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BASIC COMPARISON THEOREMS FOR WEAK AND WEAKER MATRIX SPLITTINGS*

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Abstract. The main goal of this paper is to present comparison theorems proven under natural conditions such as $N_2 \geq N_1$ and $M_1^{-1} \geq M_2^{-1}$ for weak and weaker splittings of $A = M_1 - N_1 = M_2 - N_2$ in the cases when $A^{-1} \geq 0$ and $A^{-1} \leq 0$.

Key words. Systems of linear equation, convergence conditions, comparison theorems, weak splittings, weaker splittings.

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1. Introduction. A large class of iterative methods for solving system of linear equations of the form

$$Ax = b,$$

where $A \in \mathbb{R}^{\mathsf{n}\times\mathsf{n}}$ is a nonsingular matrix and x, $b \in \mathbb{R}^{\mathsf{n}}$, can be formulated by means of the splitting

$$A = M - N \quad \text{with} \quad M \text{ nonsingular}, \tag{1.1}$$

and the approximate solution $x^{(t+1)}$ is generated as follows

$$Mx^{(t+1)} = Nx^{(t)} + b, \quad t \geq 0,$$

or equivalently,

$$x^{(t+1)} = M^{-1}Nx^{(t)} + M^{-1}b, \quad t \geq 0,$$

where the starting vector $x^{(0)}$ is given.

The above iterative method is convergent to the unique solution $x = A^{-1}b$ for each $x^{(0)}$ if and only if $\varrho(M^{-1}N) < 1$, which means that the splitting of $A = M - N$ is convergent. The convergence analysis of the above method is based on the spectral radius of the iteration matrix $\varrho(M^{-1}N)$. As is well known, the smaller is $\varrho(M^{-1}N)$, the faster is the convergence; see, e.g., [1].

The definitions of splittings, with progressively weaker conditions and consistent from the viewpoint of names, are collected in the following definition.

Definition 1.1. Let $M, N \in \mathbb{R}^{n\times n}$. Then the decomposition $A = M - N$ is called
(a) a *regular splitting of* A if $M^{-1} \geq 0$ and $N \geq 0$,

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(b) a *nonnegative splitting of* A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
(c) a *weak nonnegative splitting of* A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*),
(d) a *weak splitting of* A if M is nonsingular, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
(e) a *weaker splitting of* A if M is nonsingular and either $M^{-1}N \geq 0$ (the *first type*) or $NM^{-1} \geq 0$ (the *second type*),
(f) a *convergent splitting of* A if $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$.

The splittings defined in the successive items extend progressively a class of splittings of $A = M - N$ for which the matrices N and M^{-1} may lose the property of nonnegativity. Distinguishing both types of weak nonnegative and weaker splittings leads to further extensions allowing us to analyze cases when $M^{-1}N$ may have negative entries if only NM^{-1} is a nonnegative matrix.

Different splittings were extensively analyzed by many authors, see, e.g., [2] and the references therein.

Conditions ensuring that a splitting of a nonsingular matrix $A = M - N$ is convergent are unknown in a general case. As was pointed out in [2], the splittings defined in first three items of Definition 1.1 are convergent if and only if $A^{-1} \geq 0$, which means that both conditions $A^{-1} \geq 0$ and $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$ are equivalent. We write this formally as the following lemma.

Lemma 1.2. *Each weak nonnegative (as well as nonnegative and regular) splitting of $A = M - N$ is convergent if and only if $A^{-1} \geq 0$. In other words, if A is not a monotone matrix, it is impossible to construct a convergent weak nonnegative splitting.*

In the case of weak and weaker splittings, the assumption $A^{-1} \geq 0$ is not a sufficient condition in order to ensure the convergence of a given splitting of A; it is also possible to construct a convergent weak or weaker splitting when $A^{-1} \ngeq 0$. Moreover, as can be shown by examples the conditions $A^{-1}N \geq 0$ or $NA^{-1} \geq 0$ may not ensure that a given splitting of A will be a weak or weaker splitting.

The properties of weaker splittings are summarized in the following theorem.

Theorem 1.3. *Let $A = M - N$ be a weaker splitting of A. If $A^{-1} \geq 0$, then*
1. *If $M^{-1}N \geq 0$, then $A^{-1}N \geq M^{-1}N$ and if $NM^{-1} \geq 0$, then $NA^{-1} \geq NM^{-1}$.*
2. $\varrho(M^{-1}N) = \dfrac{\varrho(A^{-1}N)}{1 + \varrho(A^{-1}N)} = \dfrac{\varrho(NA^{-1})}{1 + \varrho(NA^{-1})}$.

Thus, we can conclude that for a convergent weaker splitting of a monotone matrix A there are three conditions $M^{-1}N \geq 0$ (or $NM^{-1} \geq 0$), $A^{-1}N \geq 0$ (or $NA^{-1} \geq 0$) and $\varrho(M^{-1}N) = \varrho(NM^{-1}) < 1$, and any two conditions imply the third.

The main goal of this paper is to present comparison theorems proven under natural conditions such as $N_2 \geq N_1$ and $M_1^{-1} \geq M_2^{-1}$ for weak and weaker splittings of $A = M_1 - N_1 = M_2 - N_2$ in the cases when $A^{-1} \geq 0$ and $A^{-1} \leq 0$.

2. Comparison theorems. When both convergent weaker splittings of a monotone matrix

$$A = M_1 - N_1 = M_2 - N_2 \tag{2.1}$$

are of the same type, the inequality

$$N_2 \geq N_1 \tag{2.2}$$

implies either

$$A^{-1}N_2 \geq A^{-1}N_1 \geq 0 \quad \text{or} \quad N_2A^{-1} \geq N_1A^{-1} \geq 0.$$

Hence, by the Perron-Frobenius theory of nonnegative matrices (see, e.g., [1]), we have $\varrho(A^{-1}N_1) \leq \varrho(A^{-1}N_2)$ or $\varrho(N_1A^{-1}) \leq \varrho(N_2A^{-1})$ and by Theorem 1.3 we can conclude the following result.

THEOREM 2.1. [2] *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weaker splittings of A of the same type, that is, either $M_1^{-1}N_1 \geq 0$ and $M_2^{-1}N_2 \geq 0$ or $N_1M_1^{-1} \geq 0$ and $N_2M_2^{-1} \geq 0$, where $A^{-1} \geq 0$. If $N_2 \geq N_1$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

This theorem, proven originally by Varga [1] for regular splittings, carries over to the case when both weaker splittings are of the same type. As is pointed out in [3] when both splittings in (2.1) are of different types, the condition (2.2) may not hold.

In the case when $A^{-1} \leq 0$, then the inequality (2.2) implies either

$$0 \leq A^{-1}N_2 \leq A^{-1}N_1 \quad \text{or} \quad 0 \leq N_2A^{-1} \leq N_1A^{-1}.$$

Hence, one can deduce the following theorem.

THEOREM 2.2. *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weaker splittings of A of the same type, that is, either $M_1^{-1}N_1 \geq 0$ and $M_2^{-1}N_2 \geq 0$ or $N_1M_1^{-1} \geq 0$ and $N_2M_2^{-1} \geq 0$, where $A^{-1} \leq 0$. If $N_2 \geq N_1$, then*

$$\varrho(M_1^{-1}N_1) \geq \varrho(M_2^{-1}N_2).$$

Similarly as in the case of $A^{-1} \geq 0$, it can be shown that when both splittings in (2.1) are of different types for $A^{-1} \leq 0$, condition (2.2) may not arise.

In the case of the weaker condition

$$M_1^{-1} \geq M_2^{-1} \tag{2.3}$$

the contrary behavior is observed. As is demonstrated on examples in [2], when both weak nonnegative splittings of a monotone matrix A are the same type, with $M_1^{-1} \geq M_2^{-1}$ (or even $M_1^{-1} > M_2^{-1}$) it may occur that $\varrho(M_1^{-1}N_1) > \varrho(M_2^{-1}N_2)$.

Let us assume that both convergent weaker splittings in (2.1) are of different types such that $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$, and let $v_1 \geq 0$ and $y_2 \geq 0$ be the eigenvectors such that

$$v_1^TM_1^{-1}N_1 = \lambda_1v_1^T \tag{2.4}$$

and

$$N_2M_2^{-1}y_2 = \lambda_2y_2, \tag{2.5}$$

where $\lambda_1 = \varrho(M_1^{-1}N_1)$ and $\lambda_2 = \varrho(M_2^{-1}N_2) = \varrho(N_2M_2^{-1})$. Multiplying (2.4) on the right by $A^{-1}y_2$ and (2.5) on the left by $v_1^TA^{-1}$, one obtains

$$v_1^TM_1^{-1}N_1A^{-1}y_2 = \lambda_1v_1^TA^{-1}y_2$$

and

$$v_1^TA^{-1}N_2M_2^{-1}y_2 = \lambda_2v_1^TA^{-1}y_2,$$

and after subtraction we obtain

$$v_1^T(A^{-1}N_2M_2^{-1} - M_1^{-1}N_1A^{-1})y_2 = (\lambda_2 - \lambda_1)v_1^TA^{-1}y_2.$$

From (1.1) we have

$$M^{-1} = (A+N)^{-1} = A^{-1}(I + NA^{-1})^{-1} = (I + A^{-1}N)^{-1}A^{-1},$$

or

$$A^{-1} = M^{-1} + M^{-1}NA^{-1} = M^{-1} + A^{-1}NM^{-1}$$

which implies that

$$A^{-1}N_2M_2^{-1} - M_1^{-1}N_1A^{-1} = M_1^{-1} - M_2^{-1}.$$

Hence, one obtains

$$v_1^T(M_1^{-1} - M_2^{-1})y_2 = (\lambda_2 - \lambda_1)v_1^TA^{-1}y_2. \tag{2.6}$$

Let us consider the following cases.

Case I. When $A^{-1} > 0$, then $v_1^TA^{-1}y_2 > 0$.

1. If $M_1^{-1} > M_2^{-1}$, then $M_1^{-1} - M_2^{-1} > 0$ and $v_1^T(M_1^{-1} - M_2^{-1})y_2 > 0$, hence $\lambda_2 - \lambda_1 > 0$ and $\lambda_2 > \lambda_1$.
2. If $M_1^{-1} \geq M_2^{-1}$, then $M_1^{-1} - M_2^{-1} \geq 0$ and

a) if $v_1^T(M_1^{-1} - M_2^{-1})y_2 > 0$, hence $\lambda_2 - \lambda_1 > 0$ and $\lambda_2 > \lambda_1$.
b) if $v_1^T(M_1^{-1} - M_2^{-1})y_2 = 0$, hence $\lambda_2 - \lambda_1 = 0$ and $\lambda_2 = \lambda_1$.

Case II. When $A^{-1} \geq 0$, then $v_1^TA^{-1}y_2 \geq 0$.

1. If $v_1^T(M_1^{-1} - M_2^{-1})y_2 > 0$, then $v_1^TA^{-1}y_2 > 0$, hence $\lambda_2 - \lambda_1 > 0$ and $\lambda_2 > \lambda_1$.
2. If $v_1^T(M_1^{-1} - M_2^{-1})y_2 = 0$, then

a) for $v_1^TA^{-1}y_2 > 0$, $\lambda_2 - \lambda_1 = 0$ and $\lambda_2 = \lambda_1$.
b) for $v_1^TA^{-1}y_2 = 0$, the relation (2.6) is satisfied for arbitrary values of λ_1 and λ_2.

The following examples of regular splittings illustrate the case II.2.b).

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = M_1 - N_1 = M_2 - N_2, \text{ where}$$

$$M_1 = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}, \ N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ M_1^{-1}N_1 = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } v_1^T = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2^{-1}N_2 = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{2}{7} \end{bmatrix} \text{ and } y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Evidently, $v_1^T(M_1^{-1} - M_2^{-1})y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{2}{35} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

and $v_1^T A^{-1} y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.$

However, a simple modification allows us to avoid this apparent difficulty appearing in the case II.2.b). Assuming a matrix $B > 0$, then instead the equations (2.4) and (2.5) the following equations may be taken in consideration

$$\widetilde{v}_1^T(\varepsilon A^{-1}B + M_1^{-1}N_1) = \widetilde{\lambda}_1 \widetilde{v}_1^T \tag{2.7}$$

and

$$(\varepsilon BA^{-1} + N_2 M_2^{-1})\widetilde{y}_2 = \widetilde{\lambda}_2 \widetilde{y}_2. \tag{2.8}$$

Since for $\varepsilon > 0$ both matrices $\varepsilon A^{-1}B + M_1^{-1}N_1$ and $\varepsilon BA^{-1} + N_2M_2^{-1}$ are irreducible, their eigenvalues $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ corresponding to spectral radii are strictly increasing functions of $\varepsilon \geq 0$ [1], and $\widetilde{\lambda}_1 = \lambda_1$, $\widetilde{\lambda}_2 = \lambda_2$, $\widetilde{v}_1^T = v_1^T$ and $\widetilde{y}_2 = y_2$ with $\varepsilon = 0$. Multiplying (2.7) on the right by $A^{-1}\widetilde{y}_2$ and (2.8) on the left by $\widetilde{v}_1^T A^{-1}$ and proceeding similarly as with the derivation of (2.6), one obtains finally

$$\widetilde{v}_1^T(M_1^{-1} - M_2^{-1})\widetilde{y}_2 = (\widetilde{\lambda}_2 - \widetilde{\lambda}_1)\widetilde{v}_1^T A^{-1}\widetilde{y}_2. \tag{2.9}$$

Since for $\varepsilon > 0$ both eigenvectors $\widetilde{v}_1$ and $\widetilde{y}_2$ are positive, it can be concluded that $\widetilde{v}_1^T(M_1^{-1} - M_2^{-1})\widetilde{y}_2 > 0$ and $\widetilde{v}_1^T A^{-1}\widetilde{y}_2 > 0$, which implies that $\widetilde{\lambda}_2 - \widetilde{\lambda}_1 > 0$ hence $\widetilde{\lambda}_2 > \widetilde{\lambda}_1$. Taking the limit for $\varepsilon \to 0$, it follows that $\widetilde{\lambda}_1 \to \lambda_1$ and $\widetilde{\lambda}_2 \to \lambda_2$ which allows us to conclude that $\lambda_2 \geq \lambda_1$.

In the case when both convergent weaker splittings are of different type but such that $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, then instead of the equations (2.4) and (2.5) we can consider the equations

$$N_1M_1^{-1}y_1 = \lambda_1 y_1 \quad \text{and} \quad v_2^T M_2^{-1}N_2 = \lambda_2 v_2^T$$

providing us the following equation

$$v_2^T(M_1^{-1} - M_2^{-1})y_1 = (\lambda_2 - \lambda_1)v_2^T A^{-1}y_1,$$

from which in a similar way we can conclude that $\lambda_2 \geq \lambda_1$.

Thus, from the above considerations we obtain the following result.

Theorem 2.3. [2] *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weaker splittings of different types, that is, either $M_1^{-1}N_1 \geq 0$ and $N_2M_2^{-1} \geq 0$ or $N_1M_1^{-1} \geq 0$ and $M_2^{-1}N_2 \geq 0$, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then*

$$\varrho(M_1^{-1}N_1) \leq \varrho(M_2^{-1}N_2).$$

In particular, if $A^{-1} > 0$ *and* $M_1^{-1} > M_2^{-1}$, *then*

$$\varrho(M_1^{-1}N_1) < \varrho(M_2^{-1}N_2).$$

Assuming now that both convergent weaker splittings of different types in (2.1) are derived from a non-monotone matrix A. Referring back to (2.6) the following cases can be analyzed.

Case III. When $A^{-1} < 0$, then $v_1^T A^{-1} y_2 < 0$.
1. If $M_1^{-1} > M_2^{-1}$, then $M_1^{-1} - M_2^{-1} > 0$ and $v_1^T (M_1^{-1} - M_2^{-1}) y_2 > 0$, hence $\lambda_2 - \lambda_1 < 0$ and $\lambda_2 < \lambda_1$.
2. If $M_1^{-1} \geq M_2^{-1}$, then $M_1^{-1} - M_2^{-1} \geq 0$ and
a) if $v_1^T (M_1^{-1} - M_2^{-1}) y_2 > 0$, hence $\lambda_2 - \lambda_1 < 0$ and $\lambda_2 < \lambda_1$.
b) if $v_1^T (M_1^{-1} - M_2^{-1}) y_2 = 0$, hence $\lambda_2 - \lambda_1 = 0$ and $\lambda_2 = \lambda_1$.

Case IV. When $A^{-1} \leq 0$, then $v_1^T A^{-1} y_2 \leq 0$.
1. If $v_1^T (M_1^{-1} - M_2^{-1}) y_2 > 0$, then $v_1^T A^{-1} y_2 < 0$, hence $\lambda_2 - \lambda_1 < 0$ and $\lambda_2 < \lambda_1$.
2. If $v_1^T (M_1^{-1} - M_2^{-1}) y_2 = 0$, then
a) for $v_1^T A^{-1} y_2 < 0$, $\lambda_2 - \lambda_1 = 0$ and $\lambda_2 = \lambda_1$.
b) for $v_1^T A^{-1} y_2 = 0$, the relation (2.6) is satisfied for arbitrary values of λ_1 and λ_2.

The following examples of weaker splittings illustrate the case IV.2.b).

$$A = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} = M_1 - N_1 = M_2 - N_2 \text{ where}$$

$$M_1 = \begin{bmatrix} -6 & 0 \\ 0 & -7 \end{bmatrix}, \; N_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \; M_1^{-1}N_1 = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{2}{7} \end{bmatrix} \text{ and } v_1^T = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}, \; N_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \; M_2^{-1}N_2 = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } y_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Evidently, $v_1^T (M_1^{-1} - M_2^{-1}) y_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{2}{35} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$

and $v_1^T A^{-1} y_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$

Assuming now a matrix $B < 0$, and repeating the same procedure as in the case of the case II.2.b), one can obtain again (2.9) from which, taking the limit for $\varepsilon \to 0$, we can conclude that $\lambda_2 \leq \lambda_1$ for the case IV.2.b). Hence, the following theorem holds.

THEOREM 2.4. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be two convergent weaker splittings of different types, that is, either* $M_1^{-1}N_1 \geq 0$ *and* $N_2 M_2^{-1} \geq 0$ *or* $N_1 M_1^{-1} \geq 0$ *and* $M_2^{-1}N_2 \geq 0$, *where* $A^{-1} \leq 0$. *If* $M_1^{-1} \geq M_2^{-1}$, *then*

$$\varrho(M_1^{-1}N_1) \geq \varrho(M_2^{-1}N_2).$$

In particular, if $A^{-1} < 0$ *and* $M_1^{-1} > M_2^{-1}$, *then*

$$\varrho(M_1^{-1}N_1) > \varrho(M_2^{-1}N_2).$$

Thus, we see that for the conditions (2.2) and (2.3) passing from the assumption $A^{-1} \geq 0$ to the assumption $A^{-1} \leq 0$ implies the change of the inequality sign in the inequalities for spectral radii.

Finally, it is evident that the following corollary holds.

COROLLARY 2.5. *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak splittings or one of them is weak and the second is weaker, then Theorems* 2.1, 2.2, 2.3, *and* 2.4 *hold.*

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