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ADDITIONAL RESULTS ON INDEX SPLITTINGS FOR DRAZIN INVERSE SOLUTIONS OF SINGULAR LINEAR SYSTEMS *

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Abstract. Given an $n \times n$ singular matrix A with $\mathrm{Ind}(A) = k$, an index splitting of A is one of the form $A = U - V$, where $R(U) = R(A^k)$ and $N(U) = N(A^k)$. This splitting, introduced by the first author, generalizes the proper splitting proposed by Berman and Plemmons. Regarding singular systems $Au = f$, the first author has shown that the iterations $u^{(i+1)} = U^{\#}Vu^{(i)} + U^{\#}f$ converge to $A^D f$, the Drazin inverse solution to the system, if and only if the spectral radius of $U^{\#}V$ is less than one. The aim of this paper is to further study index splittings in order to extend some previous results by replacing the Moore-Penrose inverse A^+ and A^{-1} with the Drazin inverse A^D. The characteristics of the Drazin inverse solution $A^D f$ are established. Some criteria are given for comparing convergence rates of $U_i^{\#}V_i$, where $A = U_1 - V_1 = U_2 - V_2$. Results of Collatz, Marek and Szyld on monotone-type iterations are extended. A characterization of the iteration matrix of an index splitting is also presented.

Key words. Index, Drazin inverse, group inverse, Moore-Penrose inverse, index splitting, proper splitting, comparison theorem, monotone iteration.

AMS subject classifications. 15A09, 65F15, 65F20

1. Introduction. It is well known that a necessary and sufficient condition for a matrix to be convergent is that all of its eigenvalues are less than one in magnitude. For linear systems with nonsingular coefficient matrices the convergence of an iterative scheme based on a splitting is equivalent to the corresponding iteration matrix being convergent. However, this is not the case for linear systems with singular coefficient matrices. This is due to fact that when we split A into $A = U - V$, we often assume that U is nonsingular.

Consider a general system of linear equations

$$Au = f, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, possibly singular, and u, f are the vectors in $\mathbb{R}^n$. Berman and Plemmons [3] consider the so-called proper splitting $A = U - V$ with

$$R(U) = R(A) \quad \text{and} \quad N(U) = N(A),$$

where $R(A)$ denotes the *range* of A and $N(A)$ denotes the *null space* of A. This is the case, for example, when U and A are nonsingular. It is shown in [3] that if U^+V is

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convergent, i.e., its spectral radius $\rho(U^+V)$ is less than one, then the iterative scheme

$$u^{(i+1)} = U^+Vu^{(i)} + U^+f \tag{1.2}$$

converges to the vector $u = (I - U^+V)^{-1}U^+f = A^+f$, which is the least squares solution of (1.1). Here U^+ denotes the Moore-Penrose inverse of U. This iterative scheme does not involve the normal equations $A^TAu = A^Tf$ and avoids the problem of A^TA being frequently ill-conditioned and influenced greatly by roundoff errors, as pointed out in [12]. Similar results have been extended to various types of generalized inverses and the corresponding solutions of (1.1); see [4].

Iterations of the type (1.2), where the system (1.1) is consistent and U is nonsingular, were studied by Keller in [17] and extended to rectangular (but still consistent) systems by Joshi [16]. In both cases the splitting is not proper.

In some situations, however, people pay more attention to the *Drazin inverse solution* A^Df of (1.1) ([6, 23, 24, 26, 28]), where A^D is the Drazin inverse of A. The Drazin inverse has various applications in the theory of finite Markov chains [5, Chapter 8], the study of singular differential and difference equations [5, Chapter 9], the investigation of Cesaro-Neumann iterations [13], cryptography [14], and iterative methods in numerical analysis [7, 9, 11, 15, 20, 27, 28]. Chen and Chen [6] presented a new splitting for singular linear systems and the computation of the Drazin inverse: Let $A = U - V$ be such that

$$R(U^k) = R(A^k) \quad \text{and} \quad N(U^k) = N(A^k). \tag{1.3}$$

It was proven in [6] that $A^D = (I - U^DV)^{-1}U^D$, where $k = \mathrm{Ind}(A)$ is the *index* of A, that is, the smallest nonnegative integer such that $R(A^{k+1}) = R(A^k)$.

For computing the Drazin inverse solution A^Df, Wei [26] proposed an *index splitting* of $A = U - V$ such that

$$R(U) = R(A^k) \ \text{ and } \ N(U) = N(A^k). \tag{1.4}$$

Then the iterative scheme (1.2) is modified into

$$u^{(i+1)} = U^{\#}Vu^{(i)} + U^{\#}f. \tag{1.5}$$

Clearly, if A and U are nonsingular, a typical splitting is an index splitting; and if $\mathrm{Ind}(A) = 1$, the index splitting reduces to a proper splitting. Note that the splitting (1.4) is easier to construct than that of (1.3); cf. Theorem 6.1.

It was shown in [26] that the iterates in (1.5) converge to A^Df if and only if $\rho(U^{\#}V) < 1$, and some sufficient conditions were given to ensure $\rho(U^{\#}V) < 1$. Partial results for proper splittings were also extended to index splittings, especially when $\mathrm{Ind}(A) = 1$.

In this paper we shall further study index splittings to establish some new results that are analogous to well-known results on regular splittings when A is nonsingular, and to proper splittings when A is singular.

The outline of this paper is as follows: In §2, we present notation used later and review briefly some preliminary results. In §3, first we give the characteristics

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of the Drazin inverse solution $A^D f$, and then present equivalent conditions for the convergence of the iterations. In §4, we discuss monotone-type iterations based on the index splitting. In §5, comparison theorems under regularity assumptions are established; a characterization of the iteration matrix of an index splitting is also developed in the last section.

2. Notation and preliminaries. Throughout the paper the following notation and definitions are used. $\mathbb{R}^n$ denotes the n-dimensional real space. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$. $\mathbb{R}^{n\times n}$ denotes the $n \times n$ real matrices. For $x,\ y \in \mathbb{R}^n$, (x, y) denotes their inner product. For L and M complementary subspaces of $\mathbb{R}^n$, $P_{L,M}$ denotes the projector on L along M.

A nonempty subset K of $\mathbb{R}^n$ is a *cone* if $\lambda \geq 0$ implies $\lambda K \subseteq K$. A cone K is *convex* if $K + K \subseteq K$, and *pointed* if $K \bigcap (-K) = \{0\}$. The *polar* of a cone K is the closed convex cone $K^* = \{y \in \mathbb{R}^n | x \in K \text{ implies } (x, y) \geq 0\}$. The *interior* K^d of a closed convex cone K is given algebraically by $K^d = \{x \in K | 0 \neq y \in K^* \text{ implies } (x, y) > 0\}$. A cone K is *solid* if K^d is nonempty. A closed convex cone K of $\mathbb{R}^n$ is *reproducing*, that is, $K + (-K) = \mathbb{R}^n$, if and only if K is solid (due to the finite-dimensionality of the space). By a *full cone* we mean a pointed, solid, closed convex cone.

DEFINITION 2.1. ([5]) Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$. The matrix $X \in \mathbb{R}^{n\times n}$ satisfying

$$AX = XA, \quad A^{k+1}X = A^k, \quad AX^2 = X \tag{2.1}$$

is called the *Drazin inverse* of A and is denoted by $X = A^D$. In particular, when $\mathrm{Ind}(A) = 1$, the matrix X in (2.1) is called the *group inverse* of A and is denoted by $X = A^{\#}$.

The Drazin inverse can be represented explicitly by the Jordan canonical form as follows. If

$$A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}, \tag{2.2}$$

where C is nonsingular and $\mathrm{rank}(C) = \mathrm{rank}(A^k)$, and N is nilpotent of order k, then

$$A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}. \tag{2.3}$$

In particular, if $\mathrm{Ind}(A) = 1$, then $N = 0$ in (2.2).

The matrix $M \in \mathbb{R}^{n\times n}$ is called K-*positive* if $MK \subseteq K$; see [4]. When $K = \mathbb{R}^n_+$, $MK \subseteq K$ is equivalent to M having nonnegative elements and denoted by $M \geq 0$. In the sequel, we denote $MK \subseteq K$ by $M \overset{K}{\geq} 0$ and $(M-N)K \subseteq K$ by $M \overset{K}{\geq} N$. Similarly, we denote $M(K\backslash\{0\}) \subseteq K^d$ by $M \overset{K}{>} 0$ and $(M - N)(K\backslash\{0\}) \subseteq K^d$ by $M \overset{K}{>} N$. For $x,\ y \in \mathbb{R}^n$, $y \overset{K}{\geq} x$ means $y - x \in K$. A sequence $\{x_i\}$ in $\mathbb{R}^n$ is called K-*monotone nondecreasing (nonincreasing)* if $x_i \overset{K}{\geq} x_{i-1}$ ($x_{i-1} \overset{K}{\geq} x_i$) for $i = 1, 2, \ldots$

Definition 2.2. ([4]) Let K be a full cone of $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$. A is called *Drazin* $\overset{K}{-}$ *monotone* if $A^D \overset{K}{\geq} 0$. In particular A is called *group* $\overset{K}{-}$ *monotone* if $A^\#$ exists and $A^\# \overset{K}{\geq} 0$.

Lemma 2.3. *Let K be a full cone of $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then $A^D \overset{K}{\geq} 0$ if and only if*

$$Ax \in K + N(A^k),\ x \in R(A^k) \Rightarrow x \in K.$$

Proof. The proof of this lemma is analogous to that of Theorem 1 in [22]. □

For the index splitting (1.4) and the corresponding iterative scheme (1.5), Wei proved the following results.

Lemma 2.4. ([26]) *Let $A = U - V$ be an index splitting of $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then*
(a) $\mathrm{Ind}(U) = 1$;
(b) *$I - U^\# V$ is nonsingular;*
(c) $A^D = (I - U^\# V)^{-1}U^\# = U^\#(I - VU^\#)^{-1}$;
(d) *$A^D f$ is the unique the solution of the system $x = U^\# V x + U^\# f$ for any $f \in \mathbb{R}^n$.*

Remark 1. It is easy to prove that $I + A^D V$ is nonsingular and $U^\# = (I + A^D V)^{-1}A^D = A^D(I + VA^D)^{-1}$.

Theorem 2.5. ([26]) *Let K be a full cone of $\mathbb{R}^n$, and let $A = U - V$ be an index splitting of A with $\mathrm{Ind}(A) = k$ such that $U^\# V \overset{K}{\geq} 0$. Then*

$$\rho(U^\# V) = \frac{\rho(A^D V)}{1 + \rho(A^D V)} < 1 \tag{2.4}$$

if and only if $A^D V \overset{K}{\geq} 0$.

Remark 2. Analogously to Theorem 2.5, we can see that if $VU^\# \overset{K}{\geq} 0$, then

$$\rho(VU^\#) = \frac{\rho(VA^D)}{1 + \rho(VA^D)} < 1 \tag{2.5}$$

if and only if $VA^D \overset{K}{\geq} 0$.

3. New convergence conditions. As aforementioned, $A^+ f$ is the minimal normal solution of (1.1) if the system is consistent, and is the minimal normal least squares solution of (1.1) if the system is not consistent (i.e., $A^+ f$ is the unique solution of the normal equations $A^T Au = A^T f$ in $R(A^T)$). Next we present two characteristics of the Drazin inverse solution $A^D f$.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then $A^D f$ is the unique solution in $R(A^k)$ of*

$$A^{k+1}u = A^k f. \tag{3.1}$$

Proof. It is clear that the system (3.1) is always consistent and $A^D f$ is a solution of it. Assume that there is another solution $u \in R(A^k)$ of (3.1). On one hand, $u - A^D f \in R(A^k)$. On the other hand, $u - A^D f \in N(A^{k+1})$ since these are all the solutions of (3.1) and $u - A^D f \in R(A^k) \bigcap N(A^k)$. Recall that if $R(A^k) \bigcap N(A^k) = \{0\}$, then $u = A^D f$. $\square$

Since (3.1) is analogous to $A^T A u = A^T f$, we shall call (3.1) the *generalized normal equations* of (1.1). The next theorem provides a better understanding of the solution $A^D f$. The P-norm is defined as $\|x\|_P = \|P^{-1}x\|_2$ for $x \in \mathbb{R}^n$, where P is a nonsingular matrix that transforms A into its Jordan canonical form (2.2).

THEOREM 3.2. *Let* $A \in \mathbb{R}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$. *Then* u^* *satisfies*

$$\|f - Au^*\|_P = \min_{u\in N(A)+R(A^{k-1})} \|f - Au\|_P$$

if and only if u^* *is the solution of*

$$A^{k+1}u = A^k f, \quad u \in N(A) + R(A^{k-1}). \tag{3.2}$$

Moreover, the Drazin inverse solution $u = A^D f$ *is the unique minimal* P*-norm solution* ([26]) *of the generalized normal equations* (3.1).

Proof. Write $f = AA^D f + (I - AA^D)f \overset{\triangle}{=} f_1 + f_2$. It follows that

$$\begin{aligned}\|f - Au\|_P^2 &= \|AA^D f - Au\|_P^2 + \|(I - AA^D)f\|_P^2 \\ &\quad + 2(AA^D f - Au)^T P^{-T} P^{-1}(I - AA^D)f.\end{aligned} \tag{3.3}$$

For any $u \in N(A) + R(A^{k-1})$, it can be easily deduced from (2.2) and (2.3) that the third term in (3.3) vanishes. Hence,

$$\begin{aligned}\|f - Au\|_P^2 &= \|AA^D f - Au\|_P^2 + \|(I - AA^D)f\|_P^2 \\ &\geq \|(I - AA^D)f\|_P^2,\end{aligned} \tag{3.4}$$

whenever $u \in N(A) + R(A^{k-1})$. The equality in (3.4) holds if and only if

$$Au = AA^D f, \quad u \in N(A) + R(A^{k-1}). \tag{3.5}$$

We are now in a position to prove the equivalence between (3.2) and (3.5). Multiplying both sides of (3.5) by A^k leads immediately to (3.2). Conversely, since $(A^D)^{k+1}A^k = A^D$, we can deduce that (3.2) is equivalent to

$$A^D A u = A^D f, \quad u \in N(A) + R(A^{k-1}). \tag{3.6}$$

Assume that u^* is the solution of (3.6). Then

$$A^D A u^* = A^D f_1. \tag{3.7}$$

It follows from (3.7) that $f_1 - Au^* \in N(A^D) = N(A^k)$. Note that $f_1 - Au^* \in R(A^k)$ since $u^* \in N(A) + R(A^{k-1})$; therefore, $f_1 - Au^* \in R(A^k) \bigcap N(A^k) = \{0\}$. Hence,

$$Au^* = f_1 = AA^D f,$$

showing that (3.2) implies (3.5).

Clearly, the general solution of the generalized normal equations (3.1) is

$$u = A^D f + z, \quad \forall z \in N(A^k).$$

It is easy to show that

$$\|u\|_P^2 = \|A^D f\|_P^2 + \|z\|_P^2 \geq \|A^D f\|_P^2.$$

Equality in the above relation holds if and only if $z = 0$, i.e., $u = A^D f$. ∎

Remark 3. In general, unlike $A^+ f$, the Drazin inverse solution $A^D f$ is not a true solution of a singular system (1.1), even if the system is consistent. However, Theorem 3.2 means that $u = A^D f$ is the unique minimal P-norm least squares solution of (1.1).

The following corollaries are obvious.

Corollary 3.3. ([26]) *Under the assumptions of Theorem 3.2, if $f \in R(A^k)$, then $u = A^D f$ is the unique minimal P-norm solution of* (1.1).

Corollary 3.4. ([27]) *Let $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{Ind}(A) = 1$. *Then, if $f \in R(A)$, $u = A^\# f$ is the unique minimal P-norm solution of* (1.1)*; if $f \notin R(A)$, $u = A^\# f$ is the unique minimal P-norm least squares solution of* (1.1).

We now give the main results of this section.

Theorem 3.5. *Let $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{Ind}(A) = k$. *Let $A = U - V$ be an index splitting of A. Let L and K be full cones of $\mathbb{R}^n$. If $U^\# L \subseteq K$ and $U^\# V \overset{K}{\geq} 0$, then the following statements are equivalent:*
(a) $A^D L \subseteq K$;
(b) $A^D V \overset{K}{\geq} 0$;
(c) $\rho(U^\# V) = \frac{\rho(A^D V)}{1+\rho(A^D V)} < 1$.

Proof. The proof of this theorem is similar to that of Theorem 3 in [3]. ∎

Similarly, we have the following result.

Theorem 3.6. *Let $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{Ind}(A) = k$ *and $A = U - V$ be an index splitting of A. Let L and K be full cones of $\mathbb{R}^n$. If $U^\# L \subseteq K$ and $VU^\# \overset{L}{\geq} 0$, then the following statements are equivalent:*
(a) $A^D L \subseteq K$;
(b) $VA^D \overset{L}{\geq} 0$;
(c) $\rho(U^\# V) = \frac{\rho(VA^D)}{1+\rho(VA^D)} < 1$.

As a result of Theorem 3.5 and Theorem 3.6, we have the following.

Corollary 3.7. *Let $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{Ind}(A) = k$ *and $A = U - V$ be an index splitting of A. Let K be a full cone of $\mathbb{R}^n$. If $U^\# \overset{K}{\geq} 0$ and either $U^\# V \overset{K}{\geq} 0$ or $VU^\# \overset{K}{\geq} 0$, then $\rho(U^\# V) < 1$ if and only if $A^D \overset{K}{\geq} 0$.*

Remark 4. Clearly, Corollary 3.7 extends the well-known results of Varga [25] and Ortega and Rheiboldt [21] for regular and weak regular splittings of a nonsingular matrix. When $\mathrm{Ind}(A) = 1$, Theorem 3.5 reduces to Theorem 4.2 in [26].

4. Monotone-type iterations. Conditions under which iterations resulting from a proper splitting of a nonsingular matrix A are monotone were given by Collatz [8]. More research on monotone iterations of proper splittings is accomplished in [3, 19]. In this section these results are partially extended and generalized to index splittings; the requirement that A be nonsingular is removed and monotonicity is replaced by K-monotonicity.

THEOREM 4.1. *Let K be a full cone of $\mathbb{R}^n$ and let $A = U - V$ be an index splitting of $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Assume that $U^{\#}V \overset{K}{\geq} 0$.*

(a) *If there exist $u^{(0)}$, $w^{(0)}$ such that $u^{(1)} \overset{K}{\geq} u^{(0)}$, $w^{(0)} \overset{K}{\geq} u^{(0)}$ and $w^{(0)} \overset{K}{\geq} w^{(1)}$, where $u^{(i)}$ and $w^{(i)}$ are computed by*

$$u^{(i+1)} = U^{\#}Vu^{(i)} + U^{\#}f, \quad w^{(i+1)} = U^{\#}Vw^{(i)} + U^{\#}f, \tag{4.1}$$

for $i = 0, 1, 2, \ldots$, then

$$u^{(0)} \overset{K}{\leq} u^{(1)} \overset{K}{\leq} \cdots \overset{K}{\leq} u^{(i)} \overset{K}{\leq} \cdots \overset{K}{\leq} A^D f \overset{K}{\leq} \cdots \overset{K}{\leq} w^{(i)} \overset{K}{\leq} \cdots \overset{K}{\leq} w^{(1)} \overset{K}{\leq} w^{(0)},$$

and for each real λ satisfying $0 \leq \lambda \leq 1$,

$$A^D f = \lambda \lim_{i\to\infty} u^{(i)} + (1-\lambda) \lim_{i\to\infty} w^{(i)}.$$

(b) *If $\rho(U^{\#}V) < 1$, then the existence of $u^{(0)}$ and $w^{(0)}$ satisfying the assumptions of clause* (a) *is assured.*

Proof. The proof of this theorem is analogous to that of Theorem 4 in [3]. ☐

REMARK 5. Clearly, if $A^D V \overset{K}{\geq} 0$, then there exist $u^{(0)}$ and $w^{(0)}$ such that the sequences $\{u^{(i)}\}$ and $\{w^{(i)}\}$ are K-monotone and converge to the Drazin inverse solution $A^D f$.

THEOREM 4.2. *Let $A = U - V$ be an index splitting of $A \in \mathbb{R}^{n\times n}$ with* $\mathrm{Ind}(A) = k$. *Let L and K be full cones of $\mathbb{R}^n$. Assume that $U^{\#}L \subseteq K$ and $U^{\#}V \overset{K}{\geq} 0$. Let $u^{(0)}$, $w^{(0)} \in R(A^k)$ satisfy*

$$u^{(0)} \overset{K}{\leq} w^{(0)} \quad \text{and} \quad Au^{(0)} \overset{L}{\leq} f \overset{L}{\leq} Aw^{(0)}.$$

Then the sequences $\{u^{(i)}\}$ and $\{w^{(i)}\}$ computed by (4.1) *converge to $A^D f$ and*

$$u^{(0)} \overset{K}{\leq} u^{(1)} \overset{K}{\leq} \cdots \overset{K}{\leq} u^{(i)} \overset{K}{\leq} \cdots \overset{K}{\leq} A^D f \overset{K}{\leq} \cdots \overset{K}{\leq} w^{(i)} \overset{K}{\leq} \cdots \overset{K}{\leq} w^{(1)} \overset{K}{\leq} w^{(0)}.$$

Proof. In accordance with Theorem 4.1, it suffices to show that $u^{(0)} \overset{K}{\leq} u^{(1)}$ and $w^{(1)} \overset{K}{\leq} w^{(0)}$. It follows form $u^{(1)} = U^{\#}Vu^{(0)} + U^{\#}f$ and $u^{(0)} \in R(A^k) = R(U)$ that

$$\begin{aligned} u^{(1)} - u^{(0)} &= U^{\#}Vu^{(0)} - U^{\#}Uu^{(0)} + U^{\#}f \\ &= U^{\#}(f - Au^{(0)}). \end{aligned}$$

Hence $u^{(1)} - u^{(0)} \in K$, because $U^{\#}L \subseteq K$ and $Au^{(0)} \overset{L}{\leq} f$. The proof of $w^{(1)} \overset{K}{\leq} w^{(0)}$ is analogous. ☐

5. Comparison theorems. In this section we present some comparison theorems on index splittings of the same matrix. These comparison theorems are generalizations of some well-known comparison theorems for nonsingular matrices. For simplicity we consider only the case $K = \mathbb{R}^n_+$.

THEOREM 5.1. *Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $A^D \geq 0$. Let $A = U_1 - V_1 = U_2 - V_2$ be index splittings of A such that $U_i^{\#} \geq 0$, $V_i \geq 0$ for $i = 1, 2$. If $V_2 \geq V_1$, then*

$$\rho(U_1^{\#}V_1) \leq \rho(U_2^{\#}V_2) < 1.$$

In particular, if $V_2 \geq V_1 \geq 0$, equality excluded, and $A^D > 0$, then

$$\rho(U_1^{\#}V_1) < \rho(U_2^{\#}V_2) < 1.$$

Proof. The proof is analogous to that of Theorems 12 and 13 in [29]. □

THEOREM 5.2. *Let $A = U_1 - V_1 = U_2 - V_2$ be index splittings of $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Let $A^D \geq 0$ and no column or row vanish. Let $U_i^{\#} \geq 0$, $U_i^{\#}V_i \geq 0$ and $V_iU_i^{\#} \geq 0$ for $i = 1, 2$. If $U_1^{\#} \geq U_2^{\#}$, then*

$$\rho(U_1^{\#}V_1) \leq \rho(U_2^{\#}V_2) < 1.$$

Moreover, if $U_1^{\#} > U_2^{\#}$ and $A^D > 0$, then

$$\rho(U_1^{\#}V_1) < \rho(U_2^{\#}V_2) < 1.$$

Proof. The proof is similar to that of Theorems 3.5 and 3.6 in [30]. □

THEOREM 5.3. *Let $A = U_1 - V_1 = U_2 - V_2$ be convergent index splittings of $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$, where $A^D \geq 0$. Assume that $U_i^{\#}V_i \geq 0$ for $i = 1, 2$. Let $y_1 \geq 0$, $y_2 \geq 0$ be such that $U_1^{\#}V_1y_1 = \rho(U_1^{\#}V_1)y_1$, $U_2^{\#}V_2y_2 = \rho(U_2^{\#}V_2)y_2$. If either*

$$V_2y_1 \geq V_1y_1 \quad \text{or} \quad V_2y_2 \geq V_1y_2,$$

with $y_2 > 0$, then

$$\rho(U_1^{\#}V_1) \leq \rho(U_2^{\#}V_2).$$

Moreover, if $A^D > 0$ and either $V_2y_1 \geq V_1y_1$, equality excluded, or $V_2y_2 \geq V_1y_2$ with $y_2 > 0$ and $V_2 \neq V_1$, then

$$\rho(U_1^{\#}V_1) < \rho(U_2^{\#}V_2).$$

Proof. The proof is similar to that of Corollary 6.3 in [30]. □

REMARK 6. The hypothesis $y_2 > 0$ is immediately satisfied if $U_2^{\#}V_2$ is irreducible.

Other comparison theorems of proper splittings for a nonsingular matrix can be similarly extended to our index splitting case; cf. [10, 18, 19].

6. Characterization of the iteration matrix. Let $A = B - C$, where B is nonsingular, and let $T = B^{-1}C$. If A is nonsingular, then $I - T$ is nonsingular. When A is singular, a necessary and sufficient condition for the existence of a splitting $A = B - C$ such that $T = B^{-1}C$ is $N(A) = N(I - T)$. Furthermore, in the latter case, if a proper splitting exists, then there are infinitely many proper splittings such that $T = B^{-1}C$; see [1]. Now it is of interest to enquire about index splittings.

An $n \times n$ matrix with $\mathrm{Ind}(A) = k$ and $\mathrm{rank}(A^k) = r$ may be factored as

$$A = P \begin{bmatrix} A_{11} & 0 \\ 0 & Q \end{bmatrix} P^{-1}, \tag{6.1}$$

where A_{11} is an $r \times r$ nonsingular matrix and Q is nilpotent of order k. The following theorem gives a sufficient and necessary condition under which an index splitting of A can be found.

THEOREM 6.1. *Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$ having the form (6.1). Then*

$$R(U) = R(A^k) \quad \textit{and} \quad N(U) = N(A^k)$$

if and only if

$$U = P \begin{bmatrix} U_{11} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \tag{6.2}$$

where U_{11} is an arbitrary nonsingular matrix of order r.

Proof. From (6.1) it follows that

$$A^k = P \begin{bmatrix} A_{11}^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

The remaining proof is analogous to that of Theorem 1 in [2]. ∎

Theorem 6.1 indicates that in order to obtain an index splitting, one should find a factorization (6.1) of A, then split A_{11} into

$$A_{11} = U_{11} - V_{11}, \tag{6.3}$$

where U_{11} is nonsingular and form U as in (6.2).

THEOREM 6.2. *Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$ having the form (6.1). Let U, U_{11} and V_{11} be as given in (6.2) and (6.3). Then*

$$\rho(U^{\#}V) = \rho(U_{11}^{-1}V_{11}).$$

Proof. The group inverse of U is

$$U^{\#} = P \begin{bmatrix} U_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Since

$$V = P \begin{bmatrix} V_{11} & 0 \\ 0 & -Q \end{bmatrix} P^{-1},$$

then

$$U^{\#}V = P \begin{bmatrix} U_{11}^{-1}V_{11} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Hence $\rho(U^{\#}V) = \rho(U_{11}^{-1}V_{11})$. ◻

Finally, we present a characterization of the iteration matrix of an index splitting.

THEOREM 6.3. *Let $A \in \mathbb{R}^{n\times n}$ with $\mathrm{Ind}(A) = k$ having the form* (6.1). *Let $T \in \mathbb{R}^{n\times n}$ be such that $I - T$ is nonsingular. Then there exists an index splitting $A = U - V$ such that $T = U^{\#}V$ if and only if T has the form*

$$T = P \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \quad G \in \mathbb{R}^{r\times r}. \tag{6.4}$$

Furthermore, if (6.4) *holds, then the index splitting $A = U - V$ with $T = U^{\#}V$ is unique.*

Proof. ($\Leftarrow$) Assume that there is an index splitting $A = U - V$ such that $T = U^{\#}V$. By Theorem 6.1 we have

$$U = P \begin{bmatrix} U_{11} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \quad V = P \begin{bmatrix} V_{11} & 0 \\ 0 & -Q \end{bmatrix} P^{-1},$$

where $U_{11} \in \mathbb{R}^{r\times r}$ is nonsingular, and $V_{11} = U_{11} - A_{11}$. So

$$T = P \begin{bmatrix} U_{11}^{-1}V_{11} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \triangleq P \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

($\Rightarrow$) Since $I - T$ is nonsingular, $I - G$ is also nonsingular. By Lemma 2.3 in [18] there exists a unique proper splitting $A_{11} = U_{11} - V_{11}$ such that $G = U_{11}^{-1}V_{11}$. Setting

$$U = P \begin{bmatrix} U_{11} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \quad V = P \begin{bmatrix} V_{11} & 0 \\ 0 & -Q \end{bmatrix} P^{-1},$$

one obtains the unique index splitting $A = U - V$ such that $T = U^{\#}V$. ◻

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