The Pentagram Map is Recurrent

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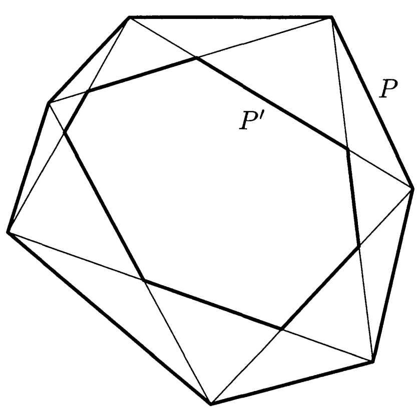
CONTENTS

The *pentagram map* is defined on the space of convex n-gons (considered up to projective equivalence) by drawing the diagonals that join second-nearest-neighbors in an n-gon and taking the new n-gon formed by the intersections. We prove that this map is recurrent; thus, for almost any starting polygon, repeated application of the pentagram map will show a near copy of the starting polygon appear infinitely often under various perspectives.



1. INTRODUCTION

Given a convex n-gon, P, one can connect every other vertex with a line segment, creating a star-like figure called the pentagram. Part of the pentagram defines a new convex n-gon, P', as shown (for $n = 7$) on the left. Iterating, one defines P'', P''', etc. As suggested by Figure 1 below, the map $P \to P''$ defines a map between labelled n-gons. We call the map $P \to P''$ the *pentagram map*. We studied the pentagram map in [Schwartz 1992].

The natural setting for the pentagram map is the projective plane, $\mathbb{RP}^2$. (See Section 2 for background information on the projective plane.) Say that two labelled strictly convex n-gons are *equivalent* if there is a projective transformation of $\mathbb{RP}^2$ which takes one to the other. Let Σ_n denote the

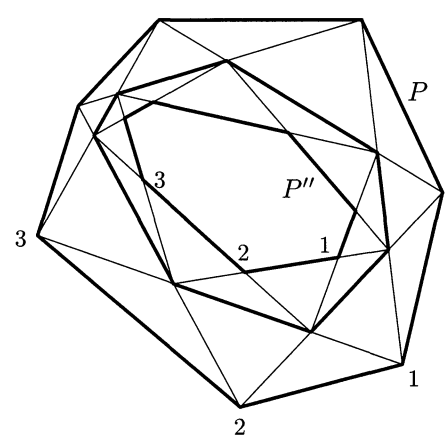


FIGURE 1. Definition of the pentagram map.

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space of equivalence classes of strictly convex n-gons. As we saw in [Schwartz 1992], the space Σ_n is diffeomorphic to $\mathbb{R}^{2n-8}$.

The pentagram map commutes with real projective transformations, and so induces a mapping $T_n : \Sigma_n \to \Sigma_n$, for all $n \geq 5$. We saw in [Schwartz 1992] that the maps T_5 and T_6^2 act trivially on Σ_5 and Σ_6 respectively. For $n \geq 7$, the map T_n does not have finite order. In this paper we verify Conjecture 4.1 of our earlier paper:

Theorem 1.1. *T_n is recurrent on Σ_n, for all $n \geq 5$.*

By *recurrent*, we mean that almost every point is an accumulation point of its own forward orbit. Our result has the following geometric interpretation. Begin with a random choice of convex polygon P, and look at the sequence $P, T_n(P), T_n^2(P), \ldots$. A near copy of P appear infinitely often, from varying perspectives.

Here is an outline of our proof of Theorem 1.1. In our earlier paper we constructed a smooth function $f : \Sigma_n \to [1, \infty)$ with the following properties:

1. $f^{-1}[1, r]$ is compact for any real number $r > 1$.
2. $f \circ T_n = f$.

To help keep this paper self-contained, and also to set up some notation needed for later steps, we will construct f in a new way and sketch proofs of the two properties above. This is done in Section 3.

The two properties above show that $f^{-1}[1, r]$ is a compact T_n-invariant set. The main step in the proof of Theorem 1.1 is:

Lemma 1.2 (Volume Lemma). *There exists a smooth volume form μ_n on Σ_n which is preserved by T_n.*

We will prove this in Section 4.

By Sard's theorem, almost every choice of r yields a smooth manifold-with-boundary $X = f^{-1}[1, r]$. The map $T = T_n$ acts as a volume preserving map on X. To deal with this situation we invoke a special case of the Poincaré Recurrence Lemma. Since the proof is short, we include it here. See [Arnol'd 1978] for more details.

Lemma 1.3 (Poincaré Recurrence). *Suppose that $X \subset \mathbb{R}^m$ is a smooth compact manifold-with-boundary. Suppose $T : X \to X$ preserves a smooth volume form, defined in a neighborhood of X. Then almost every point $x \in X$ is an accumulation point of the sequence $\{T^k(x) \mid k \in \mathbb{N}\}$.*

Proof. For any $\varepsilon > 0$, let N_ε be the Borel set of points $x \in X$ such that the sequence $\{T^k(x) \mid k \in \mathbb{N}\}$ avoids the ε-neighborhood of x. If N_ε has positive measure, we can find a δ-ball $B \subset X$ such that $\beta = B \cap N_\varepsilon$ has positive measure. Here we take $\delta < \varepsilon$. Since X has finite volume, the sets in $\{T^k(\beta)\}$ are not all pairwise disjoint. Hence, $T^i(\beta) \cap T^j(\beta)$ for some pair $i, j \in \mathbb{N}$. Setting $k = j - i$, we have $T^k(\beta) \cap \beta \neq \varnothing$. This contradiction shows that N_ε has measure zero. Since ε is arbitrary, we are done. □

Applying the Poincaré Recurrence Lemma, we see that T_n is recurrent on $f^{-1}[1, r]$ for almost every choice of real $r > 1$. We choose a sequence $r_1, r_2, \ldots$ which increases unboundedly, such that T_n is recurrent on $f^{-1}[1, r_k]$ for all k. Since these sets exhaust Σ_n, we see that T_n is recurrent on Σ_n.

The recurrence property is more general than our result suggests. Let Ω_n be the set of projective equivalence classes of n-gons. These n-gons need not be convex. T_n is defined on a full measure set of Ω_n. As we conjectured in [Schwartz 1992], it seems that $T_n : \Omega_n \to \Omega_n$ is also recurrent. Our proof here has nothing to say about this.

This paper relies on some basic projective geometry. In Section 2 we give some background information on this subject. More information on projective geometry can be found in [Hilbert and Cohn-Vossen 1950], for instance.

All the ideas for our proof, save one, came from computer experimentation. In particular, we discovered all the computations in the paper numerically. On the negative side, some of our computations are unmotivated. We don't really understand why they are true. On the positive side, we know for sure that they *are* true. For instance, the main thrust in this paper is that a certain collection of matrices always has determinant 1. We computed this determinant on millions of random samples from this family and numerically it was as close to 1 as one could expect from a finite precision calculation.

A key idea in this paper, which did not come from computer experimentation, is the notion of the corner invariant f_p, recalled here in Section 3A. We originally learned about f_p from John Conway.

2. PROJECTIVE GEOMETRY

The Projective Plane

The real projective plane, $\mathbb{RP}^2$, is the space of one-dimensional subspaces of $\mathbb{R}^3$. The ordinary plane, $\mathbb{R}^2$, can be considered as a subset of $\mathbb{RP}^2$ in the following way: The linear subspace spanned by the vector $(x, y, 1)$ is identified with the point $(x, y) \in \mathbb{R}^2$. Under this embedding, $\mathbb{RP}^2$ is a compactification of $\mathbb{R}^2$. It is not hard to see that $\mathbb{RP}^2$ naturally has the structure of a smooth manifold.

A *line* in $\mathbb{RP}^2$ is the union of all 1-dimensional subspaces contained in a given 2-dimensional subspace. Lines in $\mathbb{RP}^2$ are actually topologically equivalent to circles. The set $\mathbb{RP}^2 - \mathbb{R}^2$ is exactly a line in $\mathbb{RP}^2$, known as the *line at infinity*. Every ordinary line in $\mathbb{R}^2$ extends to a line in $\mathbb{RP}^2$ by adding in the point where it intersects the line at infinity.

The lines and points in $\mathbb{RP}^2$ are intimately related. Given any two distinct points in $\mathbb{RP}^2$ there is a unique line which contains both of them. Likewise, given any two distinct lines in $\mathbb{RP}^2$ there is unique point contained on both.

Projective Transformations

Any invertible linear transformation of $\mathbb{R}^3$ maps one-dimensional subspaces to one-dimensional subspaces, and so induces a diffeomorphism of $\mathbb{RP}^2$. This diffeomorphism is called a *projective transformation.* Projective transformations act in such a way as to map lines in $\mathbb{RP}^2$ to lines in $\mathbb{RP}^2$.

The group of projective transformations is usually denoted by $\mathrm{PGL}_3(\mathbb{R})$. It is an 8-dimensional Lie group. We say that a collection of points in $\mathbb{RP}^2$ is in *general position* if no three are contained in the same line. Say that a *quadrilateral* is a collection of 4 general position points in $\mathbb{RP}^2$. Each element of $\mathrm{PGL}_3(\mathbb{R})$ is determined by its action on a quadrilateral. Indeed, given two quadrilaterals, with points labelled, there is a unique element of $\mathrm{PGL}_3(\mathbb{R})$ that maps one quadrilateral to the other in a label preserving way.

The Cross Ratio

Suppose that p_1, p_2, p_3, p_4 are 4 points on a line $L \subset \mathbb{RP}^2$. One defines the *cross ratio* $\chi(p_1, p_2, p_3, p_4)$ in the following way. First, use an element of $\mathrm{PGL}_3(\mathbb{R})$ to identify L with (the one point extension of) the x-axis in $\mathbb{R}^2$. Let x_j be the image of p_j under this identification. Then define

$$\chi(p_1, p_2, p_3, p_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)}$$

This definition is independent of any choices used in identifying L with the x-axis. $\chi(p_1, p_2, p_3, p_4)$ is invariant under projective transformations. That is,

$$\chi(T(p_1), T(p_2), T(p_3), T(p_4)) = \chi(p_1, p_2, p_3, p_4)$$

for $T \in \mathrm{PGL}_3(\mathbb{R})$.

The Hilbert Metric

We say that a set $X \subset \mathbb{RP}^2$ is *convex* if there is a projective transformation T such that $T(X)$ is a closed compact convex subset of $\mathbb{R}^2$.

If $X \subset \mathbb{RP}^2$ is a convex set, we can define a canonical metric d_X on its interior X^o. Given unequal points $p_2, p_3 \in X^o$, let L be the line containing p_2 and p_3. Let p_1 and p_4 be the two points where L intersects the boundary ∂X. We order these points so that p_1, p_2, p_3, p_4 come in order on L. We define

$$d_X(p_2, p_3) = \log \chi(p_1, p_2, p_3, p_4).$$

Note that $d(p_2, p_3)$ approaches 0 as p_2 approaches p_3 and that $d(p_2, p_3) = d(p_3, p_2)$. The triangle inequality is also not hard to verify. d_X is known as the *Hilbert metric.*

Defined as it is, in terms of cross ratios, the Hilbert metric is natural with respect to $\mathrm{PGL}_3(\mathbb{R})$. If X and Y are convex sets and $T : X \to Y$ is a projective transformation mapping X to Y then T is an isometry as measured relative to the two Hilbert metrics.

3. THE INVARIANT FUNCTION

3A. Basic Definition

Let P be an n-gon, and give it an orientation. Let p be a vertex of P and let $a, b, \ldots, h, i$ be the points shown in Figure 2; note that a and b precede p under the given orientation. Set

$$O_p(P) = \chi(a, b, c, d),$$
$$E_p(P) = \chi(d, e, f, g),$$
$$f_p(P) = \chi(b, h, i, f).$$

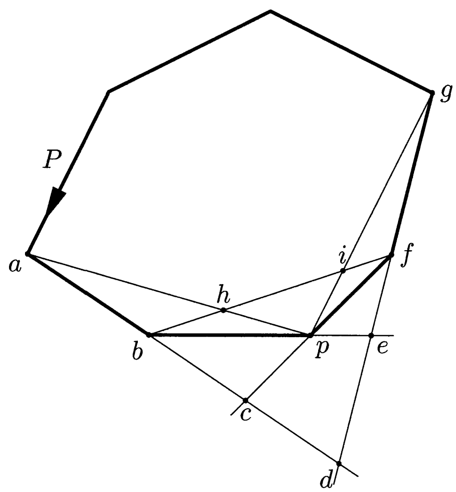


FIGURE 2. Points in the definition of the corner invariant $f_p(P) = \chi(b, h, i, f)$ and related quantities.

The quantity $f_p(P)$ is what we called the *corner invariant of P at p* in [Schwartz 1992]. Moreover, set

$$f(P) = \prod f_p(P),$$
$$O(P) = \prod O_p(P),$$
$$E(P) = \prod E_p(P),$$

where the product is taken over all vertices of P.

A short calculation shows that

$$f_p(P) = O_p(P)E_p(P).$$

To simplify the calculation, one can use the projective in variance to normalize so that the 4 vertices a, b, f, g form a unit square. We omit the details.

Taking the product of this identity over all vertices, we see that

$$f(P) = O(P)E(P).$$

Remark. Here is a geometric interpretation of $f(P)$. Let P' be the pentagram of P. Let X be the convex subset of $\mathbb{RP}^2$ whose boundary is P. Let d_X be the Hilbert metric on X. Let $p'_1, \dots, p'_n$ be the vertices of P' listed in order. By simply using the definition of the Hilbert metric, we have

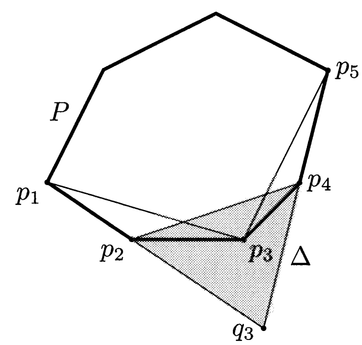
$$\log f(P) = \sum_{i=1}^{n} d_X(p'_i, p'_{i+1}),$$

where indices are taken modulo n. In other words, $\log f(P)$ is the perimeter of P' as measured in the Hilbert metric on the convex set bounded by P. This interpretation shows that $f(P) > 1$.

3B. Compactness Proof

In this section we prove that the level sets $f^{-1}[1, r] \subset \Sigma_n$ are compact. Our argument here is pretty much a repeat of what we said in [Schwartz 1992]. Given an n-gon P, the corner invariants $f_p(P)$ all lie in $[1, \infty)$. Thus, if $f(P) \in [1, r]$ then $f_p(P) \in [1, r]$ for every vertex p of P.

Let p_1, p_2, p_3, p_4, p_5 be 5 consecutive vertices of P:



The point p_3 is confined to the shaded open triangle Δ whose vertices are p_2, p_4 and q_3. Here

$$q_3 = \overline{p_1 p_2} \cap \overline{p_4 p_5}.$$

We set $f_j = f_{p_j}(P)$. Suppose that p_1, p_2, p_4, p_5 are held fixed and that p_3 (and possibly other points of P) are moved around. One observes three things:

1. If $x \in \partial\Delta$ is not on the segment $\overline{p_2 p_4}$ then $f_3 \to \infty$ as $p_3 \to x$.
2. If $x \in \overline{p_2 p_4}$ is not equal to p_2 then $f_2 \to \infty$ as $p_3 \to x$.
3. If $x \in \overline{p_2 p_4}$ is not equal to p_4 then $f_4 \to \infty$ as $p_3 \to x$.

These observations establish the following claim: If $f(P) \in [1, r]$, and p_1, p_2, p_4, p_5 are held fixed, then there is a compact set of positions, in the open triangle Δ where p_3 could be.

The map $P \to \{p_1, p_2, p_3, p_4, p_5\}$ gives a map from $\Sigma_n \to \Sigma_5$. There are n of these maps, depending on the choice of vertex p_1. From what we have just seen, the image of $f^{-1}[1, r]$, under each of these maps, is compact.

Suppose now that $\{P_k\}$ is a sequence of convex n-gons in $\mathbb{RP}^2$ such that $f(P_k) \in [1, r]$. Suppose, by induction, that we can find a sequence of projective transformations $\{T_k\}$ such that, on a subsequence, the first $m \geq 5$ points of $T_k(P_k)$ converge. Then, using the appropriate map into Σ_5 we see that, on a thinner subsequence, the $(m+1)$st points also converge. Hence, the polygons $T_k(P_k)$ converge, on a subsequence, to a fixed polygon. This proves that $f^{-1}[1,r]$ is compact.

3C. Invariance Proof

Here we sketch the proof that $f \circ T_n = f$. This proof is different from what appears in [Schwartz 1992].

Let $\Sigma_n(j)$ be the set projective classes of convex n-gons labelled by consecutive integers congruent to j mod 4. There is a canonical map from Σ_n into $\Sigma_n(1)$. A polygon whose points are labelled by integers $1, 2, 3\ldots$ is mapped to geometrically the same polygon whose points are labelled by integers $1, 5, 9, \ldots$. We denote this map by $\Sigma_n \Longrightarrow \Sigma_n(1)$. We denote the inverse map by $\Sigma_n(1) \Longrightarrow \Sigma_n$.

The map $P \to P'$, formerly defined as a map on unlabelled n-gons, can naturally be interpreted either as a map $A_n : \Sigma_n(1) \to \Sigma_n(3)$ or as a map $B_n : \Sigma_n(3) \to \Sigma_n(1)$. The two interpretations are shown, for $n = 7$, in Figure 3. The map $T_n : \Sigma_n \to \Sigma_n$ factors in the following way:

$$\Sigma_n \Longrightarrow \Sigma_n(1) \xrightarrow{A_n} \Sigma_n(3) \xrightarrow{B_n} \Sigma_n(1) \Longrightarrow \Sigma_n.$$

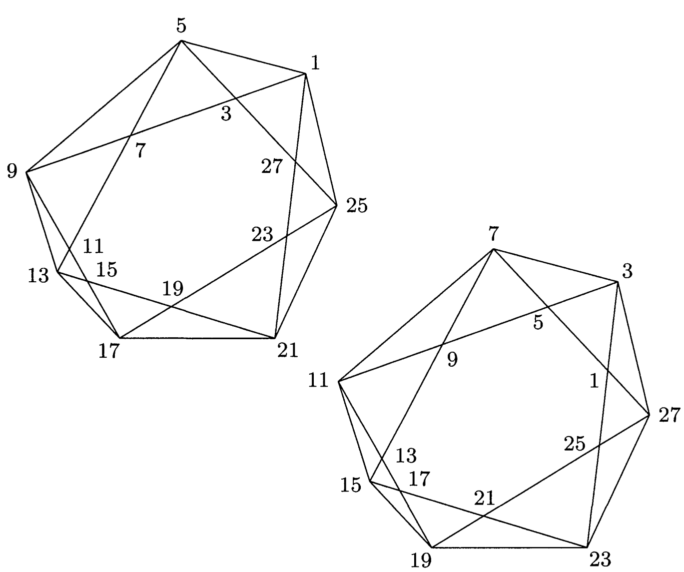


FIGURE 3. Vertex labeling for A_n (upper left) and B_n (lower right).

Let E and O be the invariants defined in the previous section. To show that $f \circ T_n = f$ we will show that

$$E \circ A_n = O, \quad O \circ A_n = E,$$
$$E \circ B_n = O, \quad O \circ B_n = E.$$

Let $P \in \Sigma_n(1)$. The vertices of P are labelled by integers congruent to 1 mod 4. We coordinatize P by the variables $(x_1, y_2, \ldots, x_{2n-1}, y_{2n})$, where

$$x_{(i+1)/2} = O_i(P), \quad y_{(i-1)/2} = E_i(P).$$

Here, for instance, $O_1(P)$ is the quantity, computed in the previous section, for the vertex 1. In these coordinates, $O(P) = \prod x_i$ and $E(P) = \prod y_i$.

We coordinatize $P' = A_n(P)$ by the variables

$$x'_{(i-1)/2} = O_i(P'), \quad y'_{(i+1)/2} = E_i(P').$$

We coordinatize $P'' = B_n(P')$ exactly as we coordinatized P, using variables x'' and y''.

A calculation shows that

$$x'_j = \left(\frac{1-x_{j+2}y_{j+3}}{1-x_{j-2}y_{j-1}}\right) y_{j+1}, \quad y'_j = \left(\frac{1-x_{j-3}y_{j-2}}{1-x_{j+1}y_{j+2}}\right) x_{j-1},$$
$$x''_j = \left(\frac{1-x'_{j-2}y'_{j-3}}{1-x'_{j+2}y'_{j+1}}\right) y'_{j-1}, \quad y''_j = \left(\frac{1-x'_{j+3}y'_{j+2}}{1-x'_{j-1}y'_{j-2}}\right) x'_{j+1}.$$

The identities in Equation $(*)$ follow immediately from these equations.

4. THE VOLUME FORM

4A. Framings and Volume Forms

Suppose that $\tilde{X}$ is a smooth manifold. A *framing* of $\tilde{X}$ is a smoothly varying choice of basis for the tangent spaces of $\tilde{X}$. That is, for each $x \in \tilde{X}$ we have a basis F_x for the tangent space $T_x\tilde{X}$. If F is a framing on $\tilde{X}$, then F canonically determines a volume form μ_F. Namely, μ_F is the volume form which assigns the value 1, at each point, to the basis given by F.

Suppose that $\tilde{X}_1$ and $\tilde{X}_2$ are two smooth manifolds, equipped with framings F_1 and F_2 respectively. If $\alpha : \tilde{X}_1 \to \tilde{X}_2$ is a diffeomorphism, and $x_1 \in \tilde{X}_1$ is a point, we define the matrix M_{x_1}, as follows: We have the differential map

$$d\alpha : T_{x_1}\tilde{X}_1 \to T_{x_2}\tilde{X}_2.$$

Here $x_2 = \alpha(x_1)$. We write out this map with respect to the bases given by F_1 and F_2. This is our matrix. We say that α is *adapted to* (F_1, F_2) if $\det(M_x) = 1$ for all $x \in \tilde{X}_1$. Note that α is adapted to (F_1, F_2) if and only if the differential $d\alpha$ maps μ_{F_1} to μ_{F_2}.

Suppose now that $G : \tilde{X} \to \tilde{X}$ is a smooth, free, proper group action. (*Free* means that every element of G acts with no fixed points, and *proper* means that $\{g \in G \mid g(K) \cap K \neq \varnothing\}$ is compact whenever K is compact.) Then the quotient $X = \tilde{X}/G$ is a smooth manifold, and any smooth map $\tilde{T} : \tilde{X} \to \tilde{X}$ that commutes with G induces a map $T : X \to X$. Here is our main technical result.

Lemma 4.1. *Suppose $G : \tilde{X} \to \tilde{X}$ is a smooth, free, proper group action. Suppose $\tilde{T} : \tilde{X} \to \tilde{X}$ is a smooth diffeomorphism which commutes with the action of G. Suppose there exists a smooth G-invariant framing F on $\tilde{X}$ such that $\tilde{T}$ is adapted to (F, F). Then there is a volume form μ on $X = \tilde{X}/G$ which is preserved by the induced map $T : X \to X$.*

Proof. We begin with a fact from linear algebra. Suppose V is an n-dimensional vector space, equipped with a volume form v. Suppose $W \subset V$ is a k-dimensional subspace, equipped with a volume form w. We shall denote the quotient map $V \to V/W$ by $x \to \bar{x}$. It is an elementary fact that there is a unique volume form q on V/W such that

$$q(\bar{x}_1 \wedge \cdots \wedge \bar{x}_{n-k}) = \frac{v(x_1 \wedge \cdots \wedge x_{n-k} \wedge y_1 \wedge \cdots \wedge y_k)}{w(y_1 \wedge \cdots \wedge y_k)}.$$

Here $x_1, \ldots, x_{n-k} \in V$ are vectors such that $\bar{x}_1, \ldots, \bar{x}_{n-k}$ is a basis for V/W, and $y_1, \ldots, y_k$ is any basis for W.

Let $x \in X$ be a point. Let $\tilde{x} \in \tilde{X}$ be a point which is mapped to X under the quotient map $\tilde{X} \to X$. Let $V = T_{\tilde{x}}\tilde{X}$ be the tangent space. Let L_G be the Lie algebra of left invariant vector fields on G. We fix a left invariant volume form on L_G. Each element of L_G defines a G-invariant vector field on $\tilde{X}$. In this way, there is a canonical embedding of L_G into V. Let W be the image of L_G under this embedding, at $\tilde{x}$. The tangent space to X at x is canonically isomorphic to the quotient V/W.

Note that V has the volume form $v = \mu_F$. Also, W has a volume form given by its identification with L_G. We use the linear algebra fact above to get a volume form μ_x on $T_xX = V/W$. This construction of μ_x does not depend on the choice of $\tilde{x}$, because everything in sight is G-invariant. Let μ be the volume form on X which restricts to μ_x at each point $x \in X$. The naturality of our construction implies that T preserves μ. $\square$

To prove Theorem 1.1 it remains to prove the Volume Lemma. Here is an outline of its proof. Let $\tilde{\Sigma}_n$ be the space of strictly convex n-gons in $\mathbb{RP}^2$. Let $G = \mathrm{PGL}_3(\mathbb{R})$. Note that G acts freely, properly and smoothly on $\tilde{\Sigma}_n$, and that $\tilde{\Sigma}_n/G = \Sigma_n$. Let $\tilde{T}_n : \tilde{\Sigma}_n \to \tilde{\Sigma}_n$ be the pentagram map, as it acts on $\tilde{\Sigma}_n$. ($\tilde{T}_n$ induces the map $T_n : \Sigma_n \to \Sigma_n$.)

We define $\tilde{\Sigma}_n(j)$ as the space of strictly convex n-gons in $\mathbb{RP}^2$, whose points are labelled by consecutive integers congruent to j mod 4. Just as we factored the map T_n in Section 3C, we factor $\tilde{T}_n$ as:

$$\tilde{\Sigma}_n \Longrightarrow \tilde{\Sigma}_n(1) \xrightarrow{\tilde{A}_n} \tilde{\Sigma}_n(3) \xrightarrow{\tilde{B}_n} \tilde{\Sigma}_n(1) \Longrightarrow \tilde{\Sigma}_n.$$

The double arrows indicate maps which just change the labels. The factorization here forms an obvious commuting diagram with the one given at the beginning of Section 3C.

Below we will construct G-invariant framings $F(1)$ and $F(3)$ on $\tilde{\Sigma}_n(1)$ and $\tilde{\Sigma}_n(3)$, respectively. Then we will show that $\tilde{A}_n$ is adapted to $(F(1), F(3))$. It follows from symmetry (or from a similar proof) that $\tilde{B}_n$ is adapted to $(F(3), F(1))$. The composition $\tilde{B}_n \circ \tilde{A}_n$ is therefore adapted to $(F(1), F(1))$. But this composition differs from $\tilde{T}_n$ only in the labels on the points of the polygons. So, there exists a G-invariant framing F on $\tilde{\Sigma}_n$ such that $\tilde{T}_n$ is adapted to (F, F). By Lemma 4.1, there exists a volume form μ_n on Σ_n which is T_n-invariant.

4B. Unit Vector in the Hilbert Metric

Suppose that L is a line in $\mathbb{R}^2$, and $A, B, C \in L$ are three points, with B separating A from C. We define

$$V(A, B, C) = \frac{(C - B)(B - A)}{C - A};$$

this is to be understood as (for example) the ratio between the collinear vectors $C - B$ and $C - A$, multiplied by $B - A$.

The Hilbert metric on the line segment $[A, C]$ is a Riemannian metric. It is just a pointwise multiple of the Euclidean metric on $[A, C]$. Thus, it makes sense to talk about the length of vectors tangent to $[A, C]$, as measured in the Hilbert metric. It is not hard to see that $V(A, B, C)$ is the tangent vector, based at B, oriented from A to C, which has unit length in the Hilbert metric on $[A, C]$. This geometric interpretation of $V(A, B, C)$ shows that it is invariant under projective transformations.

From the naturality of the construction, we see that $V(A, B, C)$ makes sense for any three collinear points in $\mathbb{RP}^2$, even if the formula breaks down—which happens if one of the points is infinite, or, more generally, if the segment $[A, C]$ intersects the line at infinity.

4C. The Framings

To construct our framing in $\tilde{\Sigma}_n(j)$ we need to construct, for each point in $\tilde{\Sigma}_n(j)$, a basis for the tangent space at that point. A point in $\tilde{\Sigma}_n(j)$ is a polygon in $\mathbb{RP}^2$. A tangent vector to the *point* is just a collection of n vectors in the plane, one per vertex of the *polygon*. To avoid using the word "tangent" too frequently, we will call the vectors in the plane *motion vectors*. Thus, a tangent vector to a polygon is a collection of n motion vectors. The intuition is that the collection of n motion vectors tells us how to move the polygon, to get a nearby polygon.

Suppose we are given a polygon P, a vertex p of P, and a single motion vector v. We can interpret v as a tangent vector by setting all the other $n-1$ motion vectors equal to 0. We call this process *extension*. The extension process starts with a motion vector and promotes it to a tangent vector by including it as the only nonzero vector in a collection of n motion vectors. This is what we will do in constructing our basis for the tangent space to $\tilde{\Sigma}_n(j)$. We will specify a collection of $2n$ motion vectors $v_1, \dots, v_{2n}$. Each motion vector is extended to a tangent vector. Thus, the $2n$ motion vectors determine $2n$ tangent vectors, a basis for the tangent space.

Here is the construction for $\tilde{\Sigma}_n(1)$. Let $P = p_1$, p_5, p_9, … be a polygon in $\tilde{\Sigma}_n(1)$. Let

$$p'_{j+2} = \overline{p_{j-4}p_j} \cap \overline{p_{j+4}p_{j+8}}.$$

For $j = 1, \dots, n$ we define the motion vectors

$$\begin{aligned} v_{2j} &= V(p_{4j+5}, p_{4j+1}, p'_{4j-1}), \\ v_{2j+1} &= V(p_{4j-3}, p_{4j+1}, p'_{4j+3}). \end{aligned} \tag{4–1}$$

The construction for $\tilde{\Sigma}_n(3)$ is similar. Let $Q = q_3$, q_7, q_{11}, … be a polygon in $\tilde{\Sigma}_n(3)$. This time, define the motion vectors

$$\begin{aligned} w_{2j} &= V(q_{4j-5}, q_{4j-1}, q'_{4j+1}), \\ w_{2j-1} &= V(q_{4j+3}, q_{4j-1}, q'_{4j-3}). \end{aligned} \tag{4–2}$$

See examples of both conventions in Figure 4.

4D. Form of The Matrix

Now consider polygons $P = (p_1, p_5, \dots) \in \Sigma_n(1)$ and $Q = \tilde{A}_n(P) = (q_3, q_7, \dots) \in \Sigma_n(3)$, and construct the respective framings $F(1) = (v_1, \dots)$ and $F(3) = (w_1, \dots)$ as in the preceding section. To show that the map

$$\tilde{A}_n : \tilde{\Sigma}_n(1) \to \tilde{\Sigma}_n(3)$$

is adapted to the pair of framings $\big(F(1), F(3)\big)$, we work out the matrix $d\tilde{A}_n$, written with respect to these two framings. For ease of exposition, we will consider the case $n = 7$.

Let λ_{ij} be the matrix entries of $d\tilde{A}_n$. The expression λ_{ij} is a function on $\tilde{\Sigma}_n(1)$. Figure 5 shows what happens when we vary P along $-v_4$, for example: as we slide p_9 along the vector $-v_4$, keeping the other vertices of P fixed, we see that only four vertices of Q move: q_3 slides along $-w_2$, and so on. In terms of the matrix, this means that $\lambda_{4j} > 0$ if and only if $j = 2, 3$, and $\lambda_{4j} < 0$ if and only if $j = 6, 7$. Likewise $\lambda_{5j} < 0$ if and only if $j = 2, 3$, and $\lambda_{5j} > 0$ if and only if $j = 6, 7$.

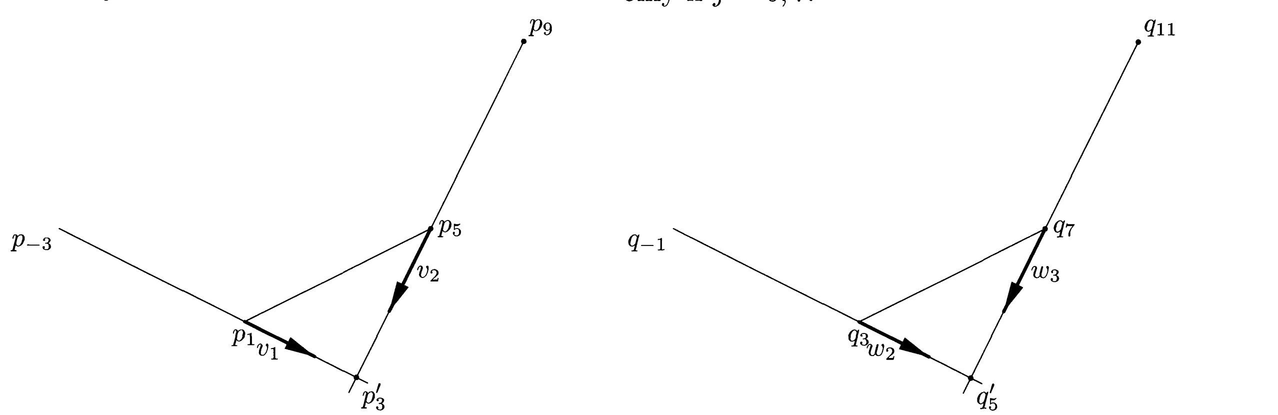


FIGURE 4. Construction of v_{2j} and v_{2j+1}, for elements of $\tilde{\Sigma}_n(1)$ (left) and of $\tilde{\Sigma}_n(3)$ (right).

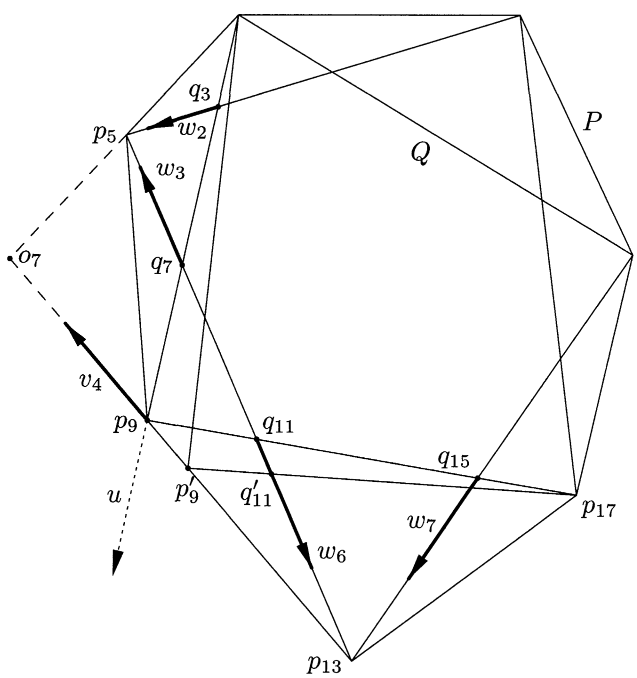


FIGURE 5. A change in P along v_4 leads to a change in Q contained in the span of w_2, w_3, w_6, and w_7.

Define

$$\Lambda^{9,5} = \begin{bmatrix} \lambda_{42} & \lambda_{52} \\ \lambda_{43} & \lambda_{53} \end{bmatrix} \qquad \Lambda^{9,13} = \begin{bmatrix} \lambda_{46} & \lambda_{56} \\ \lambda_{47} & \lambda_{57} \end{bmatrix}$$

Variation along the vector u on the line from p_1 to p_9 does not move q_3 or q_7. But u is a linear combination of v_4 and v_5; thus the linear transformation represented by $\Lambda_{9,5}$ has a nontrivial kernel, and therefore $\det(\Lambda^{9,5}) = 0$. Likewise, $\det(\Lambda^{9,13}) = 0$. The same picture occurs at each vertex. Shifting the indices in a more or less obvious way, we obtain the following expression for $d\tilde{A}_7$:

$$\begin{bmatrix} 0 & \Lambda^{5,1} & 0 & 0 & 0 & 0 & \Lambda^{25,1} \\ \Lambda^{1,5} & 0 & \Lambda^{9,5} & 0 & 0 & 0 & 0 \\ 0 & \Lambda^{5,9} & 0 & \Lambda^{13,9} & 0 & 0 & 0 \\ 0 & 0 & \Lambda^{9,13} & 0 & \Lambda^{17,13} & 0 & 0 \\ 0 & 0 & 0 & \Lambda^{13,17} & 0 & \Lambda^{21,17} & 0 \\ 0 & 0 & 0 & 0 & \Lambda^{17,21} & 0 & \Lambda^{25,21} \\ \Lambda^{1,25} & 0 & 0 & 0 & 0 & \Lambda^{21,25} & 0 \end{bmatrix}.$$

Each 0 stands for the 2×2 zero matrix. The individual 2×2 blocks $\Lambda^{i,j}$ have determinant 0. The general pattern should be clear from the case $n = 7$.

4E. Computing a Block in the Matrix

Denote by $x_1, y_2, \ldots, x_{2n-1}, y_{2n}$ the invariant coordinates for P, defined in Section 3C. We also define

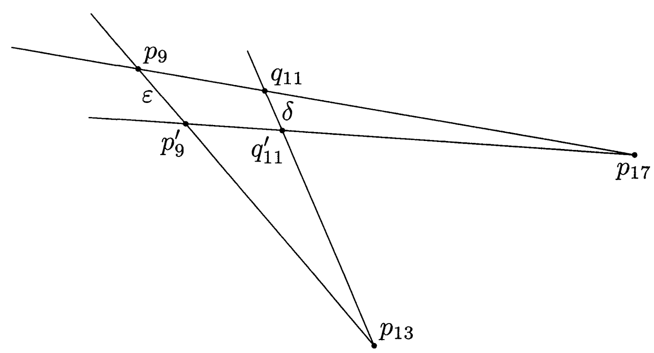
$$z_{ij} = \frac{1}{1 - y_i x_j}.$$

The entire structure of $d\tilde{A}_n$ can be deduced from symmetry and from the following formula, which we will justify shortly:

$$\Lambda^{9,13} = \begin{bmatrix} -x_5 z_{45} & z_{45} \\ -x_5 y_6 z_{89} & y_6 z_{89} \end{bmatrix}. \tag{4–3}$$

That $\det \Lambda^{9,13} = 0$ implies that the formulae for any three entries determine the fourth. We will compute $\lambda_{46} = -x_5 z_{45}$ and omit the other two calculations, which are similar.

In the following diagram, extracted from Figure 5,



let $\varepsilon = |p_9 p_9'|$ and $\delta = |q_{11} q_{11}'|$, where the bars denote Euclidean length. Basic plane geometry gives

$$\delta = \frac{|p_{13} q_{11}|\,|p_{17} q_{11}|}{|p_{13} p_9|\,|p_{17} p_9|}\,\varepsilon + O(\varepsilon^2).$$

To find λ_{46} we need to write this proportionality constant in terms of v_4 and w_6 rather than in terms of Euclidean lengths. From (4–2) we have

$$q_{11}' - q_{11} = \frac{|p_{13} q_7|}{|p_{13} q_{11}|\,|q_{11} q_7|}\,\delta\, w_6,$$

and from (4–1) we have

$$p_9' - p_9 = \frac{-|p_{13} o_7|}{|o_7 p_9|\,|p_9 p_{13}|}\,\varepsilon\, v_4.$$

Comparing the last three displayed equations we get

$$\lambda_{46} = -\frac{|p_9 o_7|\,|p_{13} q_7|\,|q_{11} p_{17}|}{|o_7 p_{13}|\,|q_7 q_{11}|\,|p_{17} p_9|},$$

which a short calculation identifies with $-x_5 z_{45}$.

4F. Computing the Determinant

The arrangement of the blocks $\Lambda^{5,9}$, $\Lambda^{13,9}$, $\Lambda^{9,5}$ and $\Lambda^{9,13}$ is

$$\begin{array}{cccccc} & & \lambda_{42} & \lambda_{52} & & \\ & & \lambda_{43} & \lambda_{53} & & \\ \lambda_{24} & \lambda_{34} & 0 & 0 & \lambda_{64} & \lambda_{74} \\ \lambda_{25} & \lambda_{35} & 0 & 0 & \lambda_{65} & \lambda_{75} \\ & & \lambda_{46} & \lambda_{56} & & \\ & & \lambda_{47} & \lambda_{57} & & \end{array}.$$

We define

$$V_2 = \frac{\det \begin{bmatrix} \lambda_{42} & \lambda_{52} \\ \lambda_{47} & \lambda_{57} \end{bmatrix}}{\lambda_{42}\lambda_{57}}, \quad H_2 = \frac{\det \begin{bmatrix} \lambda_{24} & \lambda_{74} \\ \lambda_{25} & \lambda_{75} \end{bmatrix}}{\lambda_{24}\lambda_{75}},$$

and $D_2 = \lambda_{56}\lambda_{65}$; moreover we define H_{2+i}, V_{2+i} and D_{2+i} by adding $2i$ to all of the indices in the λ_{jk}. For instance, $D_1 = \lambda_{34}\lambda_{43}$. Equation (4–3) gives

$$H_2 = V_2 = 1/z_{45}; \quad D_2 = z_{45}z_{67}.$$

Hence $\prod H_i V_i D_i = 1$. To finish the proof of the Volume Lemma we show an auxiliary result:

Lemma 4.2. $\det(d\tilde{A}_n) = \prod_{i=1}^n H_i V_i D_i$.

Proof. For ease of exposition, we take $n = 6$. The polynomial $Z = \det(d\tilde{A}_6)$ has $n!$ signed monomials. The monomials in Z are only nonzero when all variables have been chosen from the nonzero 2×2 blocks. Say that a monomial in Z is *bad* if it contains two variables picked from the same nonzero block, and otherwise *good*. The bad monomials cancel in pairs, since the determinant of each block is 0.

Figure 6 illustrates a coding for the good monomials. Each good monomial is specified by choosing one variable arbitrarily from each of the 6 above-diagonal blocks $\Lambda^{j,j-4}$. These choices, which are represented by lightly colored squares, uniquely determine the variables in the below-diagonal blocks $\Lambda^{j,4+j}$, which are represented by black squares. The monomials are signed, so that Z is a positive sum over all these monomials.

We can encode each one of these monomials by a pair of binary strings (a, b). Both a and b have length 6. The 1 bits in a indicate the columns in which the light shaded square is on the right half of the 2×2 block. The 1 bits in b indicate the rows in which the light shaded square are on the top half of the 2×2 block. For instance, the first picture is encoded by (000000, 000000). The second picture is encoded by (010100, 010010). Note that $(000000, 000000) = \prod D_k$.

If a has a 0 in the kth position, let a_k be the string obtained by changing this bit to a 1. For instance $(010010)_3 = (011010)$. We do not define a_k if a has a 1 in the kth position. We make the same definition for b. We have the following basic identity, which uses the fact that $\det \Lambda^{ij} = 0$.

$$\begin{aligned} (a, b) + (a_k, b) &= (a, b)H_k; \\ (a, b) + (a, b_k) &= (a, b)V_k. \end{aligned} \tag{4–4}$$

Let $*$ stand for either a 0 or a 1. Let S_{ij} be the set of monomials of the form $(*\cdots 0, *\cdots 0)$, such that there are i copies of $*$ in the first slot, and j copies of $*$ in the second slot. For example, S_{25} consists of the set of all monomials having the form

$$(**0000, *****0).$$

Obviously, S_{66} is the set of all monomials.

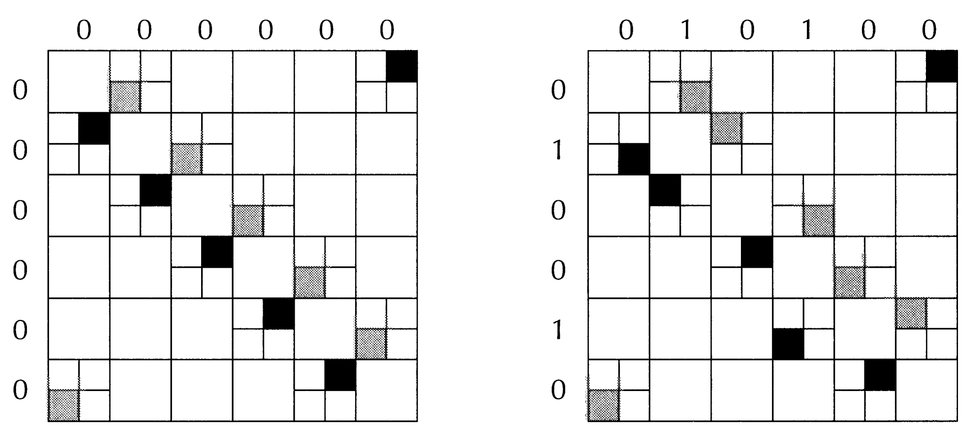


FIGURE 6. Coding for the good monomials.

Formula (4–4) gives

$$\begin{aligned}\det(d\tilde{A}_6) &= \sum_{S_{66}}(a,b) = H_5 \sum_{S_{56}}(a,b)\\ &= H_5 H_4 \sum_{S_{46}}(a,b) = \cdots = \prod H_i \sum_{S_{06}}(a,b)\\ &= V_5 \prod H_i \sum_{S_{05}}(a,b) = \cdots\\ &= \prod V_i \prod H_i \sum_{S_{00}}(a,b) = \prod H_i V_i D_i. \quad \square\end{aligned}$$

REFERENCES

[Arnol'd 1978] V. I. Arnol'd, *Mathematical methods of classical mechanics*, Graduate Texts in Math. **60**, Springer, New York, 1978. Second edition, 1989.

[Hilbert and Cohn-Vossen 1950] D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*, Chelsea, New York, 1950.

[Schwartz 1992] R. Schwartz, "The pentagram map", *Experiment. Math.* **1**:1 (1992), 71–81.

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