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ISOMETRIC TIGHT FRAMES*

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Abstract. A $d \times n$ matrix, $n \geq d$, whose columns have equal length and whose rows are orthonormal is constructed. This is equivalent to finding an isometric tight frame of n vectors in $\mathbb{R}^d$ (or $\mathbb{C}^d$), or writing the $d \times d$ identity matrix $I = \frac{d}{n}\sum_{i=1}^n P_i$, where the P_i are rank 1 orthogonal projections. The simple inductive procedure given shows that there are many such isometric tight frames.

Key words. Isometric tight frame, normalised tight frame, uniform tight frame.

AMS subject classifications. 42C15, 52B15, 42C40

1. Introduction.

Any $n \geq 3$ equally spaced unit vectors $u_1, \ldots, u_n$ in $\mathbb{R}^2$ provide the following redundant representation

$$f = \frac{2}{n}\sum_{j=1}^{n}\langle f, u_j\rangle u_j, \qquad \forall f \in \mathbb{R}^2, \tag{1.1}$$

which is the simplest example of what is called a tight frame. Such representations arose in the study of nonharmonic Fourier series in $L_2(\mathbb{R})$ (see Duffin and Schaeffer [2]) and have recently been used extensively in the theory of wavelets (see, e.g., Daubechies [1]).

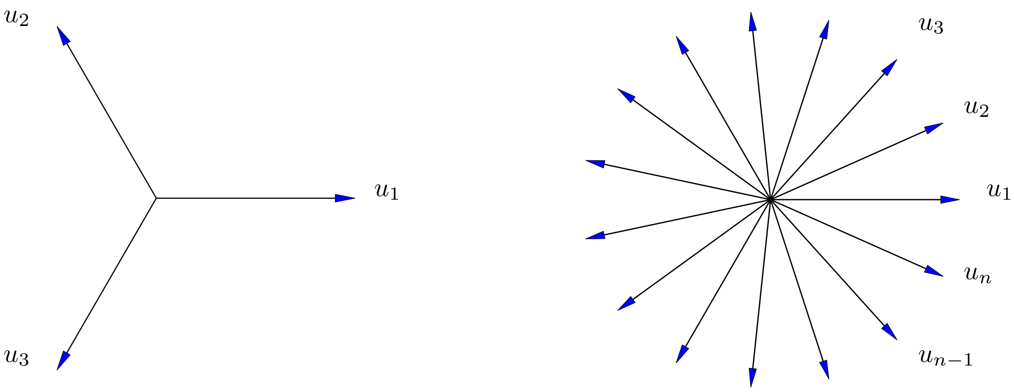

Fig. 1. Examples of n equally spaced vectors in $\mathbb{R}^2$ (which form a tight frame).

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If the vectors $u_1, \ldots, u_n$ above are taken as the columns of a matrix U, then the resulting matrix has orthogonal rows of equal length, e.g., for three and four vectors

$$U = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}. \tag{1.2}$$

We will see (Lemma 2.1) this property is equivalent to a representation of the form (1.1). The main result is that such a representation and the matrix equivalence exists for $\mathbb{F}^d$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $d \geq 1$.

THEOREM 1.1. *For $n \geq d$, there exists a $d \times n$ $\mathbb{F}$–valued matrix V whose columns $v_1, v_2, \ldots, v_n$ have equal length and whose rows have equal length and are orthogonal. This is equivalent to*

$$f = \frac{d}{n} \sum_{j=1}^{n} \langle f, u_j \rangle u_j, \qquad \forall f \in \mathbb{F}^d, \tag{1.3}$$

where $u_j := v_j / \|v_j\|$.

After proving this result, we became aware that others had considered this question with the solution being an explicit construction (most likely) first given in Goyal, Vetterli and Thao [5]. Our elementary proof, which is inductive, is entirely different. It does not require trigonometric identities and leads to many different solutions. Thus the purpose of this article is to show what are variously called isometric tight frames (Peng and Waldron [6]), normalised tight frames (Fickus [3], Zimmermann [8]) and uniform tight frames (Goyal, Kovačević and Kelner [4]) exist and are *numerous*, i.e., for given n, d many different examples can be constructed with our simple inductive procedure.

2. The construction of isometric tight frames. The key to our construction is the following matrix description of tight frames (the formal definition follows).

LEMMA 2.1. *Given vectors $v_1, \ldots, v_n \in \mathbb{F}^d$, there exists a scalar c satisfying*

$$f = c \sum_{j=1}^{n} \langle f, v_j \rangle v_j, \qquad \forall f \in \mathbb{F}^d, \tag{2.1}$$

if and only if the rows of the matrix $V := [v_1, \ldots, v_n]$ are orthogonal and have equal length. Moreover, c is given by

$$c = \frac{d}{\sum_{j=1}^{n} \|v_j\|^2}. \tag{2.2}$$

Proof. We prove the more general result (take $w_j = cv_j$) that there exists a representation of the form

$$f = \sum_{j=1}^{n} \langle f, v_j \rangle w_j, \qquad \forall f \in \mathbb{F}^d \tag{2.3}$$

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if and only if the rows of $V := [v_1, \dots, v_n]$ and $W := [w_1, \dots, w_n]$ are biorthogonal, i.e.,

$$WV^* = I. \tag{2.4}$$

This follows from the calculation $\sum_j \langle f, v_j \rangle w_j = \sum_j w_j (v_j^* f) = WV^* f$.

Now let $w_j = cv_j$, i.e., $W = cV$. Then there exists a c satisfying (2.1) if and only if

$$cVV^* = I, \tag{2.5}$$

i.e., the rows of V are orthogonal and of equal length. Taking the trace of (2.5) gives

$$c \sum_j \langle v_j, v_j \rangle = c \operatorname{trace}(V^* V) = \operatorname{trace}(cVV^*) = \operatorname{trace}(I) = d,$$

which is (2.2) □

Definition 2.2. A family $(v_j)_{j=1}^n$ in $\mathbb{F}^d$ is called a *frame* if there exist $A, B > 0$ with

$$A\|f\|^2 \le \sum_j |\langle f, v_j \rangle|^2 \le B\|f\|^2, \qquad \forall f \in \mathbb{F}^d.$$

A frame is *tight* if $A = B$. The v_j in (2.1) form a tight frame with $A = B = 1/c$, since taking the inner product of both sides of (2.1) with f gives

$$\|f\|^2 = \langle f, f \rangle = c \sum_{j=1}^n |\langle f, v_j \rangle|^2, \qquad \forall f \in \mathbb{F}^d.$$

Conversely, by the polarisation identity, each tight frame leads to a representation (2.1). By Lemma 2.1, a tight frame for $\mathbb{R}^d$ is also a tight frame for $\mathbb{C}^d$.

A tight frame is *isometric* if all the v_j have the same length, and *normalised* or *uniform* when this length is 1. We now restate Theorem 1.1 in terms of frames.

Theorem 2.3. *(Isometric tight frames). For $n \ge d$, there exists an isometric tight frame of n vectors in $\mathbb{F}^d$, i.e., there exist $v_1, \dots, v_n \in \mathbb{F}^d$ (of equal length) with*

$$f = \frac{d}{n} \sum_{j=1}^n \langle f, u_j \rangle u_j, \qquad \forall f \in \mathbb{F}^d, \tag{2.6}$$

where $u_j := v_j / \|v_j\|$. This is equivalent to there being a $d \times n$ matrix $V := [v_1, \dots, v_n]$ with columns of equal length and rows which are orthogonal and of equal length.

Our construction depends on two simple consequences of Lemma 2.1.

I. Unions of frames. If $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are isometric tight frames with the same common length, then their union is also, since the matrix

$$[v_1, \dots, v_n, w_1, \dots, w_m]$$

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has columns of equal length and rows which are orthogonal and of equal length.

II. Complementary frames. Given an isometric tight frame of n vectors in $\mathbb{F}^d$ we can construct (many) isometric tight frames of n vectors in $\mathbb{F}^{n-d}$ as follows. Let $v_1, \ldots, v_n$ be an isometric tight frame for $\mathbb{F}^d$. Then the $d \times n$ matrix $V := [v_1, \ldots, v_n]$ has orthogonal rows of equal length, say ℓ. Using the Gram–Schmidt process we can add an additional $n - d$ orthogonal rows of length ℓ to V to obtain an $n \times n$ matrix

$$A = \begin{bmatrix} v_1 & \cdots & v_n \\ w_1 & \cdots & w_n \end{bmatrix}, \qquad w_j \in \mathbb{F}^{n-d}.$$

Since the rows of A/ℓ are orthonormal, it is a unitary matrix, and so its columns have equal length. Thus $w_1, \ldots, w_n$ are of equal length (as the v_j are), and so form an isometric tight frame of n vectors in $\mathbb{F}^{n-d}$, which we will refer to as complementary to $v_1, \ldots, v_n$. For $n-d \geq 2$, there are infinitely many different complementary isometric tight frames, e.g., for three vectors in $\mathbb{R}$ one could take $V = [1, 1, -1]$, and then

$$(2.7) \qquad A = \begin{bmatrix} 1 & 1 & -1 \\ \frac{\sqrt{3}}{\sqrt{2}} & -\frac{\sqrt{3}}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & -1 \\ \frac{\sqrt{3}}{\sqrt{14}} & \frac{2\sqrt{3}}{\sqrt{14}} & \frac{3\sqrt{3}}{\sqrt{14}} \\ \frac{5}{\sqrt{14}} & \frac{-4}{\sqrt{14}} & \frac{1}{\sqrt{14}} \end{bmatrix}$$

giving the following complementary isometric tight frames for $\mathbb{R}^2$:

$$(w_j)_{j=1}^3 = \left(\begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{bmatrix} \right), \quad (w_j)_{j=1}^3 = \left(\begin{bmatrix} \frac{\sqrt{3}}{\sqrt{14}} \\ \frac{5}{\sqrt{14}} \end{bmatrix}, \begin{bmatrix} \frac{2\sqrt{3}}{\sqrt{14}} \\ \frac{-4}{\sqrt{14}} \end{bmatrix}, \begin{bmatrix} \frac{3\sqrt{3}}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{bmatrix} \right).$$

Proof of Theorem 2.3. By Lemma 2.1, it suffices to prove (by induction) the existence of a V with the desired properties.

Case $n = d \geq 1$. Take $V = U$ a unitary matrix, since unitary matrices have orthonormal rows and columns.

Case $n > d$. Write $n = kd + d + r$, where $0 \leq r < d$. We seek a matrix V of the form $V = [V_1, V_2, \cdots, V_k, W]$, where $V_j = cU_j$, $c > 0$, with U_j unitary. By the remark on the union of frames, this has the desired properties if (and only if) the $d \times (d + r)$ matrix W has columns of length c and orthogonal rows of equal length, i.e., its columns are an isometric tight frame of $d + r$ vectors in $\mathbb{F}^d$ (of length c). Such a frame is obtained by taking (an appropriate scalar multiple of) any frame which is complementary to an isometric tight frame of $d + r$ vectors in $\mathbb{F}^r$ (which we can construct by (strong) induction since $r < d$). □

We now illustrate the construction of the proof with an example.

Example. We construct an isometric tight frame of 8 vectors in $\mathbb{R}^3$. For simplicity, we take the unitary matrices U_j in the proof to be the identity I, and identify an isometric tight frame $(v_j)_{j=1}^n$ with the matrix $V = [v_1, \ldots, v_n]$. Thus, we seek an isometric tight frame of the form $V = [c_1 I, W_1]$ where W_1 is 3×5. This W_1 can be obtained by taking the complement of an isometric tight frame W_2 which is 2×5,

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say $W_2 = [c_2 I, W_3]$ where W_3 is 2×3. For W_3 we take the complement of W_4 a 1×3 isometric tight frame. Hence, e.g., taking $W_4 = [1,1,-1]$ and W_3 to be the first complement in (2.7), one can obtain

$$W_4 = [1,1,-1] \quad \overset{\text{take complement}}{\longrightarrow} \quad W_3 = \begin{bmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\overset{\text{add columns}}{\longrightarrow} \quad W_2 = \begin{bmatrix} 2 & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ 0 & 2 & 1 & 1 & 2 \end{bmatrix}$$

$$\overset{\text{take complement}}{\longrightarrow} \quad W_1 = \begin{bmatrix} 3 & 0 & -\sqrt{3} & \sqrt{3} & 0 \\ 0 & 3 & -1 & -1 & -2 \\ 0 & 0 & \sqrt{5} & \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

$$\overset{\text{add columns}}{\longrightarrow} \quad V = \begin{bmatrix} 3 & 0 & 0 & 3 & 0 & -\sqrt{3} & \sqrt{3} & 0 \\ 0 & 3 & 0 & 0 & 3 & -1 & -1 & -2 \\ 0 & 0 & 3 & 0 & 0 & \sqrt{5} & \sqrt{5} & -\sqrt{5} \end{bmatrix},$$

giving the isometric tight frame

$$\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{3} \\ -1 \\ \sqrt{5} \end{bmatrix}, \begin{bmatrix} \sqrt{3} \\ -1 \\ \sqrt{5} \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -\sqrt{5} \end{bmatrix} \right). \tag{2.8}$$

3. Discussion. For n odd, say $n = 2k+1$, the following explicit construction was given in [5]. Consider the following $n\times n$ matrix which has orthogonal rows of equal length.

$$\begin{bmatrix} 1 & \cos(2\pi\frac{1}{n}) & \cos(2\pi\frac{2}{n}) & \cdots & \cos(2\pi\frac{n-1}{n}) \\ 0 & \sin(2\pi\frac{1}{n}) & \sin(2\pi\frac{2}{n}) & \cdots & \sin(2\pi\frac{n-1}{n}) \\ 1 & \cos(2\pi\frac{2}{n}) & \cos(2\pi\frac{4}{n}) & \cdots & \cos(2\pi\frac{2(n-1)}{n}) \\ 0 & \sin(2\pi\frac{2}{n}) & \sin(2\pi\frac{4}{n}) & \cdots & \sin(2\pi\frac{2(n-1)}{n}) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos(2\pi\frac{k}{n}) & \cos(2\pi\frac{2k}{n}) & \cdots & \cos(2\pi\frac{(n-1)k}{n}) \\ 0 & \sin(2\pi\frac{k}{n}) & \sin(2\pi\frac{2k}{n}) & \cdots & \sin(2\pi\frac{(n-1)k}{n}) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{3.1}$$

Since $\sin^2 + \cos^2 = 1$, any $d\times n$ submatrix obtained by taking d rows consisting of pairs of rows $2i-1$ and $2i$, together with row n if d is odd, has columns of equal

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length, and hence gives an isometric tight frame of n vectors in $\mathbb{R}^d$ (or $\mathbb{C}^d$). A similar example for n even is given in [4] and [8].

Our construction, which can be extended by allowing the unitary matrices U_j to be replaced by matrices for (smaller) isometric tight frames, provides many examples of isometric tight frames. For example, in (2.8) the first three vectors could be replaced by any orthogonal basis of vectors (with length 3), and the fourth and fifth by any orthogonal basis for their span – not to mention taking different complementary frames.

Spherical equidistribution. Motivated by the example of equally spaced vectors in $\mathbb{R}^2$, it was hoped that isometric tight frames in $\mathbb{R}^d$ would correspond to points which were equally spaced on the sphere. Indeed, Fickus [3] has shown that normalised tight frames correspond to families of unit vectors which minimise the so called *frame potential*

$$\mathrm{FP}((v_j)_{j=1}^n) := \sum_{j=1}^n \sum_{k=1}^n |\langle v_j, v_k\rangle|^2,$$

corresponding to the *frame force*

$$\mathrm{FF}(a,b) := \langle a,b\rangle(a-b).$$

However, unlike the electrostatic analogue, the frame force between vectors which are equal is zero (rather than infinite), as is evidenced by the repeated vectors in example (2.8). In Vale and Waldron [7], it is shown that examples such as those given by (3.1) have a higher degree of symmetry than a generic frame obtained from our construction, and so are to be considered 'more' equally spaced. Thus, our construction does not give the 'best' isometric tight frames. The symmetry groups involved are finite, and so optimal isometric tight frames can be determined.

Terminology. Here we used the term 'isometric tight frame' (introduced in [6]) for what is afterall a tight frame for which the vectors have equal length. To us the epithet 'normalised' (which seems to have superseded 'uniform') suggests some standard presentation (which can be obtained by scaling), and indeed has been used some in the case when the (tight) frame bounds are (chosen to be) $A = B = 1$.

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