tag{3-2}$$

where the sum runs over all cubic characters of conductor  $\leq X$ . As the family  $\mathcal{F}_E$  of such an  $L$ -function has symmetry type  $U(N)$ , we have

**Conjecture 3.2. (Keating and Snaith Conjecture for cubic twists.)**  $M_E(s, X) \sim a_E(s/2) M_U(s, N)$ , where  $N \sim 2 \log X$  and  $a_E(s/2)$  is an arithmetic factor depending only on the curve  $E$ .

In the conjecture, the relation between  $N$  and  $X$  is obtained by equating the mean density of eigenangles of matrices in the unitary group, and the mean density of nontrivial zeroes of the twisted  $L$ -functions  $L_E(s, \chi)$  at a fixed height. More precisely, let

$$N(T, \chi) = \#\{s \in \mathbb{C} : 0 < \operatorname{Re}(s) < 2, 0 < \operatorname{Im}(s) < T \text{ and } L_E(s, \chi) = 0\}$$

be the number of zeroes of  $L_E(s, \chi)$  in the critical strip up to height  $T$ . Then, using the Argument Principle, one proves that

$$N(T, \chi) = \frac{T}{\pi} \log \left( \frac{\sqrt{N_E} \mathfrak{f} T}{2\pi} \right) - \frac{T}{\pi} + O(\log T).$$

Equating the densities of zeroes at a fixed height  $T$ , one gets

$$\frac{N}{2\pi} \sim \frac{1}{\pi} \log \left( \frac{\sqrt{N_E} \mathfrak{f} T}{2\pi} \right) \Rightarrow N \sim 2 \log \mathfrak{f}$$

as stated in Conjecture 3.2. The arithmetic factor  $a_E(s/2)$  captures the arithmetic missing from the random matrix theory, and we can compute it for the family of cubic twists  $\mathcal{F}_E$  in the next section. The conjectural moments can then be compared with the empirical ones (see Figure 4), and our data is consistent with the Keating and Snaith Conjecture for the family  $\mathcal{F}_E$ .

From Conjecture 3.2, the probability density function for the distribution of the special values  $|L_E(1, \chi)|$  for  $L$ -functions  $L_E(s, \chi) \in \mathcal{F}_E$  is

$$\begin{aligned} P_E(x, X) &= \frac{1}{2\pi i} \int_{(c)} M_E(s, X) x^{-s-1} ds \\ &\sim \frac{1}{2\pi i} \int_{(c)} a_E(s/2) M_U(s, N) x^{-s-1} ds \\ &\sim a_E(-1/2) R(N) \quad \text{for small } x \quad (3-3) \\ &\sim a_E(-1/2) G^2(1/2) N^{1/4} \quad \text{for large } N. \quad (3-4) \end{aligned}$$

Figure 5 compares the empirical distribution with the probability density function  $P_U(x, N)$ .

Let  $k_E = 1$  when  $\omega_E = 1$ , and  $k_E = \sqrt{3}$  when  $\omega_E = -1$ . From (2-1) and Theorem 2.1, we have

$$\begin{aligned} |L_E(1, \chi)| &= \left| \frac{\Omega \tau(\chi) k_E n_\chi}{2 \mathfrak{f}} \right| \\ &= \frac{|\Omega k_E| n_\chi}{2 \sqrt{\mathfrak{f}}} = n_\chi \frac{c_E}{\sqrt{\mathfrak{f}}}, \quad (3-5) \end{aligned}$$

where  $c_E$  is a constant depending only on the curve  $E$ . We now use the properties of the integers  $n_\chi$  to give the

measure of the interval of vanishing for  $|L_E(1, \chi)|$ , i.e., we write

$$\operatorname{Prob} \{ |L_E(1, \chi)| = 0 \} = \operatorname{Prob} \{ |L_E(1, \chi)| < B(\mathfrak{f}) \}$$

for some function  $B(\mathfrak{f})$  of the conductor of the character. In view of Theorem 2.1 and Conjecture 2.2, we set

$$B(\mathfrak{f}) = \begin{cases} \frac{c_E 3^{\nu(\mathfrak{f})-1}}{\sqrt{\mathfrak{f}}} & \text{if } E \text{ has rational 3-torsion;} \\ \frac{c_E}{\sqrt{\mathfrak{f}}} & \text{otherwise,} \end{cases}$$

which completely determines our probabilistic model. Using the probability density function  $P_E(x, X) \sim a_E(-1/2) R(N)$  for small  $x$ , we have

$$\begin{aligned} \operatorname{Prob} \{ |L_E(1, \chi)| = 0 \} &= \int_0^{B(\mathfrak{f})} a_E(-1/2) R(N) dx \\ &= a_E(-1/2) R(N) B(\mathfrak{f}). \end{aligned}$$

We first consider the case where  $E$  does not have rational 3-torsion. Summing the probabilities, this gives

$$N_E(X) = c_E a_E(-1/2) R(N) \sum_{\mathfrak{f} \leq X} \frac{1}{\sqrt{\mathfrak{f}}}.$$

As

$$N(X) = \sum_{\mathfrak{f} \leq X} 1 \sim c_3 X \quad \text{as } X \rightarrow \infty$$

for some constant  $c_3$  (see Corollary 5.3), we obtain using partial summation

$$\begin{aligned} N_E(X) &\sim 2 c_3 c_E a_E(-1/2) R(N) X^{1/2} \\ &\sim 2^{5/4} G^2(1/2) c_3 c_E a_E(-1/2) X^{1/2} \log^{1/4} X \\ &\sim b_E X^{1/2} \log^{1/4} X \quad \text{as } X \rightarrow \infty. \end{aligned}$$

Similarly, if  $E$  has rational 3-torsion,

$$N_E(X) = c_E a_E(-1/2) R(N) \sum_{\mathfrak{f} \leq X} \frac{3^{\nu(\mathfrak{f})-1}}{\sqrt{\mathfrak{f}}}.$$

As

$$\sum_{\mathfrak{f} \leq X} 3^{\nu(\mathfrak{f})} \sim c_3' X \log^2 X \quad \text{as } X \rightarrow \infty$$

for some constant  $c_3'$  (see Corollary 5.6), we obtain using partial summation

$$\begin{aligned} N_E(X) &= \frac{2}{3} c_3' c_E a_E(-1/2) R(N) \sqrt{X} \log^2 X \\ &\sim \frac{2^{5/4}}{3} G^2(1/2) c_3' c_E a_E(-1/2) \sqrt{X} \log^{9/4} X \\ &\sim b_E X^{1/2} \log^{9/4} X \quad \text{as } X \rightarrow \infty. \end{aligned}$$

Hence the nature of the logarithmic factor seems to depend subtly on the arithmetic of the curve  $E$ . On the other hand, the heuristic model points to a growth rate satisfying

$$\log N_E(X) \sim \frac{1}{2} \log X.$$

This is supported by the empirical data in Section 6, and is consistent with the lower bounds for curves with rational 3-torsion proved in [Fearnley and Kisilevsky 00]. In fact, the empirical data seems to indicate a more refined conclusion of the type conjectured in [Conrey et al. 02],

$$N_E(X) \sim b_E X^{1/2} \log^{e_E} X$$

for some constants  $b_E$  and  $e_E$  depending on  $E$  (see Figures 2 and 3).

#### 4. MOMENTS

As mentioned in the last section, the work of Keating and Snaith led to some remarkable conjectures for the moments of special values in families of  $L$ -functions. Their conjectures agree with the known results for the first few integral moments of the Riemann zeta-function (see [Hardy and Littlewood 18, Ingham 26]), and with the known results for the first few integral moments of twists by quadratic Dirichlet characters (see [Goldfeld and Viola 79, Jutila 81, Soundararajan 00]). They also agree with the number-theoretic heuristics of [Conrey and Ghosh 92, Conrey and Gonek 01]. In order to verify that our empirical data also provide support for the Keating and Snaith conjectures, we need to compute the arithmetical factor  $a_E(s/2)$  of Conjecture 3.2.

Let  $k$  be a positive integer. We now consider the  $2k$ th moments

$$M_E(2k, X) = \frac{1}{N(X)} \sum_{f \leq X} |L_E(1, \chi)|^{2k},$$

where the sum runs over cubic characters of conductor less than  $X$ . In this special case, the Keating and Snaith conjectures can be stated as

**Conjecture 4.1. (Keating and Snaith Conjecture for cubic twists.)** *Let  $k$  be a positive integer. Then,*

$$M_E(2k, X) \sim a_E(k) g_k (2 \log X)^{k^2},$$

where

$$g_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

and  $a_E(k)$  is some arithmetical factor related to the curve  $E$ .

The arithmetical factor  $a_E(k)$  cannot be obtained from the random matrix model which contains no arithmetic, but can be computed using a number-theoretic heuristic as explained in [Conrey et al. 02]. We consider

$$L(s) = \frac{1}{N(X)} \sum_{f \leq X} |L_E(s, \chi)|^{2k}$$

in some half plane  $\text{Re}(s) > c$ . Following [Conrey et al. 02], one keeps only the diagonal terms, and neglects all error terms to write  $L(s)$  as  $\zeta(s)^{k^2} f(s)$  for some function  $f(s)$  analytic at  $s = 1$ . Then, specialising at  $s = 1$ ,  $\zeta(s)^{k^2}$  corresponds to  $(\log X)^{k^2}$  and  $f(s)$  to  $a_E(k)$ . One can then evaluate  $a_E(k)$  at any  $k \in \mathbb{C}$ , and in particular at  $k = -1/2$  as in Section 3.

We write

$$\begin{aligned} L(s) &= \frac{1}{N(X)} \sum_{f \leq X} |L_E(s, \chi)|^{2k} \\ &= \frac{1}{N(X)} \sum_{f \leq X} L_E(s, \chi)^k L_E(s, \bar{\chi})^k \\ &= \frac{1}{N(X)} \sum_{f \leq X} \sum_{n_1, \dots, n_{2k}} \frac{a_{n_1} \dots a_{n_{2k}}}{(n_1 \dots n_{2k})^s} \\ &\quad \times \chi(n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}) \\ &= \sum_{n_1, \dots, n_{2k}} \frac{a_{n_1} \dots a_{n_{2k}}}{(n_1 \dots n_{2k})^s} \frac{1}{N(X)} \\ &\quad \times \sum_{f \leq X} \chi(n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}). \end{aligned}$$

If  $n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}$  is a rational cube, the inner sum is

$$\frac{1}{N(X)} \sum_{\substack{f \leq X \\ (n_1 \dots n_{2k}, f)=1}} 1 \sim c_3(d)$$

as  $X \rightarrow \infty$ , where for  $d = n_1 \dots n_{2k}$  and  $c_3(d)$  as defined in Corollary 5.5.

For integers  $n_1, \dots, n_{2k}$ , let  $c(n_1, \dots, n_{2k}) = c_3(d)$  for  $d = n_1 \dots n_{2k}$ , and let  $\psi(n_1, \dots, n_{2k}) = 1$  when  $n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}$  is a rational cube, and  $\psi(n_1, \dots, n_{2k}) = 0$  otherwise. Considering only the contribution from the terms where  $n_1 \dots n_k n_{k+1}^{-1} \dots n_{2k}^{-1}$  is a rational cube, we obtain

$$\begin{aligned} L(s) &\sim \sum_{n_1, \dots, n_{2k}} \frac{a_{n_1} \dots a_{n_{2k}}}{(n_1 \dots n_{2k})^s} c(n_1, \dots, n_{2k}) \psi(n_1, \dots, n_{2k}) \\ &= \sum_{n_1, \dots, n_{2k}} f(n_1, \dots, n_{2k}), \end{aligned}$$

where  $f(n_1, \dots, n_{2k})$  is a multiplicative function of the  $2k$  variables. Then,  $L(s)$  has the Euler product

$$\begin{aligned} L(s) &= \prod_p \sum_{\substack{e_1, \dots, e_{2k} \\ e_1 + \dots + e_k \equiv e_{k+1} + \dots + e_{2k} \pmod{3}}} \frac{a_p^{e_1} \dots a_p^{e_{2k}}}{(p^{e_1 + \dots + e_{2k}})^s} c(p^{e_1}, \dots, p^{e_{2k}}) \\ &= \prod_{p \equiv 2 \pmod{3}} \sum_{\substack{e_1, \dots, e_{2k} \\ e_1 + \dots + e_k \equiv e_{k+1} + \dots + e_{2k} \pmod{3}}} \frac{a_p^{e_1} \dots a_p^{e_{2k}}}{(p^{e_1 + \dots + e_{2k}})^s} \\ &\quad \times \prod_{p \equiv 1 \pmod{3}} \left( 1 + \frac{p}{p+2} \right) \\ &\quad \times \sum_{\substack{e_1, \dots, e_{2k} \\ e_1 + \dots + e_k \equiv e_{k+1} + \dots + e_{2k} \pmod{3}}}^* \frac{a_p^{e_1} \dots a_p^{e_{2k}}}{(p^{e_1 + \dots + e_{2k}})^s} \\ &\quad \times \prod_{p=3} \left( 1 + \frac{9}{11} \sum_{\substack{e_1, \dots, e_{2k} \\ e_1 + \dots + e_k \equiv e_{k+1} + \dots + e_{2k} \pmod{3}}}^* \frac{a_p^{e_1} \dots a_p^{e_{2k}}}{(p^{e_1 + \dots + e_{2k}})^s} \right) \\ &= \prod_p E(p, s) \end{aligned} \tag{4-1}$$

where  $*$  indicates that the term  $e_1 = \dots = e_{2k} = 0$  is missing from the sum.

**Lemma 4.2.** *Let  $E(p, s)$  be the Euler factor defined by Equation (4-2). For any  $\epsilon > 0$ ,*

$$E(p, s) = 1 + k^2 \frac{a_p^2}{p^{2s}} + O_k(p^{-3s+\epsilon}).$$

*Proof:* Suppose  $p \equiv 2 \pmod{3}$ . Then,

$$E(p, s) = \sum_{\substack{e_1, \dots, e_{2k} \\ e_1 + \dots + e_k \equiv e_{k+1} + \dots + e_{2k} \pmod{3}}} \frac{a_p^{e_1} \dots a_p^{e_{2k}}}{(p^{e_1 + \dots + e_{2k}})^s}.$$

Using  $n = \sum_{i=1}^{2k} e_i$ ,  $n_1 = \sum_{i=1}^k e_i$ , and  $n_2 = \sum_{i=k+1}^{2k} e_i$ , and collecting the terms with the same  $n$ , we write the above sum as

$$\sum_{n=0}^{\infty} \frac{c_n}{p^{ns}}.$$

Clearly,  $c_0 = 1$  and  $c_1 = 0$ . For  $n = 2$ , the only choice with  $n_1 \equiv n_2 \pmod{3}$  is  $n_1 = n_2 = 1$ . There are  $k^2$  tuples  $(e_1, \dots, e_{2k})$  with  $n_1 = n_2 = 1$  and for each such tuple,

$$\frac{a_p^{e_1} \dots a_p^{e_{2k}}}{(p^{e_1 + \dots + e_{2k}})^s} = \frac{a_p^2}{p^{2s}},$$

and  $c_2 = k^2 a_p^2$ . In general, there are  $O(n^k)$  tuples with  $\sum_{i=1}^{2k} e_i = n$ , and for each such tuple  $a_p^{e_1} \dots a_p^{e_{2k}}$  is at

most  $O(p^{2kn\epsilon})$  for any  $\epsilon > 0$ . This gives

$$\begin{aligned} E(p, s) &= 1 + k^2 \frac{a_p^2}{p^{2s}} + O_k \left( \sum_{n=3}^{\infty} (p^{-s+\epsilon})^n \right) \\ &= 1 + k^2 \frac{a_p^2}{p^{2s}} + O_k(p^{-3s+\epsilon}) \end{aligned}$$

for any  $\epsilon > 0$ . The proof for  $p \equiv 0, 1 \pmod{3}$  is similar.  $\square$

From Lemma 4.2,  $L(s)$  has a pole of order  $k^2$  at  $s = 1$  as does the Rankin-Selberg convolution

$$L(E \otimes E, s) = \sum_{n=1}^{\infty} \left( \frac{a_n}{\sqrt{n}} \right)^2 \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{a_n^2}{n^{s+1}}$$

(see [Iwaniec 97, Section 13.8] for more details). Then,

$$L(s) = \zeta(s)^{k^2} \prod_p \left( 1 - \frac{1}{p^s} \right)^{k^2} E(p, s),$$

where

$$\prod_p \left( 1 - \frac{1}{p^s} \right)^{k^2} E(p, s)$$

is analytic at  $s = 1$ . We then set

$$a_E(k) = \prod_p \left( 1 - \frac{1}{p} \right)^{k^2} E(p, 1). \tag{4-3}$$

We now write the Euler factors  $E(p, 1)$  in a more suitable form using the multiplicativity of the  $a_p$ .

**Lemma 4.3.** *Let  $\rho$  be a primitive third root of 1, and let*

$$F(p) = \sum_{\substack{e_1, \dots, e_{2k} \\ e_1 + \dots + e_k \equiv e_{k+1} + \dots + e_{2k} \pmod{3}}} \frac{a_p^{e_1} \dots a_p^{e_{2k}}}{p^{e_1 + \dots + e_{2k}}}.$$

Then, as a formal series,  $F(p)$  is

$$\left\{ \begin{array}{l} \frac{1}{3} \left( 1 - \frac{a_p}{p} + \frac{1}{p} \right)^{-2k} + \frac{2}{3} \left( 1 - \frac{\rho a_p}{p} + \frac{\rho^2}{p} \right)^{-k} \\ \quad \times \left( 1 - \frac{\rho^{-1} a_p}{p} + \frac{\rho^{-2}}{p} \right)^{-k} \quad \text{for } p \nmid N_E; \\ \frac{1}{3} \left( 1 - \frac{a_p}{p} \right)^{-2k} + \frac{2}{3} \left( 1 - \frac{\rho a_p}{p} \right)^{-k} \\ \quad \times \left( 1 - \frac{\rho^{-1} a_p}{p} \right)^{-k} \quad \text{for } p \mid N_E. \end{array} \right.$$

*Proof:* Using  $n_1 = \sum_{i=1}^k e_i$ ,  $n_2 = \sum_{i=k+1}^{2k} e_i$ , and the characteristic function

$$\frac{1}{3} (1 + \rho^{n_1 - n_2} + \rho^{-n_1 + n_2}) = \begin{cases} 1 & \text{if } n_1 \equiv n_2 \pmod{3}; \\ 0 & \text{otherwise,} \end{cases}$$

we have the formal equalities

$$\begin{aligned} & \sum_{\substack{e_1, \dots, e_{2k} \\ n_1 \equiv n_2 \pmod{3}}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{p^{e_1 + \dots + e_{2k}}} = \\ & \frac{1}{3} \sum_{e_1, \dots, e_{2k}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{p^{e_1 + \dots + e_{2k}}} + \frac{2}{3} \sum_{e_1, \dots, e_{2k}} \frac{a_{p^{e_1}} \dots a_{p^{e_{2k}}}}{p^{e_1 + \dots + e_{2k}}} \rho^{n_1 - n_2} \\ & = \frac{1}{3} \left( \sum_{e=1}^{\infty} \frac{a_{p^e}}{p^e} \right)^{2k} + \frac{2}{3} \left( \sum_{e=1}^{\infty} \frac{a_{p^e} \rho^e}{p^e} \right)^k \left( \sum_{e=1}^{\infty} \frac{a_{p^e} \rho^{-e}}{p^e} \right)^k. \end{aligned} \tag{4-4}$$

Using the multiplicativity of the Fourier coefficients  $a_n$ , we get for any  $\alpha \in \mathbb{C}^*$

$$\sum_{e=1}^{\infty} \frac{a_{p^e} \alpha^e}{p^e} = \begin{cases} \left(1 - \frac{\alpha a_p}{p} + \frac{\alpha^2}{p}\right)^{-1} & \text{if } p \nmid N_E; \\ \left(1 - \frac{\alpha a_p}{p}\right)^{-1} & \text{if } p \mid N_E. \end{cases}$$

Replacing in (4-4), this proves the lemma. □

Using the above lemma in (4-1), we can write the Euler factors as

$$E(p, 1) = \begin{cases} \frac{p}{p+2} F(p) + \frac{2}{p+2} & \text{for } p \equiv 1 \pmod{3}; \\ F(p) & \text{for } p \equiv 2 \pmod{3}; \\ \frac{9}{11} F(p) + \frac{2}{11} & \text{for } p = 3. \end{cases}$$

This expression is now valid for all  $k \in \mathbb{C}$ , and not only integers. This value of  $a_E(k)$  is used to compute the conjectural moments of Figure 4.

### 5. NUMBER OF CUBIC CONDUCTORS

We give in this section asymptotics for

$$N(X) = \# \{ \text{cubic characters of conductor } \mathfrak{f} \leq X \}$$

$$N_d(X) = \# \{ \text{cubic characters of conductor } \mathfrak{f} \leq X$$

$$\text{with } (\mathfrak{f}, d) = 1 \}$$

$$S(X) = \sum_{\mathfrak{f} \leq X} 3^{\nu(\mathfrak{f})}$$

which are needed in the rest of the paper. The estimate for  $N(X)$  can also be found in [Cohen et al. 02].

**Lemma 5.1.** *Let  $\chi$  be a cubic character of conductor  $\mathfrak{f}$ . Then,  $\mathfrak{f} = (9)^\alpha p_1 \dots p_t$ , where  $p_1, \dots, p_t$  are distinct primes congruent to 1 modulo 3, and  $\alpha = 0$  or 1. Furthermore, for each such conductor, there are  $2^{t+\alpha} = 2^{\nu(\mathfrak{f})}$  distinct cubic characters with conductor  $\mathfrak{f}$ .*

*Proof:* A cubic Dirichlet character of conductor  $\mathfrak{f}$  can be written uniquely as a product of cubic Dirichlet characters of prime power conductor. Since the prime power conductors of cubic characters are either 9 or a prime  $p$  congruent to 1 modulo 3, the first statement of the lemma follows. Furthermore, writing  $\mathfrak{f} = (9)^\alpha p_1 \dots p_t$ , we see that there are  $2^{\alpha+t} = 2^{\nu(\mathfrak{f})}$  cubic characters with conductor  $\mathfrak{f}$  since there are two characters of order 3 for each such prime power conductor. □

Let  $a(n)$  be the number of cubic characters of conductor  $n$ . Then, it follows from the above lemma that

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \pmod{3}} \left(1 + \frac{2}{p^s}\right),$$

and the above series converges for  $\text{Re}(s) > 1$ . We then have to analyse the analytic behavior of  $L(s)$  at  $s = 1$ . We find out that

**Proposition 5.2.**  $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$

has a simple pole at  $s = 1$  with residue

$$c_3 = \frac{11\sqrt{3}}{18\pi} \prod_{p \equiv 1 \pmod{3}} \left(1 - \frac{2}{p(p+1)}\right).$$

*Proof:*

$$\begin{aligned} L(s) &= \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \pmod{3}} \left(1 + \frac{2}{p^s}\right) \\ &= g(s) \prod_{p \equiv 1 \pmod{3}} \left(1 - \frac{1}{p^s}\right)^{-2}, \end{aligned}$$

where

$$\begin{aligned} g(s) &= \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \pmod{3}} \left(1 - \frac{1}{p^s}\right)^2 \left(1 + \frac{2}{p^s}\right) \\ &= \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \pmod{3}} \left(1 - \frac{3}{p^{2s}} + \frac{2}{p^{3s}}\right) \end{aligned}$$

is analytic at  $s = 1$ .



Let  $K$  be the field obtained by adding a third root of 1. Then,  $K = \mathbb{Q}(\sqrt{-3})$  and the Dedekind zeta function

$$\zeta_K(s) = \left(1 - \frac{1}{3^s}\right)^{-1} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^s}\right)^{-2} \times \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^{2s}}\right)^{-1}$$

has a simple pole at  $s = 1$  with residue

$$\rho = \frac{2^{r+s} \pi^s \text{reg}(K) h_K}{\omega_K |\Delta_K|^{1/2}} = \frac{\pi}{3\sqrt{3}}.$$

Using this fact, we get

$$\begin{aligned} L(s) &= g(s) \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{1}{p^s}\right)^{-2} \\ &= g(s) \left(1 - \frac{1}{3^s}\right) \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^{2s}}\right) \zeta_K(s) \\ &= h(s) \zeta_K(s), \end{aligned}$$

where

$$\begin{aligned} h(s) &= \left(1 + \frac{2}{9^s}\right) \left(1 - \frac{1}{3^s}\right) \\ &\quad \times \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{3}{p^{2s}} + \frac{2}{p^{3s}}\right) \\ &\quad \times \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^{2s}}\right) \end{aligned}$$

is analytic at  $s = 1$ . One computes

$$\begin{aligned} h(1) &= \frac{11}{9} \frac{2}{3} \prod_{p \equiv 1,2 \pmod 3} \left(1 - \frac{1}{p^2}\right) \\ &\quad \times \prod_{p \equiv 1 \pmod 3} \frac{(1 - 3p^{-2} + 2p^{-3})}{(1 - p^{-2})} \\ &= \frac{11}{12\zeta(2)} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{2}{p(p+1)}\right). \end{aligned}$$

Then,  $L(s)$  has a simple pole at  $s = 1$  with residue

$$\begin{aligned} c_3 &= \frac{\pi}{3\sqrt{3}} h(1) = \frac{11\sqrt{3}}{18\pi} \prod_{p \equiv 1 \pmod 3} \left(1 - \frac{2}{p(p+1)}\right) \\ &= 0.3170564\dots \quad \square \end{aligned}$$

**Corollary 5.3.**  $N(X) \sim c_3 X$  as  $X \rightarrow \infty$ .

*Proof:* Using Proposition 5.2 and the Tauberian Theorem (see, for example, [Murty 01]), we have

$$N(X) = \sum_{n \leq X} a(n) \sim c_3 X. \quad \square$$

**Remark 5.4.** The constant  $c_{\mathbb{Q}}(C_3)$  on [Cohen et al. 02, page 104] is half of our constant as there are two characters per cyclic cubic field.

**Corollary 5.5.** Let  $d$  be a positive integer. Then,

$$N_d(X) \sim c_3(d) N(X) \quad \text{as } X \rightarrow \infty,$$

where

$$c_3(d) = \begin{cases} \prod_{p \equiv 1 \pmod 3} \frac{p}{p+d} & \text{for } 3 \nmid d; \\ \frac{9}{11} \prod_{p \equiv 1 \pmod 3} \frac{p}{p+d} & \text{for } 3 \mid d. \end{cases}$$

*Proof:* Suppose that  $3 \nmid d$ . Let  $b(n)$  be the number of cubic characters of conductor  $n$  when  $(n, d) = 1$ , and  $b(n) = 0$  otherwise. We consider the  $L$ -function

$$\begin{aligned} L_2(s) &= \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \\ &= \left(1 + \frac{2}{9^s}\right) \prod_{p \equiv 1 \pmod 3} \left(1 + \frac{2}{p^s}\right)^{-1} \\ &\quad \times \prod_{p \equiv 1 \pmod 3} \left(1 + \frac{2}{p^s}\right) \\ &= f(s) L(s), \end{aligned}$$

where

$$f(s) = \prod_{p \equiv 1 \pmod 3} \left(1 + \frac{2}{p^s}\right)^{-1} = \prod_{p \equiv 1 \pmod 3} \frac{p^s}{p^s + 2}$$

is analytic at  $s = 1$ . Then, using Proposition 5.2 and the Tauberian Theorem, this gives

$$\sum_{n \leq X} b(n) \sim f(1) c_3 X,$$

and the result follows. The proof for  $3 \mid d$  is similar.  $\square$

**Corollary 5.6.**

$$S(X) = \sum_{f \leq X} 3^{\nu(f)} \sim c_3' X \log^2 X \quad \text{as } X \rightarrow \infty$$

for some constant  $c_3'$ .

*Proof:* Using Lemma 5.1, we write

$$\sum_{f \leq X} 3^{\nu(f)} = \sum_{n \leq X} a(n),$$

where

$$a(n) = \begin{cases} 6^{\nu(n)} & \text{if } n \text{ is the conductor} \\ & \text{of cubic character;} \\ 0 & \text{otherwise.} \end{cases}$$

Now, working exactly as above, consider the  $L$ -function

$$\begin{aligned} L(s) &= \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \left(1 + \frac{6}{9^s}\right) \prod_{p \equiv 1 \pmod 3} \left(1 + \frac{6}{p^s}\right) \\ &= \zeta_K(s)^3 g(s), \end{aligned}$$

where  $g(s)$  is analytic at  $s = 1$ . Then,  $L(s)$  has a pole of order 3 with residue  $c'_3$  (say) at  $s = 1$ , and it follows from the Tauberian Theorem that

$$S(X) = \sum_{n \leq X} a(n) \sim c'_3 X \log^2 X. \quad \square$$

### 6. NUMERICAL DATA

In order to effectively compute twisted  $L$ -functions, we use the series representation

$$\begin{aligned} L_E(1, \chi) &= \sum_{n=1}^{\infty} \frac{a_n}{n} \exp\left(-\frac{2\pi n}{f\sqrt{N_E}}\right) \\ &\times \left(\chi(n) + \omega_E \chi(N_E) \frac{\tau(\chi)^2}{f} \bar{\chi}(n)\right) \end{aligned}$$

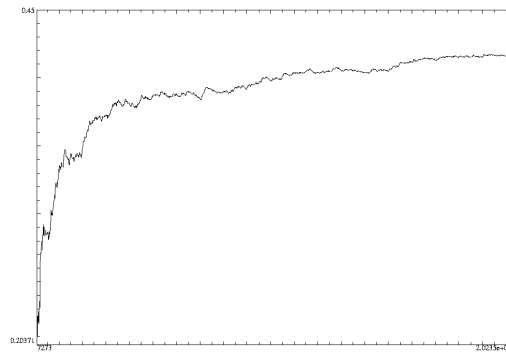
derived from the functional equation (1-2). This series is rapidly convergent for small values of  $f\sqrt{N_E}$  and has an easily computable (though conservative) bound on the truncation error after  $k$  terms, namely

$$\frac{4}{1-q} q^k \quad \text{where } q = \exp\left(-\frac{2\pi}{f\sqrt{N_E}}\right).$$

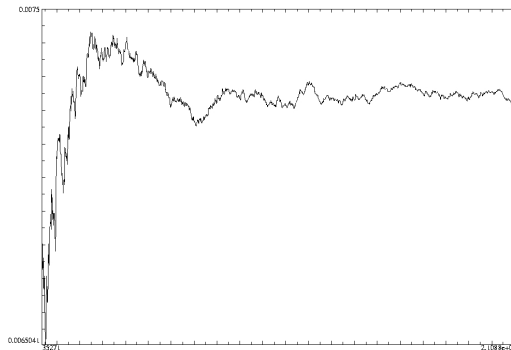
A small sample of eight elliptic curves was selected and computer runs of varying lengths were performed to establish a database of cubic twists. The curves were chosen to represent a variety of torsion and rank. Curves of small conductor are chosen in order to maintain precision in the calculations; in the case of E11A and E14A, up to 16,000,000 terms were summed for the highest conductor twists. The computations were greatly assisted by the fact that  $n_\chi$  is an integer. At least four-decimal-place accuracy was maintained in these integers throughout the calculations. The empirical results are shown in Figures 1-6.

| Curve | Torsion | Rank | Maximal conductor | Number of characters | Number of vanishing |
|-------|---------|------|-------------------|----------------------|---------------------|
| E11A  | 5       | 0    | 2,023,513         | 320,795              | 1152                |
| E14A  | 6       | 0    | 2,108,767         | 260,001              | 4347                |
| E15A  | 8       | 0    | 399,979           | 51,890               | 807                 |
| E32A  | 4       | 0    | 300,217           | 47,577               | 117                 |
| E36A  | 6       | 0    | 283,051           | 36,718               | 346                 |
| E37A  | 1       | 1    | 279,211           | 41,991               | 559                 |
| E37B  | 3       | 0    | 364,723           | 54,830               | 1899                |
| E389A | 1       | 2    | 99,991            | 15,851               | 408                 |

**FIGURE 1.** The eight elliptic curves selected for this study with the sample sizes used. The number of characters is the number of characters  $\chi$  with conductor  $f$  smaller than the maximal conductor and such that  $(f, N_E) = 1$ . For each conductor  $f$ , there are 2 conjugate cubic characters  $\chi, \bar{\chi}$  with  $L_E(1, \bar{\chi}) = \overline{L_E(1, \chi)}$ , and only one of them is counted. The number of vanishing is the number of such characters with  $L_E(1, \chi) = 0$ .



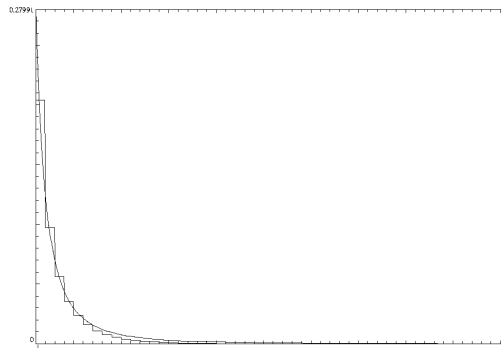
**FIGURE 2.** Ratio of the empirical  $N_E(X)$  with  $\sqrt{X} \log^{1/4} X$  for the curve E11A and  $1 \leq X \leq 2,023,513$ .



**FIGURE 3.** Ratio of the empirical  $N_E(X)$  with  $\sqrt{X} \log^{9/4} X$  for the curve E14A and  $1 \leq X \leq 2,108,767$ .

| Curve |             | $s = 1/2$ | $s = 1$ | $s = 3/2$ | $s = 2$ | $s = 3$ | $s = 4$ |
|-------|-------------|-----------|---------|-----------|---------|---------|---------|
| E11A  | Empirical   | 1.420     | 2.878   | 7.349     | 22.02   | 274.3   | 4617.   |
|       | Conjectural | 1.436     | 2.962   | 7.621     | 22.34   | 227.7   | 2288.   |
|       | Ratio       | 0.990     | 0.972   | 0.964     | 0.985   | 1.205   | 2.017   |
| E14A  | Empirical   | 1.268     | 2.196   | 4.696     | 11.76   | 104.6   | 1302.   |
|       | Conjectural | 1.282     | 2.243   | 4.796     | 11.66   | 83.18   | 599.8   |
|       | Ratio       | 0.990     | 0.979   | 0.979     | 1.008   | 1.257   | 2.171   |
| E15A  | Empirical   | 1.384     | 2.609   | 5.995     | 15.86   | 149.4   | 1874.   |
|       | Conjectural | 1.400     | 2.677   | 6.175     | 15.87   | 117.9   | 816.9   |
|       | Ratio       | 0.989     | 0.974   | 0.971     | 1.000   | 1.266   | 2.294   |
| E32A  | Empirical   | 1.221     | 1.928   | 3.641     | 7.863   | 49.23   | 407.2   |
|       | Conjectural | 1.225     | 1.946   | 3.629     | 7.468   | 35.42   | 154.8   |
|       | Ratio       | 0.996     | 0.991   | 1.003     | 1.052   | 1.389   | 2.630   |
| E36A  | Empirical   | 1.184     | 1.792   | 3.202     | 6.491   | 35.34   | 253.6   |
|       | Conjectural | 1.193     | 1.814   | 3.188     | 6.101   | 24.29   | 86.90   |
|       | Ratio       | 0.992     | 0.988   | 1.004     | 1.063   | 1.454   | 2.919   |
| E37A  | Empirical   | 1.468     | 3.196   | 8.935     | 29.50   | 441.3   | 8592.   |
|       | Conjectural | 1.483     | 3.280   | 9.197     | 29.40   | 341.3   | 3547.   |
|       | Ratio       | 0.990     | 0.974   | 0.972     | 1.003   | 1.292   | 2.421   |
| E37B  | Empirical   | 1.119     | 1.656   | 2.946     | 6.060   | 36.15   | 311.3   |
|       | Conjectural | 1.127     | 1.646   | 2.829     | 5.395   | 22.69   | 93.66   |
|       | Ratio       | 0.993     | 1.006   | 1.041     | 1.123   | 1.593   | 3.323   |
| E389A | Empirical   | 1.594     | 3.960   | 13.08     | 52.36   | 1210.   | 38636.  |
|       | Conjectural | 1.614     | 4.088   | 13.68     | 53.95   | 1015.   | 17901.  |
|       | Ratio       | 0.988     | 0.969   | 0.956     | 0.971   | 1.192   | 2.158   |

**FIGURE 4.** Moments of cubic twists for the eight selected elliptic curves. The empirical moments are the moments (3–2) for various values of  $s$  and up to  $X$  given in Figure 1. The conjectural moments are computed following Conjecture 3.2 with the arithmetic factor  $a_E(s)$  of Section 4. For small values of  $s$ , our data supports Conjecture 3.2. The divergence between the conjectural and empirical data for higher moments can be explained by the asymptotic nature of the moments. We use only the leading order asymptotic for the conjectural moments, but there are several other terms which will contribute strongly when the sample size is relatively small [Conrey et al. 02].



**FIGURE 5.** Histogram of the empirical values  $|L_E(1, \chi)|$  for the curve  $E14$  and the sample size of Figure 1 superimposed with the probability distribution function  $P_U(x, N)$  with  $N = 12$ . The probability distribution is computed using the approximations of [Keating and Snaith 00a].

| Curve         | E11A  | E14A | E15A | E32A | E36A | E37A | E37B | E389A |
|---------------|-------|------|------|------|------|------|------|-------|
| Factor        | 10    | 6    | 8    | 4    | 2    | 4    | 12   | 12    |
| $n_\chi = 0$  | 1152  | 4347 | 807  | 117  | 346  | 559  | 1899 | 408   |
| $n_\chi = 1$  | 1662  | 344  | 287  | 695  | 118  | 1096 | 150  | 962   |
| $n_\chi = 2$  | 1117  | 440  | 229  | 509  | 108  | 645  | 136  | 493   |
| $n_\chi = 3$  | 2676  | 1379 | 414  | 209  | 683  | 1264 | 1419 | 761   |
| $n_\chi = 4$  | 1328  | 336  | 660  | 201  | 54   | 799  | 171  | 521   |
| $n_\chi = 5$  | 1069  | 288  | 219  | 515  | 97   | 657  | 147  | 427   |
| $n_\chi = 6$  | 1711  | 2707 | 470  | 194  | 535  | 715  | 785  | 374   |
| $n_\chi = 7$  | 1827  | 390  | 327  | 789  | 161  | 879  | 188  | 542   |
| $n_\chi = 8$  | 1125  | 442  | 504  | 768  | 107  | 527  | 105  | 330   |
| $n_\chi = 9$  | 2578  | 2365 | 378  | 209  | 959  | 836  | 2853 | 414   |
| $n_\chi = 10$ | 631   | 293  | 174  | 329  | 58   | 376  | 88   | 223   |
| $n_\chi = 11$ | 1336  | 299  | 227  | 534  | 95   | 533  | 118  | 325   |
| $n_\chi = 12$ | 2183  | 2188 | 872  | 66   | 260  | 702  | 993  | 301   |
| $n_\chi = 13$ | 1607  | 365  | 274  | 666  | 122  | 624  | 149  | 327   |
| $n_\chi = 14$ | 1015  | 489  | 229  | 559  | 119  | 429  | 98   | 225   |
| $n_\chi = 15$ | 1625  | 1044 | 288  | 164  | 466  | 618  | 810  | 252   |
| $n_\chi = 16$ | 1182  | 330  | 790  | 516  | 71   | 388  | 103  | 216   |
| $n_\chi = 17$ | 1262  | 273  | 217  | 519  | 104  | 439  | 99   | 199   |
| $n_\chi = 18$ | 1624  | 5605 | 353  | 138  | 766  | 385  | 1489 | 187   |
| $n_\chi = 19$ | 1433  | 331  | 256  | 605  | 106  | 459  | 151  | 221   |
| $n_\chi = 20$ | 770   | 272  | 388  | 131  | 44   | 349  | 60   | 162   |
| $n_\chi = 21$ | 2562  | 1439 | 409  | 219  | 679  | 686  | 1173 | 250   |
| $n_\chi = 22$ | 786   | 255  | 139  | 327  | 65   | 237  | 75   | 116   |
| $n_\chi = 23$ | 1193  | 276  | 203  | 467  | 80   | 400  | 84   | 167   |
| $n_\chi = 24$ | 1634  | 2304 | 721  | 211  | 487  | 405  | 606  | 138   |
| $n_\chi = 25$ | 952   | 241  | 203  | 478  | 73   | 323  | 116  | 136   |
| $n_\chi = 26$ | 852   | 385  | 187  | 477  | 90   | 306  | 67   | 112   |
| $n_\chi = 27$ | 2169  | 2571 | 319  | 182  | 919  | 450  | 3111 | 176   |
| $n_\chi = 28$ | 1199  | 315  | 526  | 183  | 43   | 346  | 107  | 138   |
| $n_\chi = 29$ | 1119  | 236  | 156  | 408  | 91   | 330  | 85   | 128   |
| $n_\chi = 30$ | 920   | 1719 | 218  | 80   | 256  | 254  | 461  | 87    |
| Maximal value | 10139 | 9872 | 4250 | 1867 | 2322 | 1968 | 1935 | 1443  |

**FIGURE 6.** Frequency distribution for  $n_\chi$ . Each line of the table is the number of incidences of  $n_\chi = 0, 1, \dots, 30$  for all characters with conductor  $1 \leq f \leq X$  for the sample sizes given in Figure 1. The maximal value is the largest  $n_\chi$  in this sample. The factor of the first line is the multiple of the period  $\Omega_E$  used to make the values  $n_\chi$  integral without a common factor.

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