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NON-TRIVIAL SOLUTIONS TO CERTAIN MATRIX EQUATIONS*

AIHUA LI† AND DUANE RANDALL†

Abstract. The existence of non-trivial solutions X to matrix equations of the form $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ over the real numbers is investigated. Here F and G denote monomials in the $(n \times n)$-matrix $\mathbf{X} = (x_{ij})$ of variables together with $(n \times n)$-matrices $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ for $s \geq 1$ and $n \geq 2$ such that F and G have different total positive degrees in $\mathbf{X}$. An example with $s = 1$ is given by $F(\mathbf{X}, \mathbf{A}) = \mathbf{X}^2\mathbf{A}\mathbf{X}$ and $G(\mathbf{X}, \mathbf{A}) = \mathbf{A}\mathbf{X}\mathbf{A}$ where $deg(F) = 3$ and $deg(G) = 1$. The Borsuk-Ulam Theorem guarantees that a non-zero matrix $\mathbf{X}$ exists satisfying the matrix equation $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ in $(n^2 - 1)$ components whenever F and G have different total odd degrees in $\mathbf{X}$. The Lefschetz Fixed Point Theorem guarantees the existence of special orthogonal matrices $\mathbf{X}$ satisfying matrix equations $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ whenever $deg(F) > deg(G) \geq 1$, $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ are in $SO(n)$, and $n \geq 2$. Explicit solution matrices $\mathbf{X}$ for the equations with $s = 1$ are constructed. Finally, nonsingular matrices $\mathbf{A}$ are presented for which $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ admits no non-trivial solutions.

Key words. Polynomial equation, Matrix equation, Non-trivial solution.

AMS subject classifications. 39B42, 15A24, 55M20, 47J25, 39B72

1. Matrix equations involving special monomials. Given monomials $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ and $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ in the $(n \times n)$-matrix $\mathbf{X} = (x_{ij})$ of variables with $n \geq 2$ and with total degrees $deg(F) > deg(G) \geq 1$ in $\mathbf{X}$, we investigate the existence of non-trivial solutions $\mathbf{X}$ to the matrix equation

$$F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s). \tag{1.1}$$

For example, $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ is such an equation. We note that in this equation, $F(\mathbf{X}, \mathbf{A}) = \mathbf{X}^2\mathbf{A}\mathbf{X}$ and $G(\mathbf{X}, \mathbf{A}) = \mathbf{A}\mathbf{X}\mathbf{A}$ both contain products $\mathbf{A}\mathbf{X}$ and $\mathbf{X}\mathbf{A}$. We first record a sufficient condition for non-trivial solutions to the equation (1.1).

PROPOSITION 1.1. *Suppose that the monomials $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ and $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ both contain the product $\mathbf{A}_i\mathbf{X}$ or both contain $\mathbf{X}\mathbf{A}_i$, for some i with $1 \leq i \leq s$. Whenever $\mathbf{A}_i$ is a singular matrix, the matrix equation* (1.1) *admits non-trivial solutions $\mathbf{X}$.*

Proof. Let $\mathbf{X}$ be any non-zero $(n \times n)$-matrix whose columns belong to the null space of $\mathbf{A}_i$ whenever both F and G contain $\mathbf{A}_i\mathbf{X}$. Similarly, let $\mathbf{X}$ be any non-zero matrix whose rows belong to the null space of $\mathbf{A}_i^T$ in case both F and G contain $\mathbf{X}\mathbf{A}_i$. □

Our principal result affirms the existence of non-trivial solutions $\mathbf{X}$ to matrix equations $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) = G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ whenever $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ belong to the special orthogonal group $SO(n)$ for any integer $n \geq 2$. We first construct explicit non-trivial solutions for such matrix equations with $s = 1$.

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†Department of Mathematics and Computer Science, Loyola University New Orleans, New Orleans, LA 70118, USA (ali@loyno.edu, randall@loyno.edu). The first author was supported by the BORSF under grant LEQSF(2000-03)RD-A-24.

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PROPOSITION 1.2. *Every matrix equation $F(\mathbf{X},\mathbf{A}) = G(\mathbf{X},\mathbf{A})$ for monomials F and G with different total odd degrees in $\mathbf{X}$ admits a non-trivial solution $\mathbf{X}$ of the form $\mathbf{A}^{p/q}$ whenever $\mathbf{A}$ belongs to $SO(n)$ for $n \geq 2$.*

Proof. We may assume that $deg(F) > deg(G) \geq 1$. We seek a solution $\mathbf{X} = \mathbf{A}^{p/q}$ to the matrix equation $F(\mathbf{X},\mathbf{A}) \cdot (G(\mathbf{X},\mathbf{A}))^{-1} = \mathbf{I}_n$. The classical Spectral Theorem for $SO(n)$ in [3] affirms that $\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}\mathbf{C}$ for matrices $\mathbf{B}$ and $\mathbf{C}$ in $SO(n)$ where $\mathbf{B}$ consists of blocks of non-trivial rotations $R(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}$ along the diagonal together with an identity submatrix $\mathbf{I}_l$. A solution $\mathbf{X}$ commuting with powers of $\mathbf{A}$ reduces the matrix equation $F(\mathbf{X},\mathbf{A})\cdot(G(\mathbf{X},\mathbf{A}))^{-1} = \mathbf{I}_n$ to $\mathbf{X}^{deg(F)-deg(G)} = \mathbf{A}^p$ for some integer p. Setting $q = deg(F) - deg(G)$, we obtain $\mathbf{X} = \mathbf{A}^{p/q} = \mathbf{C}^{-1}\mathbf{B}^{p/q}\mathbf{C}$ where $\mathbf{B}^{p/q}$ consists of blocks of rotations $R(p\theta_i/q)$ along the diagonal together with $\mathbf{I}_l$. □

We now establish the existence of non-trivial solutions to many matrix equations via the Lefschetz Fixed Point Theorem. For example, the matrix equation $\mathbf{X}^2\mathbf{A}_1\mathbf{A}_2^2\mathbf{X}\mathbf{A}_2^3\mathbf{A}_1^2 = \mathbf{A}_1^3\mathbf{A}_2\mathbf{A}_1^2\mathbf{X}\mathbf{A}_2^3$ admits rotation matrices as solutions whenever $\mathbf{A}_1$ and $\mathbf{A}_2$ belong to $SO(n)$ for any $n \geq 2$.

THEOREM 1.3. *There is a solution $\mathbf{X}$ in $SO(n)$ to any matrix equation $F(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s) = G(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s)$, i.e., equation (1.1), with $deg(F) > deg(G) \geq 1$ and $n \geq 2$ whenever the $(n \times n)$-matrices $\mathbf{A}_i$ belong to $SO(n)$ for $1 \leq i \leq s$.*

Proof. Solutions $\mathbf{X}$ in $SO(n)$ to the matrix equation (1.1) are precisely the fixed points of the continuous function $H : SO(n) \longrightarrow SO(n)$ defined by $H(\mathbf{X}) = \mathbf{X} \cdot F(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s) \cdot [G(\mathbf{X},\mathbf{A}_1,\mathbf{A}_2,\cdots,\mathbf{A}_s)]^{-1}$. The existence of fixed points for the map H follows from its non-zero Lefschetz number $L(H)$. We affirm that $L(H) = (deg(G) - deg(F))^m$ where $n = 2m$ or $n = 2m+1$.

Brown in [1, p.49], calculated the Lefschetz number $L(\rho_k)$ for the k^{th} power map $\rho_k : G \longrightarrow G$ defined by $\rho_k(g) = g^k$ on any compact connected topological group G which is an ANR (absolute neighborhood retract). He proved that $L(\rho_k) = (1-k)^\lambda$ where λ denotes the number of generators for the primitively generated exterior algebra $H^*(G\,;\mathbb{Q})$. For $G = SO(n)$, $\lambda = m$ where $n = 2m$ or $n = 2m+1$; see [4, p.956]. It suffices to show that H is homotopic to $\rho_k : SO(n) \longrightarrow SO(n)$ where $k = deg(F) - deg(G) + 1$.

For each i with $1 \leq i \leq s$, let $g_i : [0,1] \longrightarrow SO(n)$ denote any path in $SO(n)$ from $\mathbf{A}_i = g_i(0)$ to the identity matrix $\mathbf{I}_n = g_i(1)$. Replacing each matrix $\mathbf{A}_i$ by the function g_i in $H : SO(n) \longrightarrow SO(n)$ produces a homotopy $H_t : SO(n) \longrightarrow SO(n)$ for $0 \leq t \leq 1$ with $H_0 = H$ and $H_1 = \rho_k$. Thus $L(H) = (1-k)^m = (deg(G)-deg(F))^m \neq 0$ so H has a fixed point. □

We now establish the existence of non-trivial solutions $\mathbf{X}$ to all matrix equations of the form (1.1) in any (n^2-1) components whenever F and G have different odd degrees in $\mathbf{X}$ for any $s \geq 1$ and $n \geq 1$. For example, given any $(n \times n)$-matrix $\mathbf{A}$, there is a non-zero matrix $\mathbf{X}$ such that $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ in at least (n^2-1)-components. This is a best possible result, since we shall construct matrices $\mathbf{A}$ for which $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ admits only the trivial solution. We use the Borsuk-Ulam Theorem following the paper of Lam [2] to prove the following.

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THEOREM 1.4. *Given any monomials* $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ *and* $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ *in the* $(n \times n)$*-matrix* $\mathbf{X} = (x_{ij})$ *together with arbitrary matrices* $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ *in* $\mathrm{M}_n(\mathbb{R})$ *for* $n \geq 2$ *such that* $deg(F)$ *and* $deg(G)$ *are different odd integers, the matrix equation* (1.1) *admits a non-trivial solution* $\mathbf{X}$ *in* $(n^2 - 1)$ *components.*

Proof. Set each component of the matrix $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) - G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ equal to zero, except for one fixed component. We obtain $n^2 - 1$ polynomial equations in the n^2 variables x_{ij}. Now each component of $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ and $G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s)$ is a homogeneous polynomial whose degree is given by $deg(F)$ or $deg(G)$ respectively. Consequently, every monomial in the $(n^2 - 1)$ polynomial equations has an odd degree, either $deg(F)$ or $deg(G)$. Suppose that the system of $n^2 - 1$ polynomial equations in the n^2 variables had no non-zero solution. As $\mathbf{X}$ ranges over the unit sphere S^{n^2-1} in $\mathbb{R}^{n^2}$, normalization of the non-zero vectors $F(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) - G(\mathbf{X}, \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s) \in \mathbb{R}^{n^2-1}$ produces a continuous function $P : S^{n^2-1} \longrightarrow S^{n^2-2}$. Since $deg(F)$ and $deg(G)$ are distinct odd integers, P commutes with the antipodal maps on the spheres. But the classical Borsuk-Ulam Theorem [5, p.266] affirms that no such function P can exist. □

2. The special matrix equation $\mathbf{X}^2\mathbf{AX} - \mathbf{AXA} = \mathbf{0}$. Given any non-zero $(n \times n)$-matrix $\mathbf{A}$, consider the matrix equation

$$\mathbf{X}^2\mathbf{AX} - \mathbf{AXA} = \mathbf{0}\ . \tag{2.1}$$

In this section we discuss solution types of the equation (2.1). We list a few obvious facts about solutions.

LEMMA 2.1.
1. *If* $\mathbf{X} \in \mathrm{M}_n(\mathbb{R})$ *is a solution to* (2.1)*, then* $-\mathbf{X}$ *is a solution too;*
2. *If* $|\mathbf{A}| < 0$*, then* (2.1) *has no nonsingular solutions.*
3. *If* $\mathbf{A} = \mathbf{B}^2$ *for some* $\mathbf{B} \in \mathrm{M}_n(\mathbb{R})$*, then* $\mathbf{X} = \mathbf{B}$ *is a non-trivial solution.*
4. *If* $\mathbf{A}^m = \mathbf{I}_n$ *and* m *is odd, then* $\mathbf{X} = \mathbf{A}^{\frac{m+1}{2}}$ *is a non-trivial solution.*
5. *If* $\mathbf{A}^3 = \mathbf{0}$*, then* $\mathbf{X} = k\mathbf{A}$ *is a solution to* (2.1) *for all* $k \in \mathbb{R}$.
6. *Suppose* $\mathbf{P}$ *is a nonsingular matrix and* $\mathbf{B} = \mathbf{PAP}^{-1}$*. Then a matrix* $\mathbf{X}$ *satisfies the equation* $\mathbf{X}^2\mathbf{AX} - \mathbf{AXA} = \mathbf{0}$ *if and only if* $\mathbf{Y} = \mathbf{PXP}^{-1}$ *satisfies* $\mathbf{Y}^2\mathbf{BY} - \mathbf{BYB} = \mathbf{0}$.

By Lemma 2.1(6.), when the matrix $\mathbf{A}$ is diagonalizable, the equation (2.1) can be reduced to the diagonal case. We first characterize all solutions for scalar matrices $\mathbf{A}$.

THEOREM 2.2. *Let* $\mathbf{A} = a\mathbf{I}_n \in \mathrm{M}_n(\mathbb{R})$*, where* $n > 1$ *and* $a \neq 0$*. Then the equation* (2.1) *has non-trivial solutions. Furthermore, the solution set (over the real numbers) consists of matrices in* $\mathrm{M}_n(\mathbb{R})$ *of the form*

$$\mathbf{X} = \mathbf{Q}^{-1} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \mathbf{Q},$$

where $\mathbf{Q}$ *is a nonsingular matrix with complex entries and* $\lambda_i = 0$, $\sqrt{a}$*, or* $-\sqrt{a}$ *for* $i = 1, 2, \ldots, n$*. In particular, nonsingular solutions are those with* $\lambda_1\lambda_2 \cdots \lambda_n$ *not*

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equal to zero. In summary,
1. *If* $a^n > 0$ *with* $n > 2$, *then* (2.1) *has both singular solutions and nonsingular solutions;*
2. *If* $a^n < 0$ *and* $n > 2$, *then* (2.1) *has only singular solutions;*
3. *In case of* $a < 0$ *and* $n = 2$, *there are nonsingular solutions, but no non-trivial singular solutions to* (2.1).

Proof. Suppose $\mathbf{X}$ is a solution to (2.1). Then

$$\mathbf{X}^2\mathbf{AX} - \mathbf{AXA} = a\mathbf{X}^3 - a^2\mathbf{X} = \mathbf{0} \Longleftrightarrow \mathbf{X}^3 = a\mathbf{X}.$$

Every matrix $\mathbf{X}$ satisfying $\mathbf{X}^3 = a\mathbf{X}$ is diagonalizable over the complex numbers. Suppose $\mathbf{X}$ is similar to a diagonal matrix $\mathbf{D} = diag(\lambda_i)$, then $\mathbf{X}^3 = a\mathbf{X} \Longleftrightarrow \mathbf{D}^3 = a\mathbf{D}$. This implies $\lambda_i^2 = a$ or $\lambda_i = 0$ for $i = 1, 2, \ldots, n$. Thus all the solutions to (2.1) are the real matrices similar to these diagonal matrices. Claim 1. is obvious by choosing appropriate (real) λ_i's. For 2., $|\mathbf{A}| < 0$. By Lemma 2.1(2.), equation (2.1) has no nonsingular solutions. The existence of singular solutions over the real numbers is based on the fact that every 2×2 diagonal matrix of the form $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$, where λ is a non-real complex number, can be realized by a complex nonsingular matrix $\mathbf{Q}$. Assume $\lambda = \sqrt{-a} \cdot i$, one can check that $\mathbf{Q} = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ gives $\mathbf{Q}^{-1} \begin{bmatrix} \sqrt{-a} \cdot i & 0 \\ 0 & -\sqrt{-a} \cdot i \end{bmatrix} \mathbf{Q} = \begin{bmatrix} 0 & \sqrt{-a} \\ -\sqrt{-a} & 0 \end{bmatrix} \in \mathrm{M}_2(\mathbb{R})$. Since $n > 2$, we always can choose at least one diagonal block of $\mathbf{D}$ to be $\begin{bmatrix} \sqrt{-a} \cdot i & 0 \\ 0 & -\sqrt{-a} \cdot i \end{bmatrix}$ and extend it to a singular solution by choosing at least one zero diagonal element. In case of $a < 0$ and $n = 2$, nonsingular solutions are similar to $\begin{bmatrix} 0 & \sqrt{-a} \\ -\sqrt{-a} & 0 \end{bmatrix}$. We show by contradiction that in this case (2.1) has no non-trivial singular solutions. Assume $\mathbf{0} \neq \mathbf{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ is a non-trivial solution to (2.1) and $|\mathbf{X}| = 0$. Then $\mathbf{X}^2 = (x_1 + x_4)\mathbf{X} \Longrightarrow (x_1 + x_4)^2\mathbf{X} = a\mathbf{X} \Longrightarrow a = (x_1 + x_4)^2 \geq 0$, a contradiction. □

By Lemma 2.1(6.), if $\mathbf{A}$ is diagonalizable, we only need to consider the solvability of the equation (2.1) for the similar diagonal matrix. Now let us treat diagonal matrices.

THEOREM 2.3. *Suppose* $\mathbf{A}$ *is a non-zero diagonal matrix which has at least one positive entry. Then the equation* $\mathbf{X}^2\mathbf{AX} - \mathbf{AXA} = \mathbf{0}$ *has non-trivial solutions.*

Proof. Let $\mathbf{A} = diag(\lambda_i)$. Without loss of generality, let $\lambda_1 > 0$. Then the diagonal matrix $\mathbf{X} = diag(\alpha_i)$ will give non-trivial solutions, where $\alpha_1 = \sqrt{\lambda_1}$ and for $i > 1$, $\alpha_i = 0$ or $\sqrt{\lambda_i}$ if $\lambda_i > 0$. When $\lambda_i \geq 0$ for all i, we obtain non-trivial solutions $\mathbf{X} = diag(\sqrt{\lambda_i})$. □

COROLLARY 2.4. *For* $n > 1$, *the equation* (2.1) *has non-trivial solutions for all* $n \times n$ *positive definite and all positive semidefinite matrices* $\mathbf{A}$.

We end this section with the following proposition.

PROPOSITION 2.5. *Suppose* $\mathbf{A} \in \mathrm{M}_n(\mathbb{R})$ *is similar to a block matrix, i.e., there*

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exists a nonsingular matrix $\mathbf{P}$ *such that*

$$\mathbf{PAP}^{-1} = \begin{bmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_m \end{bmatrix},$$

where each $\mathbf{A}_i$ *is a square matrix. Suppose* $\mathbf{Y}_i$ *satisfies* $\mathbf{Y}_i^2\mathbf{A}_i\mathbf{Y}_i - \mathbf{A}_i\mathbf{Y}_i\mathbf{A}_i = \mathbf{0}$, *for* $i = 1, 2, \cdots, m$. *Then the matrix* $\mathbf{X} = \mathbf{P}^{-1}\mathbf{BP}$ *is a solution to* $\mathbf{X}^2\mathbf{AX} - \mathbf{AXA} = \mathbf{0}$, *where* $\mathbf{B}$ *is a block matrix with blocks* $\mathbf{B}_i = \mathbf{Y}_i$ *or* $\mathbf{0}$. *Thus, if at least one of the solutions* $\mathbf{Y}_i$*'s is not zero, we can extend it to non-trivial solutions for the equation* $\mathbf{X}^2\mathbf{AX} = \mathbf{AXA}$.

THEOREM 2.6. *Let* $\mathbf{A}$ *be a real* $n \times n$ *matrix with distinct negative eigenvalues. Then the equation* $\mathbf{X}^2\mathbf{AX} = \mathbf{AXA}$ *admits only the trivial solution.*

Proof. Suppose first that $\mathbf{X}$ is an invertible solution. Then we have

$$\mathbf{A}^{-1}\mathbf{X}^2\mathbf{A} = \mathbf{XAX}^{-1}.$$

Thus the eigenvalues of $\mathbf{X}^2$ are the same as those of $\mathbf{A}$. Since the eigenvalues of $\mathbf{A}$ are negative and distinct, the eigenvalues of $\mathbf{X}$ are all pure imaginary and of distinct modulus. This is impossible.

If $\mathbf{X}$ is a singular solution, let $\mathbf{v}$ be a null vector of $\mathbf{X}$ and observe that $\mathbf{0} = \mathbf{AXAv} = \mathbf{XAv}$. Thus the null space of $\mathbf{X}$ is $\mathbf{A}$-invariant. Then there exists an invertible matrix $\mathbf{B}$ such that

$$\mathbf{X} = \mathbf{B}\begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\mathbf{B}^{-1} \quad \text{and} \quad \mathbf{A} = \mathbf{B}\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}\mathbf{B}^{-1}.$$

By Lemma 2.1(6.),

$$\begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^2 \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}.$$

This yields $\mathbf{Y}^2\mathbf{PY} = \mathbf{PYP}$ and by induction $\mathbf{Y} = \mathbf{0}$. (See Theorem 3.3 for the 2×2 case.) This means that

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^2 = \mathbf{0} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{ECP} & \mathbf{0} \end{bmatrix},$$

which gives $\mathbf{ECP} = \mathbf{0}$. Since $\mathbf{E}$ and $\mathbf{P}$ are invertible, $\mathbf{C} = \mathbf{0}$, so $\mathbf{X}$ is the trivial solution. ∎

3. The special case $n = 2$**.** In this section, we focus on the equation (2.1) for 2×2 matrices. Denote

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

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We first consider the existence of non-trivial solutions to (2.1) when $\mathbf{A}$ is an orthogonal matrix. When $\mathbf{A}$ is orthogonal with $|\mathbf{A}| = 1$, the existence of a non-trivial (orthogonal) solution $\mathbf{X} = \mathbf{A}^{1/2}$ is given in Proposition 1.2.

PROPOSITION 3.1. *Let* $\mathbf{A}$ *be an orthogonal matrix in* $\mathrm{M}_2(\mathbb{R})$ *with* $|\mathbf{A}| = -1$. *A non-trivial singular solution to (2.1) is given by* $\mathbf{X} = \frac{1}{2}\begin{bmatrix} 1+a_1 & a_2 \\ a_2 & 1-a_1 \end{bmatrix}$.

Proof. When $|\mathbf{A}| = -1$, $\mathbf{A}$ is a symmetric matrix with two distinct eigenvalues 1 and -1. Thus $\mathbf{A}$ is diagonalizable to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. By Lemma 2.1(6.) and Theorem 2.3, (2.1) has a non-trivial solution. A matrix of the form $\mathbf{X} = \mathbf{P}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{P}^{-1}$ is a non-trivial singular solution to (2.1) when $\mathbf{P}$ satisfies $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The solution $\mathbf{X} = \frac{1}{2}\begin{bmatrix} 1+a_1 & a_2 \\ a_2 & 1-a_1 \end{bmatrix}$ is obtained by finding such a matrix $\mathbf{P}$ made of two linearly independent eigenvectors of $\mathbf{A}$ via linear algebra (refer to the proof of Theorem 2.2). □

Now we discuss more general cases. In the next theorem, we show constructively that the equation (2.1) has non-trivial solutions for a large group of two by two matrices $\mathbf{A}$ (over the real numbers).

THEOREM 3.2. *Consider* $\mathbf{0} \neq \mathbf{A} \in \mathrm{M}_{\mathbf{2}}(\mathbb{R})$. *The equation* (2.1) *has non-trivial solutions in the following cases:*
1. $\mathbf{A}$ *has two distinct real eigenvalues, not both negative.*
2. $\mathbf{A}$ *is a scalar matrix.*
3. $\mathbf{A}$ *is a non-scalar matrix with a repeated non-negative eigenvalue.*

Proof. By Lemma 2.1 and Theorem 2.3, the first is true. The second claim is from Theorem 2.2. For the third, without loss of generality, we may assume

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ a_3 & a_1 \end{bmatrix},$$

where $0 \leq a_1$ and $a_3 \neq 0$. If $a_1 = 0$, the matrix $\mathbf{X} = \begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix}$ gives a non-trivial solution to (2.1) for any real number $x_3 \neq 0$. If $a_1 \neq 0$, the lower triangular matrix $\mathbf{X} = \begin{bmatrix} \sqrt{a_1} & 0 \\ a_3/(2\sqrt{a_1}) & \sqrt{a_1} \end{bmatrix}$ gives a non-trivial solution to (2.1). □

We note that by Proposition 2.5, we can extend solutions to (2.1) for the 2×2 case to solutions for $(n \times n)$-matrices. Finally, we construct non-zero matrices $\mathbf{A}$ for which $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ admits only the trivial solution.

THEOREM 3.3. *The equation* $\mathbf{X}^2\mathbf{A}\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}$ *admits only the trivial solution for any* $\mathbf{A} \in \mathrm{M}_2(\mathbb{R})$ *having two distinct negative eigenvalues or having a single negative eigenvalue of geometric multiplicity 1.*

Proof. For the first case, it is sufficient to assume $\mathbf{A} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$, where $\lambda_1 > \lambda_2 > 0$. Suppose $\mathbf{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ is a solution. Then $|\mathbf{X}| = 0$ or $\pm\sqrt{\lambda_1\lambda_2}$ since

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$\mathbf{A}$ is nonsingular. By comparing the non-diagonal entries of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we obtain the following two equations

$$\left\{\begin{array}{l} x_2(\lambda_1 x_1^2+\lambda_1 x_2 x_3+\lambda_2 x_1 x_4+\lambda_2 x_4^2+\lambda_1\lambda_2)=0 \\ x_3(\lambda_1 x_1^2+\lambda_1 x_1 x_4+\lambda_2 x_2 x_3+\lambda_2 x_4^2+\lambda_1\lambda_2)=0. \end{array}\right. \tag{3.1}$$

First we assume $0\neq|\mathbf{X}|=\sqrt{\lambda_1\lambda_2}$. Then $x_2x_3=x_1x_4-\sqrt{\lambda_1\lambda_2}$. Thus (3.1) becomes

$$\left\{\begin{array}{l} x_2(\lambda_1 x_1^2+(\lambda_1+\lambda_2)x_1 x_4+\lambda_2 x_4^2+\lambda_1\lambda_2-\lambda_1\sqrt{\lambda_1\lambda_2})=0 \\ x_3(\lambda_1 x_1^2+(\lambda_1+\lambda_2)x_1 x_4+\lambda_2 x_4^2+\lambda_1\lambda_2-\lambda_2\sqrt{\lambda_1\lambda_2})=0. \end{array}\right. \tag{3.2}$$

If $x_2x_3\neq 0$, then equations in (3.2) imply $\lambda_1\sqrt{\lambda_1\lambda_2}=\lambda_2\sqrt{\lambda_1\lambda_2}\Longrightarrow\lambda_1=\lambda_2$, a contradiction. If $x_2x_3=0$, we compare the (1,1) entries of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$ to obtain $-\lambda_1x_1^3=\lambda_1^2x_1\Longrightarrow x_1=0\Longrightarrow|\mathbf{X}|=0$, a contradiction again. Therefore $|\mathbf{X}|\neq\sqrt{\lambda_1\lambda_2}$. The same argument shows that $|\mathbf{X}|\neq-\sqrt{\lambda_1\lambda_2}$.

Now consider the case $|\mathbf{X}|=0$, i.e., $x_1x_4=x_2x_3$. By matrix multiplication, we have

$$\mathbf{X}^2\mathbf{AX}=-(x_1+x_4)(\lambda_1x_1+\lambda_2x_4)\begin{bmatrix} x_1 & x_2\\ x_3 & x_4\end{bmatrix}=\begin{bmatrix}\lambda_1^2x_1 & \lambda_1\lambda_2x_2\\ \lambda_1\lambda_2x_3 & \lambda_2^2x_4\end{bmatrix}=\mathbf{AXA}.$$

If $x_2\neq 0$ or $x_3\neq 0$, then $(x_1+x_4)(\lambda_1x_1+\lambda_2x_4)=-\lambda_1\lambda_2$ by comparing the non-diagonal entries. Apply this to the diagonal entries, we obtain $\lambda_1\lambda_2x_1=-\lambda_1^2x_1$ and $\lambda_1\lambda_2x_4=-\lambda_2^2x_4\Longrightarrow x_1=x_4=0$. Thus $\mathbf{X}^2\mathbf{AX}=\mathbf{0}\Longrightarrow\mathbf{AXA}=\mathbf{0}\Longrightarrow\mathbf{X}=\mathbf{0}$, since $\mathbf{A}$ is invertible. This gives only a trivial solution to (2.1). At last, consider the case of $x_2=0=x_3$. Since $x_1x_4=x_2x_3$, x_1 or $x_4=0$. If $x_1=0$, compare the (2,2)-entry of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we have $\lambda_2x_4^3=-\lambda_2^2x_4\Longrightarrow x_4=0$. Similarly, $x_4=0\Longrightarrow x_1=0$. Therefore $x_1=x_2=x_3=x_4=0$ and $\mathbf{X}$ is a trivial solution.

Now assume $\mathbf{A}$ has a single negative eigenvalue of geometric multiplicity 1. Let $\mathbf{A}=\begin{bmatrix} a_1 & 0\\ a_3 & a_1\end{bmatrix}$ where $a_1<0$ and $a_3\neq 0$. Assume $\mathbf{0}\neq\begin{bmatrix} x_1 & x_2\\ x_3 & x_4\end{bmatrix}$ is a solution to (2.1). We first claim that $x_2\neq 0$. If not, the diagonal entries of $\mathbf{X}^2\mathbf{AX}-\mathbf{AXA}$ are $a_1x_1(x_1^2-a_1)$ and $a_1x_4(x_4^2-a_1)$. Since a_1 is negative, it forces $x_1=x_4=0$ and then $x_3=0$. Now assume $\mathbf{X}$ is a singular solution. Then the second row of $\mathbf{X}$ is k times the first row for some real number $k\neq 0$. By equating the second row minus k times the first row of both $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we obtain a contradiction. When $\mathbf{X}$ is a nonsingular solution, $|\mathbf{X}|=a_1$ or $-a_1$. Since $x_2\neq 0$, $x_3=\frac{x_1x_4\pm a_1}{x_2}$. Then by equating the components of $\mathbf{X}^2\mathbf{AX}$ and $\mathbf{AXA}$, we obtain the following two equations:

$$\left\{\begin{array}{c} (x_1+x_4)x_2(a_1x_1\pm a_3x_2+a_1x_4)=0 \\ (x_1+x_4)(a_1x_1x_4\pm a_3x_2x_4\pm a_1^2+a_1x_4^2)=0. \end{array}\right.$$

This implies $x_1+x_4=0$. Then the $(1,1)$-component of $\mathbf{X}^2\mathbf{AX}-\mathbf{AXA}$ is $\pm a_1x_2a_3$ which can not be zero, a contradiction.

In conclusion, the equation (2.1) has no non-trivial solutions. □

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