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MINIMAL CP RANK*

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Abstract. For every completely positive matrix A, $\text{cp-rank}\,A \geq \text{rank}\,A$. Let $\text{cp-rank}\,G$ be the maximal cp-rank of a CP matrix realization of G. Then for every graph G on n vertices, $\text{cp-rank}\,G \geq n$. In this paper the graphs G on n vertices for which equality holds in the last inequality, and graphs G such that $\text{cp-rank}\,A = \text{rank}\,A$ for every CP matrix realization A of G, are characterized.

Key words. Completely positive matrices, cp-rank, rank 1 representation.

AMS subject classifications. 15A23, 15F48

1. Introduction. An $n \times n$ matrix A is *completely positive* (CP) if there exists an entrywise nonnegative (not necessarily square) matrix B such that $A = BB^T$. Equivalently, A is completely positive if it can be represented as a sum of rank 1 symmetric entrywise nonnegative matrices

$$A = \sum_{i=1}^{m} b_i b_i^T. \tag{1.1}$$

Such a representation is called a *rank* 1 *representation* of A. The vectors b_i are the columns of a matrix B satisfying $A = BB^T$. The minimal m for which there exists an $m \times n$ nonnegative matrix B satisfying $A = BB^T$ (or a rank 1 representation with m summands) is called the *cp-rank* of A, and denoted by $\text{cp-rank}\,A$.

Every CP matrix is doubly nonnegative (DNN), that is, it is both nonnegative and positive semidefinite. The converse is not true, and the problem of determining which DNN matrices are CP is an open one. Computing the cp-rank of a given CP matrix is another open problem. For surveys of work done on these two problems, see [1], [4], [10], and also [7, pp. 304–306].

The definition clearly implies that $\text{cp-rank}\,A \geq \text{rank}\,A$ for every CP matrix A. It is known that equality holds when $n \leq 3$, or when $\text{rank}\,A \leq 2$. But for every $n \geq 4$ there exists an $n \times n$ CP matrix such that $\text{cp-rank}\,A > \text{rank}\,A$. For 4×4 CP matrices $\text{cp-rank}\,A \leq 4$. For $n \geq 5$, there exist $n \times n$ CP matrices with cp-rank greater than n [15, 5, 6].

An upper bound on $\text{cp-rank}\,A$ in terms of $\text{rank}\,A$ was established in [12] and sharpened in [3] to the following tight inequality (when $\text{rank}\,A \geq 2$):

$$\text{cp-rank}\,A \leq \frac{\text{rank}\,A(\text{rank}\,A + 1)}{2} - 1.$$

In particular this implies that for every $n \times n$ CP matrix A ($n \geq 2$),

$$\text{cp-rank}\,A \leq \frac{n(n+1)}{2} - 1.$$

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However, it seems that the least upper bound on the cp-ranks of all $n \times n$ CP matrices may be much smaller. In [11], Drew, Johnson and Loewy conjectured that for every $n \times n$ CP matrix A ($n \geq 4$),

$$\text{cp-rank}\, A \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

So far, the conjecture has been proved only for matrices with special graphs, or for special matrices.

Here, a *graph* means a simple undirected graph. If A is an $n \times n$ symmetric matrix, the *graph of* A, denoted by $G(A)$, is the graph on vertices $1, 2, \ldots, n$ with $\{i, j\}$ an edge iff $i \neq j$ and $a_{ij} \neq 0$. If A is a CP matrix and $G(A) = G$, we say that A is a *CP matrix realization of* G.

Definition 1.1. Let G be a graph on n vertices. The *cp-rank of* G, denoted by cp-rank G, is the maximal cp-rank of a CP matrix realization of G, that is,

$$\text{cp-rank}\, G = \max\{\text{cp-rank}\, A \mid A \text{ is CP and } G(A) = G\}.$$

We may rephrase the Drew-Johnson-Loewy conjecture: For every graph G on $n \geq 4$ vertices, cp-rank $G \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. It was proved for triangle free graphs in [11], for graphs which contain no odd cycle on 5 or more vertices in [10], for all graphs on 5 vertices which are not the complete graph in [14], and for nonnegative matrices with a positive semidefinite comparison matrix (and any graph) in [8]. But the conjecture is still open.

Clearly, for any graph G on n vertices, cp-rank $G \geq n$. In this paper we characterize all graphs which attain this lower bound.

Remark 1.2. A graph G on n vertices satisfies cp-rank $G = n$ if and only if for every nonsingular CP matrix A such that $G(A) = G$, cp-rank A = rank A. To see that, note that if cp-rank $G = n$, A is nonsingular and CP, and $G(A) = G$, then $n = \text{rank}\, A \leq \text{cp-rank}\, A \leq n$ and equality follows. For the reverse implication, note that each CP matrix A satisfying $G(A) = G$ is a limit of nonsingular CP matrices with the same graph: $A = \lim_{\varepsilon \to 0^+}(A + \varepsilon I)$. If for each $\varepsilon > 0$ cp-rank $(A + \varepsilon I) = n$, then cp-rank $A \leq n$. This implies cp-rank $G = n$.

The remark shows that our problem is related to the question: Which CP matrices A satisfy cp-rank A = rank A?

Definition 1.3. We say that a graph G is *of type I* if every nonsingular CP matrix A with graph G satisfies cp-rank A = rank A. We say that G is *of type II* if every CP matrix A with graph G has cp-rank A = rank A.

Of course, if G is of type II, then G is of type I. In this paper we characterize all graphs of type I and all graphs of type II. We show that a graph is of type I iff it does not contain a triangle free graph with more edges than vertices, and a graph is of type II iff it contains no even cycle and no triangle free graph with more edges than vertices.

We denote the vertex set of a graph G by $V(G)$, the edge set by $E(G)$. We assume that the reader is familiar with basic graph theoretic terms, such as a cycle, a path

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and a complete graph. The notations and terminology we use are mostly as in [9]. We mention here some of them: A graph H is a *subgraph of* G (notation: $H \subseteq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is an *induced subgraph* if $E(H)$ contains all edges of G which have both ends in $V(H)$. We denote the cycle on n vertices by C_n, and the complete graph on n vertices by K_n (K_2 is a single edge, $K_3 = C_3$ is a triangle). A *clique* in a graph G is a subset of $V(G)$ that induces a complete subgraph. We denote by $d_G(u, v)$ the distance in G between the vertices u and v. A *chord* of a cycle C is an edge $\{u, v\}$ connecting vertices u, v of C, where $d_C(u, v) \geq 2$. A vertex v of a connected graph G is a *cut vertex* if deleting it, together with its adjacent edges, disconnects G. A connected graph is a *block* if it has no cut vertices. A *block of* a connected graph G is a subgraph of G which is a block and is maximal with respect to this property. We will use the fact that in a block on 3 vertices or more, any two vertices are connected by at least two paths, which have no vertex in common except for the first and the last; see [9, p. 44]. A graph G is *triangle free* if it has no clique of size 3 or more. We will say that cliques $V_1, V_2, \ldots, V_k$ *cover* G, if the subgraphs of G they induce, $G_1, G_2, \ldots, G_k$, cover G in the following sense: $V(G) = \cup_{i=1}^{k} V_i$ and $E(G) = \cup_{i=1}^{k} E(G_i)$. We denote by $c(G)$ the minimal number of cliques in a clique cover of G.

Other notations and terminology: We denote the cardinality of a set E by $|E|$. The *support* of a nonnegative vector $a \in \mathbb{R}^n$, $a^T = (a_1, \ldots, a_n)$ is

$$\operatorname{supp} a = \{1 \leq i \leq n \,|\, a_i \neq 0\}.$$

For every n, we denote by $e_1, \ldots, e_n$ the vectors of the standard basis of $\mathbb{R}^n$, and by E_{ij} the $n \times n$ matrix whose only nonzero entry is 1 in the ij position. J_n is the $n \times n$ matrix of all ones, I_n is the $n \times n$ identity matrix, and 0_n is the $n \times n$ zero matrix. If A is an $n \times n$ matrix and $\alpha \subseteq \{1, \ldots, n\}$, we denote by $A[\alpha]$ the principal submatrix of A on rows and columns α. For a CP matrix, a *minimal rank 1 representation* is a rank 1 representation of A which has cp-rank A summands. Note that if (1.1) is any rank 1 representation of A, then $\operatorname{supp} b_1, \ldots, \operatorname{supp} b_m$ are m cliques covering $G(A)$. Hence

Observation 1.4. *For every CP matrix A,* cp-rank $A \geq c(G(A))$.

A graph is *completely positive* if every DNN matrix realization of the graph is CP. We will use the following two results. The first was obtained in a series of papers:

Theorem 1.5. [15, 5, 6, 13] *A graph G is completely positive if and only if it has no subgraph which is an odd cycle of length greater than* 4.

The second theorem we use is obtained by combining a result of [11] with a result of [5], and the above observation.

Theorem 1.6. [11, 5] *If G is a triangle free graph on $n \geq 4$ vertices and A is a CP matrix realization of G, then*

(a) *If G is a tree,* cp-rank A = rank A.

(b) *If G is not a tree,* cp-rank $A = |E(G)|$.

Finally, a permutation similarity preserves both complete positivity and cp-rank. Thus we may number (and renumber) the vertices of any graph as we please. Also, it is easy to see that if $A = A_1 \oplus A_2$, then A is CP iff A_1 and A_2 are, rank A = rank A_1 + rank A_2, and cp-rank A = cp-rank A_1 + cp-rank A_2. Hence, a graph G is of

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type I (type II) iff every connected component of G is of type I (type II), and we may restrict our attention to connected graphs and irreducible matrices.

2. Graphs of Types I and II.

The main results are Theorems 2.12 and 2.16. We prove them through a series of propositions. We begin with several simple examples of graphs of both types.

Example 2.1. By the results mentioned in the introduction, every graph on 3 vertices or less is of type II. Every graph on 4 vertices is of type I, but there exist graphs on 4 vertices which are not of type II. By a continuity argument this implies that K_4 itself is not of type II: Suppose A is any 4×4 CP matrix such that $\text{cp-rank}\, A > \operatorname{rank} A$. A necessarily has nonzero entries in each row. Let e be the vector of all ones, and define $A_\varepsilon = A + \varepsilon(Ae)(Ae)^T$. For each $\varepsilon > 0$, A_ε is a CP matrix, $G(A_\varepsilon) = K_4$ and $\operatorname{rank} A_\varepsilon = \operatorname{rank} A$. There exists an ε such that $\text{cp-rank}\, A_\varepsilon > \operatorname{rank} A_\varepsilon$. Otherwise, $\text{cp-rank}\, A_\varepsilon = \operatorname{rank} A_\varepsilon = \operatorname{rank} A$ for every ε. But since $A = \lim_{\varepsilon\to 0+} A_\varepsilon$, this would imply that $\text{cp-rank}\, A \le \operatorname{rank} A$, which contradicts the choice of A.

By Theorem 1.6, every tree is of type II.

C_n is of type I for each $n \ge 3$. For $n \ge 4$ this follows from Theorem 1.6.

We show that C_n is of type II iff $n \ge 3$ is odd: By Theorem 1.6, the cp-rank of each CP matrix realization of C_n, $n \ge 4$, is n. If $n \ge 4$ is even, then there exists a CP matrix A with graph C_n and rank $n-1$; see [6]. For such A, $\text{cp-rank}\, A = n > n-1 = \operatorname{rank} A$. But if $n \ge 5$ is odd, every CP matrix realization of C_n is necessarily of rank n. To see that, note that if A is such a matrix, and (1.1) is a rank 1 representation of A, then each edge of $G(A)$ is the support of at least one of the vectors $b_1, \dots, b_m$. So suppose $b_1, \dots, b_n$ are nonnegative vectors such that $\operatorname{supp} b_i = \{i, i+1\}$ for $i = 1, \dots, n-1$ and $\operatorname{supp} b_n = \{n, 1\}$. By the patterns of the vectors

$$b_1 b_2 \dots b_{n-1} b_n = \begin{array}{ccccccc} + & 0 & 0 & \cdots & \cdots & 0 & + \\ + & + & 0 & \cdots & \cdots & 0 & 0 \\ 0 & + & + & & & \vdots & \vdots \\ 0 & 0 & + & \ddots & & \vdots & \vdots \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & & & + & 0 \\ 0 & 0 & 0 & & \cdots & + & + \end{array}$$

it is clear that $b_1, \dots, b_{n-1}$ are linearly independent and, since n is odd, that b_n cannot be a linear combination of the first $n-1$ vectors. Hence these are n linearly independent vectors in the column space of A, $\operatorname{cs} A$; $\operatorname{rank} A = n$; see also [2]. Since C_n is of type I, $\text{cp-rank}\, A = \operatorname{rank} A = n$ for every CP matrix realization of C_n.

Remark 2.2. By Theorem 1.6, a triangle free graph G that has more edges than vertices is not of type I: Such a graph is not a tree. Hence if A is a nonsingular CP matrix and $G(A) = G$, then $\operatorname{rank} A = |V(G)|$, while $\text{cp-rank}\, A = |E(G)| > |V(G)|$.

Proposition 2.3. *Let G be a connected graph with a cut vertex. If $G = G_1 \cup G_2$ where $G_1 \cap G_2$ is a single vertex, and G_1 is either one edge or a triangle, then*

(a) *G is of type I iff G_2 is of type I,*

and

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(b) *G is of type II iff G_2 is of type II.*

Proof. Assume G_1 is the complete graph on vertices $1, \ldots, k$, $k = 2$ or 3, and G_2 is a graph on vertices $k, \ldots, n$. If A is a CP matrix realization of G, then

$$A = (A_1 \oplus 0_{n-k}) + (0_{k-1} \oplus A_2),$$

where A_1 and A_2 are completely positive, $G(A_1) = G_1$ and $G(A_2) = G_2$.

$$\operatorname{rank} A_1 + \operatorname{rank} A_2 - 1 \leq \operatorname{rank} A \leq \operatorname{rank} A_1 + \operatorname{rank} A_2$$

and equality on the left occurs iff $e_k \in \operatorname{cs}(A_1 \oplus 0_{n-k}) \cap \operatorname{cs}(0_{k-1} \oplus A_2)$. But if $e_k \in \operatorname{cs}(A_1 \oplus 0_{n-k})$, then there exists $\delta > 0$ such that $(A_1 \oplus 0_{n-k}) - \delta e_k e_k^T = (A_1 \oplus 0_{n-k}) - \delta E_{kk}$ is positive semidefinite. Let δ_0 be a maximal such δ, and let $A_1' \oplus 0_{n-k} = (A_1 \oplus 0_{n-k}) - \delta_0 E_{kk}$ and $0_{k-1} \oplus A_2' = (0_{k-1} \oplus A_2) + \delta_0 E_{kk}$. Then A_1' is DNN and $G(A_1') = G_1$. Since G_1 is a completely positive graph (Theorem 1.5), A_1' is CP. Clearly A_2' is also CP, $G(A_2') = G_2$, and $A = (A_1' \oplus 0_{n-k}) + (0_{k-1} \oplus A_2')$. Hence we may assume that $A = (A_1 \oplus 0_{n-k}) + (0_{k-1} \oplus A_2)$, where A_1 and A_2 are completely positive, $G(A_1) = G_1$ and $G(A_2) = G_2$, and $e_k \notin \operatorname{cs}(A_1 \oplus 0_{n-k})$, so that $\operatorname{rank} A = \operatorname{rank} A_1 + \operatorname{rank} A_2$. Also, since A_1 is 2×2 or 3×3, $\text{cp-rank}\, A_1 = \operatorname{rank} A_1$.

Now if G_2 is of type II, then $\text{cp-rank}\, A_2 = \operatorname{rank} A_2$, and therefore

$$\operatorname{rank} A \leq \text{cp-rank}\, A \leq \text{cp-rank}\, A_1 + \text{cp-rank}\, A_2 = \operatorname{rank} A_1 + \operatorname{rank} A_2 = \operatorname{rank} A.$$

Suppose G_2 is of type I and $\operatorname{rank} A = n$. Since $e_k \notin \operatorname{cs}(A_1 \oplus 0_{n-k})$, $\operatorname{rank} A_1 \leq k - 1$. And this inequality together with the equality $\operatorname{rank} A_1 + \operatorname{rank} A_2 = n$ implies that $\operatorname{rank} A_2 \geq n - k + 1$, and therefore $\operatorname{rank} A_2 = n - k + 1$. But $G(A_2) = G_2$, and G_2 is of type I, hence $\text{cp-rank}\, A_2 = \operatorname{rank} A_2$, and we deduce as above that $\text{cp-rank}\, A = \operatorname{rank} A = n$. □

Corollary 2.4. *If H is a block of a connected graph G, and every other block of G is either an edge or a triangle, then G is of type I (type II) iff H is of type I (respectively, type II).*

Proof. This follows easily from the previous proposition by induction on the number of blocks other than H in G. In a graph G with two or more blocks, there exists a block which has exactly one cut vertex of G among its vertices. As a matter of fact, there exist at least two such blocks. Thus if G is a graph that fits the above description, and G has at least two blocks, we may assume that $G = G_1 \cup G_2$, where $G_1 \cap G_2$ consists of a single vertex, G_1 is either an edge or a triangle, and G_2 fits the same description as G, but has one less triangle or edge block than G. □

Corollary 2.5. *If in a connected graph G each block is either an edge or a triangle, then G is of type II.*

Proposition 2.6. *If G is a graph of type I, then every subgraph of G is also of type I.*

Proof. First suppose that $V(H) = V(G) = \{1, \ldots, n\}$ and $E(H)$ is a proper subset of $E(G)$. Let A be a rank n CP matrix realization of H. For each edge $e \in E(G) \setminus E(H)$, denote by 1_e the $0 - 1$ vector supported by e. For every $\varepsilon > 0$ let

$$A_\varepsilon = A + \sum_{e \in E(G) \setminus E(H)} \varepsilon 1_e 1_e^T.$$

Then A_ε is clearly CP, $G(A_\varepsilon) = G$ and (since $\operatorname{rank} A_\varepsilon \geq \operatorname{rank} A = n$) $\operatorname{rank} A_\varepsilon = n$. Hence for each $\varepsilon > 0$ $\text{cp-rank}\, A_\varepsilon = n$. Since $A = \lim_{\varepsilon \to 0+} A_\varepsilon$, this implies that $\text{cp-rank}\, A \leq n$, and therefore $\text{cp-rank}\, A = n$.

Next suppose that $V(H)$ is a proper subset of $V(G)$. Assume w.l.o.g. that $V(H) = \{1, \ldots, k\}$ for some $k < n$. Given a rank k CP matrix A such that $G(A) = H$, let

$$A_1 = A \oplus I_{n-k}.$$

Clearly A_1 is a CP matrix, $\operatorname{rank} A_1 = n$, $V(G(A_1)) = \{1, \ldots, n\}$, and $E(G(A_1))$ is a subset of $E(G)$. By the first part of the proof, $\text{cp-rank}\, A_1 = n$. It is easy to see that every minimal rank 1 representation of A_1 is of the form

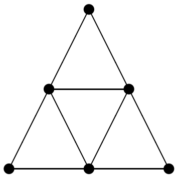
$$\sum_{i=1}^{k} b_i b_i^T + \sum_{i=k+1}^{n} e_i e_i^T,$$

where $\sum_{i=1}^{k} b_i b_i^T$ is a rank 1 representation of A. Hence $\text{cp-rank}\, A = k$. $\square$

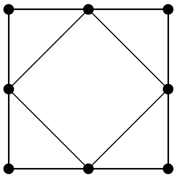
Combining the last proposition together with remark 2.2 we get the following result.

Corollary 2.7. *If a graph G has a triangle free subgraph with more edges than vertices, then G is not of type I.*

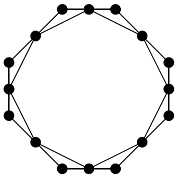
We intend to show that the converse of Corollary 2.7 holds also, and in the process describe all graphs which have no triangle free subgraph with more edges than vertices. First we introduce several blocks which contain no such triangle free subgraph. For $n \geq 3$ denote by S_{2n} the cycle on $2n$ vertices with chords connecting each even vertex to the next even vertex (assuming the cycle vertices are numbered consecutively).



S_6



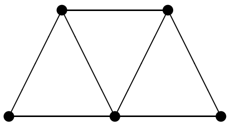
S_8



S_{16}

Note that the chords generate an n-cycle (on the n even vertices). We will call this cycle the *inner cycle* of S_{2n}

We denote by S_5 the graph



S_5

PROPOSITION 2.8. *For every $n \geq 3$ the graph S_{2n} is of type I.*

Proof. We first consider the case $n \geq 4$. Renumber S_{2n}'s vertices so that 1 is a vertex of degree 1, adjacent to vertices 2 and 3. Let A be a CP matrix with $G(A) = S_{2n}$ and $\operatorname{rank} A = 2n$. Take any rank 1 representation of A of the form (1.1), and let

$$\Omega_1 = \{1 \leq i \leq m \mid \operatorname{supp} b_i \subseteq \{1,2,3\}\}\ ,\ \Omega_2 = \{1,2,\dots,m\} \setminus \Omega_1,$$
$$B = \sum_{i \in \Omega_1} b_i b_i^T\ \ ,\ \ C = \sum_{i \in \Omega_2} b_i b_i^T.$$

Then B and C are CP, $B = B' \oplus 0_{2n-3}$, $C = 0_1 \oplus C'$. $G(B')$ is a triangle, and $G(C')$ is a graph on $2n-1$ vertices which is a "chain"of $n-1$ triangles; every block of $G(C'))$ is a triangle. Of course, $A = B + C$. Note that

$$B' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & \alpha_0 & a_{23} \\ a_{13} & a_{23} & \beta_0 \end{bmatrix},$$

where $a_{12}^2/a_{11} \leq \alpha_0 \leq a_{22}$. If $\alpha_0 = a_{12}^2/a_{11}$, then $a_{23} = (a_{12}a_{13})/a_{11}$ and

$$B' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & \alpha_0 & a_{23} \\ a_{13} & a_{23} & a_{13}^2/a_{11} \end{bmatrix} + \delta E_{33},$$

for some $\delta \geq 0$. In this case, denote $B'' = B' - \delta E_{33}$ and $C'' = C' + \delta E_{33}$. B'' is CP, and clearly so is C''; $A = (B'' \oplus 0_{2n-3}) + (0_1 \oplus C'')$, and

$$2n = \operatorname{rank} A \leq \text{cp-rank}\, A \leq \text{cp-rank}\, B'' + \text{cp-rank}\, C'' \leq 1 + (2n-1) = 2n.$$

Now consider the case $a_{12}^2/a_{11} < \alpha_0 \leq a_{22}$. For every $a_{12}^2/a_{11} < \alpha \leq a_{22}$ denote

$$B'(\alpha) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & \alpha & a_{23} \\ a_{13} & a_{23} & f(\alpha) \end{bmatrix},$$

where $f(\alpha)$ is the unique real number for which $B'(\alpha)$ is singular, i.e., $f(\alpha) = [a_{23}(a_{11}a_{23} - a_{12}a_{13}) - a_{13}(a_{12}a_{23} - \alpha a_{13})]/(a_{11}\alpha - a_{12}^2)$. Since $a_{11} > 0$ and $a_{11}\alpha - {a_{12}}^2 > 0$, $B'(\alpha)$ is a positive semidefinite matrix. In particular $f(\alpha) \geq 0$, and since $B'(\alpha)$ is nonnegative it is CP (Theorem 1.5). Let $B(\alpha) = B'(\alpha) \oplus 0_{2n-3}$, and $C(\alpha) = A - B(\alpha)$. $C(\alpha) = 0_1 \oplus C'(\alpha)$. Now if B', B, C' and C are as above, then $\beta_0 \geq f(\alpha_0)$, $B(\alpha_0) = B - (\beta_0 - f(\alpha_0))E_{33}$, and $C(\alpha_0) = C + (\beta_0 - f(\alpha_0))E_{33}$. Hence $C(\alpha_0)$ is CP. On the other hand, it is clear that $C(a_{22})$ is not positive semidefinite. By the continuity of the eigenvalues of $C(\alpha)$, it follows that there exists a $\alpha_0 < \alpha_1 < a_{22}$ such that $C(\alpha_1)$ is a singular positive semidefinite matrix. In particular, $C(\alpha_1)_{22} > 0$, $C(\alpha_1)_{33} > 0$, so $C(\alpha_1)$ is DNN. By Theorem 1.5, $G(C(\alpha_1))$ is a completely positive graph, hence $C(\alpha_1)$ is a CP matrix. Thus we have

$$2n = \operatorname{rank} A \leq \text{cp-rank}\, A \leq \text{cp-rank}\, B'(\alpha_1) + \text{cp-rank}\, C'(\alpha_1) \leq 2 + (2n-2) = 2n.$$

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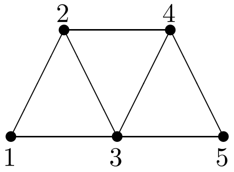
The last inequality follows from the fact that both $B'(\alpha_1)$ and $C'(\alpha_1)$ are singular and their graphs are of type II.

For the proof that S_6 is also of type I we need to show first that S_5 is of type I. Number the vertices of S_5 as follows:



Let A be a CP matrix with $G(A) = S_5$ and $\operatorname{rank} A = 5$. We use a technique of [14] to show that cp-rank $A \leq 5$. Let (1.1) be a rank 1 representation of A, and denote

$$\Omega_1 = \{1 \leq i \leq m \,|\, 1 \in \operatorname{supp} b_i\}\ ,\ \Omega_2 = \{1, \ldots, m\} \setminus \Omega_1$$
$$A_1 = \sum_{i \in \Omega_1} b_i b_i^T \quad , \quad A_2 = \sum_{i \in \Omega_2} b_i b_i^T .$$

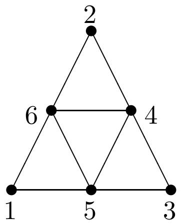
Both matrices are CP and $A = A_1 + A_2$. Rows 4 and 5 of A_1 are zero, and $A_2 = 0_1 \oplus A_2'$. The support of row 5 in A_2 is contained in that of row 4. Let

$$a = \min \left\{ \frac{(A_2)_{4j}}{(A_2)_{5j}} \,\middle|\, j = 3, 4, 5 \right\} .$$

(a is attained at some $j \neq 4$, since $(A_2)_{44}(A_2)_{55} \geq (A_2)_{45}^2$). Let $S = I_5 - aE_{45}$. Then SA_2S^T is a is a DNN matrix, and at least one of its entries in positions 4,3 and 4,5 is zero. Since SA_2S^T is a DNN matrix with just four nonzero rows, it is CP. Since row 3 of A_1 is zero, we have $SA_1S^T = A_1$. Hence $SAS^T = SA_1S^T + SA_2S^T = A_1 + SA_2S^T$ is CP, rank $(SAS^T) = \operatorname{rank} A = 5$, and cp-rank $A \leq$ cp-rank (SAS^T) (since S^{-1} is a nonnegative matrix). Therefore it suffices to show that cp-rank $(SAS^T) = 5$. But the graph of SAS^T is contained in one of the following graphs

The graph on the left is a subgraph of S_8; the one on the right has two blocks, a K_2 and a block on 4 vertices. By the beginning of this proof, Example 2.1, and Proposition 2.3, both graphs are of type I, hence cp-rank $(SAS^T) \leq 5$.

We now show by similar arguments that S_6 is also of type I. Renumber its vertices as follows:



Let A be a rank 6 CP matrix realization of S_6. Let (1.1) be a rank 1 representation of A and denote

$$\begin{aligned}\Omega_1 &= \{1 \le i \le m \,|\, 1 \in \operatorname{supp} b_i\},\\ \Omega_2 &= \{1 \le i \le m \,|\, 2 \in \operatorname{supp} b_i\},\\ \Omega_3 &= \{1, \dots, m\} \setminus (\Omega_1 \cup \Omega_2).\end{aligned}$$

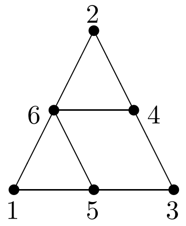
Note that $\Omega_1 \cap \Omega_2 = \emptyset$. Let

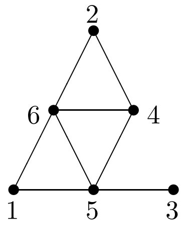
$$A_1 = \sum_{i \in \Omega_1} b_i b_i^T \quad , \quad A_2 = \sum_{i \in \Omega_2} b_i b_i^T \quad , \quad A_3 = \sum_{i \in \Omega_3} b_i b_i^T .$$

These are three CP matrices, and $A = A_1 + A_2 + A_3$. Rows $2, 3, 4$ of A_1 are zero, rows $1, 3, 5$ of A_2 are zero, and rows 1 and 2 of A_3 are zero. In A_3, the support of row 3 is contained in that of row 4. Let

$$a = \min \left\{ \frac{(A_3)_{4j}}{(A_3)_{3j}} \,\middle|\, j = 3, 4, 5 \right\},$$

and $S = I_6 - aE_{43}$. Then SA_3S^T is a CP matrix, and at least one of its entries in positions 4,3 or 4,5 is zero; $SAS^T = SA_1S^T + SA_2S^T + SA_3S^T = A_1 + A_2 + SA_3S^T$ is CP, $\operatorname{rank}(SAS^T) = \operatorname{rank} A$, and $\text{cp-rank}\, A \le \text{cp-rank}\,(SAS^T)$. Finally, the graph of SAS^T is contained in one of the following graphs





The graph on the left is a subgraph of S_8, and the graph on the right has two blocks: a K_2 and an S_5. By the first parts of this proof and Proposition 2.3 both graphs are of type I. Hence $G(SAS^T)$ is of type I, and $\text{cp-rank}\,(SAS^T) \le 6$. $\square$

PROPOSITION 2.9. *If a connected graph G has no triangle free subgraph with more edges than vertices, then each block of G has one of the following forms:*

(i) *a single edge*

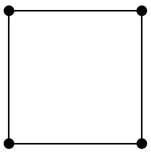
(ii) *a triangle*
(iii) *a* K_4
(iv) *a block which is contained in* S_{2m} *for some* $m \geq 3$,

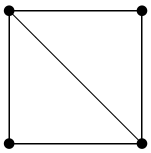
and at most one of G*'s blocks has more than* 3 *vertices.*

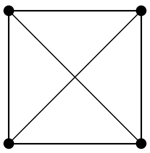
Before presenting the proof, we state and prove the following lemma.

LEMMA 2.10. *Let* G *be a connected graph. If* H *is a block which has no triangle free subgraph with more edges than vertices, then* H *has one of the forms* (i), (ii), (iii) *or* (iv) *of Proposition 2.9.*

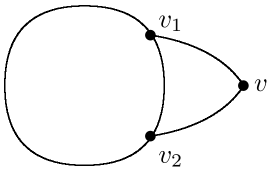
Proof. We need to show that if that H is a block on $n \geq 4$ vertices, which has no such triangle free subgraph, then H has form (iii) or (iv). The only blocks on 4 vertices are







The first two are subgraphs of S_6. Hence we consider the case $n \geq 5$. Let C be a cycle subgraph of H of maximal length. We first show that C is a cycle on all of H's n vertices. If this is not the case, assume w.l.o.g. that C is the cycle on vertices $1, 2, \ldots, k$, $k < n$, with edges $\{i, i+1\}$, $i = 1, \ldots, k-1$, and $\{k, 1\}$. Let v be a vertex in H which is not a vertex of C. Since H is connected, there exists a path P from v to 1. Let v_1 be the first vertex on this path which is a vertex of C. Let P_1 be the section of P which is a path from v to v_1. We now show that there is also a similar path from v to a different vertex of C. If $v_1 \neq 1$, take a path Q from v to 1 which has no internal vertex in common with P. Let v_2 be the first vertex on Q which is a vertex of C. Let P_2 be the section of Q which is a path from v to v_2. If $v_1 = 1$, then if we add to P the edge $\{1, 2\}$, we get a path from v to 2. There is another path from v to 2, which has no internal vertex in common with the first path, say Q. Let v_2 be the first vertex on Q which is a vertex of C, and let P_2 be the section of Q which is a path from v to v_2. In any case, $v_1 \neq v_2$, v_i is the only vertex on P_i which is a vertex of C, $i = 1, 2$, and v is the only vertex which is both a vertex of P_1 and a vertex of P_2. Denote by l_i the number of edges in P_i, $i = 1, 2$.



If v_1 and v_2 are adjacent in C, then the graph consisting of all edges of C other than $\{v_1, v_2\}$, the edges of P_1 and the edges of P_2, is a cycle on $k + l_1 + l_2 - 1 \geq k + 1$ vertices, in contradiction to the choice of C. If v_1 and v_2 are not adjacent in C, then $k > 3$, and the graph consisting of all of C's edges, and the edges of P_1 and P_2 is a triangle free graph on $k+l_1+l_2-1$ vertices, with $k+l_1+l_2$ edges, and this contradicts

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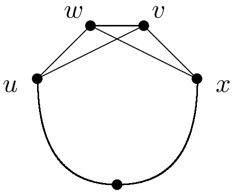
the initial assumption regarding H. Hence there is no such vertex v, and the maximal cycle C is a spanning cycle of H.

We now consider which chords of C may be edges of H. There is no edge in H which is a chord between two vertices u, v such that the $d_C(u,v) \geq 3$, since the graph consisting of C and such a chord would be a triangle free subgraph of H which has n vertices and $n+1$ edges. Hence, if there exists an edge of H which is a chord of C, it would be $\{u,v\}$ where $d_C(u,v) = 2$. If w is the vertex such that $d_C(u,w) = 1$ and $d_C(v,w) = 1$, then there is no edge $\{w,x\}$ in H which is a chord of C. Otherwise, $d_C(w,x) = 2$, so either u or v is halfway between w and x. Assume w.l.o.g. that v is. Then the graph consisting of the two chords, and all of G's edges except for $\{w,v\}$, is a triangle free graph on n vertices with $2+(n-1) = n+1$ edges. ($n \geq 5$ guarantees an additional vertex on the cycle C, between u and x.)



This completes the proof that H is subgraph of some S_{2m} (If H consists of the cycle C and k such chords, then $H \subseteq S_{2(n-k)}$). □

Proof of Proposition 2.9. If G has no triangle free subgraph with more edges than vertices, then no block of G has such subgraph. By Lemma 2.10, each block of G has one of the forms (i) – (iv). Suppose two of these blocks had 4 vertices or more. By the proof of Lemma 2.10, each of these large blocks has a spanning cycle. Let C_1 and C_2 be spanning cycles of these two blocks. The cycles may share a vertex, or, if they don't, there is a path in G from a vertex of C_1 to a vertex of C_2. A graph consisting of two cycles on 4 or more vertices sharing a vertex, or two such cycles and a path connecting them, is a triangle free graph with more edges than vertices. But this contradicts our assumption on G, hence there cannot be two blocks on 4 or more vertices in G. □

PROPOSITION 2.11. *If G is a connected graph of the form described in Proposition 2.9, then G is of type I.*

Proof. Combine Propositions 2.6 and 2.8 with Corollary 2.4. □

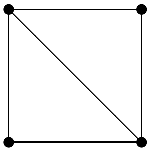
All these propositions add up to the first of our main results.

THEOREM 2.12. *Let G be a connected graph, then the following are equivalent:*

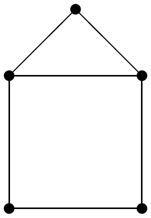
(a) *G is of type I.*
(b) *G contains no triangle free graph with more edges than vertices.*
(c) *Each block of G is an edge, or a triangle, or a K_4, or a subgraph of S_{2m} for some $m \geq 3$, and at most one of G's blocks has more than 3 vertices.*

We now characterize graphs of type II. Since every type II graph is of type I, it suffices to check which of the graphs described in Theorem 2.12 is of type II.

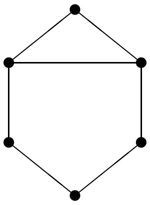
We first consider some specific blocks. Denote by H_n the graph on n vertices, $n \geq 4$, which consists of a cycle C on n vertices and exactly one chord, joining vertices u and v of the cycle, where $d_C(u,v) = 2$.



H_4



H_5



H_6

Proposition 2.13. *If $n \geq 5$, then H_n is not of type II.*

Proof. Let the edges of H_n be $\{i, i+1\}$, $i = 1, \ldots, n-1$, $\{1, n-1\}$, and $\{1, n\}$. We construct a CP matrix with graph H_n, rank $n-1$, and cp-rank n. Actually, we use one construction for the case that $n \geq 5$ is odd, and another for the case that $n \geq 6$ is even.

For odd n: Let R be the following $n \times n$ matrix:

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 1 & & & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & & \vdots & \vdots \\ & & & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

If n is even, let R be the following $n \times n$ matrix:

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 1 & 2 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 1 & & & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & & \vdots & \vdots \\ & & & \ddots & \ddots & & \\ 0 & 0 & 0 & & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

In both cases let $A = RR^T$. Then for odd n,

$$A = \begin{bmatrix} 3 & 1 & 0 & \cdots & \cdots & 2 & 1 \\ 1 & 2 & 1 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & & \vdots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 2 & 1 & 0 \\ 2 & 0 & 0 & & 1 & 3 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

For even n,

$$A = \begin{bmatrix} 6 & 1 & 0 & \cdots & \cdots & 1 & 3 \\ 1 & 2 & 1 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & & \vdots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 2 & 1 & 0 \\ 1 & 0 & 0 & & 1 & 2 & 1 \\ 3 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

In both cases it is clear that A is CP, $G(A) = H_n$, and $\text{cp-rank}\, A \leq n$. Also, it is easy to see that $\text{rank}\, A = n-1$. Denote the ith column of R by R_i, then for odd n, columns $1, \ldots, n-2, n$ of R are linearly independent and $R_{n-1} = \sum_{i=1}^{n-2} (-1)^{i+1} R_i$. For even n the first $n-1$ columns of R are linearly independent and $R_n = \sum_{i=1}^{n-1} (-1)^{i+1} R_i$. We show that in both cases $\text{cp-rank}\, A \geq n$.

Let (1.1) be a minimal rank 1 representation of A. Let

$$\Omega_1 = \{1 \leq i \leq m \,|\, 1 \in \text{supp}(b_i)\} \quad , \quad \Omega_2 = \{1, \ldots, m\} \setminus \Omega_1.$$

(both are nonempty) and let

$$B = \sum_{i \in \Omega_1} b_i b_i^T \quad , \quad C = \sum_{i \in \Omega_2} b_i b_i^T.$$

Then B and C are CP. $C = C' \oplus 0_1$ and in B only rows $1, n-1, n$ are nonzero. By the minimality of the representation, $\text{cp-rank}\, B = |\Omega_1|$, $\text{cp-rank}\, C = \text{cp-rank}\, C' = |\Omega_2|$, and $\text{cp-rank}\, A = \text{cp-rank}\, B + \text{cp-rank}\, C$. If $|\Omega_1| = 1$, B is of rank 1, and since its last row equals that of A, we necessarily have

$$B[1, n-1, n] = J_3 \quad \text{(for odd } n\text{)}, \quad B[1, n-1, n] = \begin{bmatrix} \frac{9}{2} & \frac{3}{2} & 3 \\ \frac{3}{2} & \frac{1}{2} & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{(for even } n\text{)}.$$

For odd n we get that $C = A - B$ is a CP matrix whose graph is an even cycle on $n-1$ vertices, hence $\text{cp-rank}\, C = n-1$, and $\text{cp-rank}\, A = \text{cp-rank}\, B + \text{cp-rank}\, C = 1 + (n-1) = n$. In the case that n is even it turns out that $|\Omega_1| = 1$ is impossible — in this case the $1, n-1$ element of $C = A - B$ is $-1/2$, a contradiction since C is CP.

If $|\Omega_1| \geq 2$, then (in both cases) $\text{cp-rank}\, B \geq 2$. $G(C')$ is either the cycle on $n-1$ vertices or the path from vertex 1 to vertex $n-1$. Hence at least $n-2$ cliques are needed to cover all its edges. Thus $\text{cp-rank}\, C' \geq n-2$, and we get

$$\text{cp-rank}\, A = \text{cp-rank}\, B + \text{cp-rank}\, C \geq 2 + (n-2) = n. \qquad \square$$

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In Proposition 2.15 we will see that H_4 is not of type II either. But first we want to consider blocks which are subgraphs of S_{2k} for some $k \geq 4$. Any such block H, which is not an edge or a triangle, contains the inner cycle C of S_{2k}, and if $H \neq C$, then H contains also $1 \leq r \leq k$ triangles, each consisting of an edge e of C, and a vertex not in C which is joined by edges to e's ends.

PROPOSITION 2.14. *Any block H contained in S_{2k}, $k \geq 4$, which is not an edge or an odd cycle, is not of type II.*

Proof. Let C be the inner cycle of S_{2k}. As mentioned above, any block $H \subseteq S_{2k}$ which is not an edge or a triangle, is a graph on $k+r$ vertices containing C and $0 \leq r \leq k$ triangles. If $r = 0$, then $H = C$. If C is an even cycle, it is not of type II (Example 2.1). For $r \geq 1$ we prove by induction on r that there exists a CP matrix A such that $G(A) = H$, $\operatorname{rank} A = k + r - 1$ and $\text{cp-rank}\, A = k + r$. If $r = 1$, $H = H_{k+1}$, and by Proposition 2.13 there exists such matrix A.

Suppose the proposition holds for subgraphs of S_{2k} which are blocks with $r-1 \geq 1$ triangles, and let $H \subseteq S_{2k}$ be a block which has r triangles. Then H has $n = k + r$ vertices, and we may assume that the vertices $n-2$ and $n-1$ are adjacent vertices in C, that the vertex n is not a vertex of C, and that n is joined by edges to $n-2$ and $n-1$. We denote by H' the graph obtained from H by deleting vertex n and the two edges adjacent to it. By the induction hypothesis there exists a CP matrix A' such that $G(A') = H'$, $\operatorname{rank} A' = k + (r-1) - 1$ and $\text{cp-rank}\, A' = k + (r-1)$. Let

$$A = (A' \oplus 0_1) + (0_{n-2} \oplus J_3).$$

Then A is CP, and clearly $\operatorname{rank} A = \operatorname{rank} A' + 1 = k + r - 1$. We show that $\text{cp-rank}\, A = \text{cp-rank}\, A' + 1 = k + r$. Let (1.1) be a minimal rank 1 representation of A. Let

$$\Omega_1 = \{1 \leq i \leq m \mid \operatorname{supp}(b_i) \subseteq \{n-2, n-1, n\}\} \quad , \quad \Omega_2 = \{1, \ldots, m\} \setminus \Omega_1,$$
$$B = \sum_{i \in \Omega_1} b_i b_i^T \quad , \quad C = \sum_{i \in \Omega_2} b_i b_i^T.$$

Then $B = 0_{n-3} \oplus B'$, $C = C' \oplus 0_1$, $A = B + C$, and by the minimality of the representation $\text{cp-rank}\, A = \text{cp-rank}\, B + \text{cp-rank}\, C$. Note that

$$B' = \begin{bmatrix} + & \alpha+1 & 1 \\ \alpha+1 & + & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{2.1}$$

where $+$ denotes a positive entry and $\alpha > 0$ is the $n-2, n-1$ entry of A'. We may assume that B' is singular. (If B' is not singular, let δ_0 be the maximal $\delta > 0$ such that $B - \delta e_{n-1} e_{n-1}^T$ is positive semidefinite. We may replace B by $B_0 = B - \delta_0 e_{n-1} e_{n-1}^T$ and C by $C_0 = C + \delta_0 e_{n-1} e_{n-1}^T$. We have $\operatorname{rank} B_0 = \operatorname{rank} B - 1$ and (since B and B_0 have only three nonzero rows each) $\text{cp-rank}\, B_0 = \text{cp-rank}\, B - 1$. Also, $\text{cp-rank}\, C_0 \leq \text{cp-rank}\, C + 1$. By the minimality of the original representation, there is actually an equality in the last inequality, and $\text{cp-rank}\, A = \text{cp-rank}\, B_0 + \text{cp-rank}\, C_0$. Note also that $B_0[n-2, n-1, n]$ is of the form (2.1).) By (2.1), $\operatorname{rank} B' \geq 2$. Hence $\operatorname{rank} B' = 2$. From (2.1) it is also easy to deduce that matrix $B' - J_3$ is a rank 1 positive semidefinite

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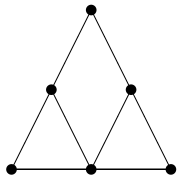
matrix and has a zero last row, and nonnegative off-diagonal entries. Because of the positive semidefiniteness of $B' - J_3$, its diagonal entries are also nonnegative. Hence it is CP. Thus $B' = J_3 + (C'' \oplus 0_1)$, where C'' is a 2×2 rank 1 CP matrix. As above, we may replace B by $0_{n-3} \oplus J_3$ and C by $C + (0_{n-3} \oplus C'' \oplus 0_1)$ without destroying the minimality of the representation. But $C' + (0_{n-3} + C'') = A'$. We therfore have

$$\text{cp-rank}\, A = \text{cp-rank}\, J_3 + \text{cp-rank}\, A' = 1 + k + (r-1). \qquad \square$$

The same holds also for blocks which are subgraphs of S_6.

PROPOSITION 2.15. *If H is a block contained in S_6, and H is not an edge or an odd cycle, then H is not of type II.*

Proof. H is one of the following: C_4, H_4, H_5, S_5, H_6, C_6, S_6 itself, or the following graph:



By Example 2.1 and Proposition 2.14, we only need to show that H_4, S_5 and S_6 are not of type II.

$$A_1 = \begin{bmatrix} 6 & 3 & 3 & 0 \\ 3 & 5 & 1 & 3 \\ 3 & 1 & 5 & 3 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$

is a CP matrix realization of H_4 and rank $A_1 = 3$. Let $A_1 = \sum_{i=1}^{m} b_i b_i^T$ be a minimal rank 1 representation of A_1. If $1 \in \operatorname{supp} b_i$ for exactly one i, say $i = 1$, then the first row of $b_1 b_1^T$ is equal to that of A_1, and hence

$$b_1 b_1^T = \begin{bmatrix} 6 & 3 & 3 & 0 \\ 3 & \frac{3}{2} & \frac{3}{2} & 0 \\ 3 & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

But then $(A_1 - b_1 b_1^T)_{23} < 0$, a contradiction to $A_1 - b_1 b_1^T$ being CP. Hence the vertex 1 belongs to at least two of the supports $\operatorname{supp} b_i$, and by the same reasoning so does the vertex 4. Since 1 and 4 cannot be in the same supports, this shows that cp-rank $A_1 = m \geq 4$. Hence H_4 is not of type II.

Next let $A_2 = (J_3 \oplus 0_2) + (0 \oplus A_1)$. A_2 is a CP matrix realization of S_5, and rank $A_2 = 4$. Let $A_2 = \sum_{i=1}^{m} b_i b_i^T$ be a minimal rank 1 representation of A_2. If $1 \in$

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$\operatorname{supp} b_i$ for exactly one i, say $i = 1$, then necessarily $b_1 b_1^T = J_3 \oplus 0_2$, and cp-rank $A_2 =$ cp-rank $b_1 b_1^T +$ cp-rank $A_1 = 1 + 4 = 5$. Suppose 1 belongs to two of the supports, say $\operatorname{supp} b_1$ and $\operatorname{supp} b_2$. We argue as in the case of H_4 that 5 also belongs to two supports, say $\operatorname{supp} b_3, \operatorname{supp} b_4$. But if $1 \in \operatorname{supp} b_i$ or $5 \in \operatorname{supp} b_i$, then $\{2, 4\} \not\subseteq \operatorname{supp} b_i$. Thus there is a fifth vector b_5 such that $\{2, 4\} \subseteq \operatorname{supp} b_5$. Hence cp-rank $A_2 = m \geq 5$; S_5 is not of type II. By a similar argument

$$A_3 = (A_1 \oplus 0_1) + \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 4 & 0 & 1 \\ 3 & 5 & 1 & 3 & 0 \\ 4 & 1 & 6 & 3 & 1 \\ 0 & 3 & 3 & 6 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

is another CP matrix realization of S_5. rank $A_3 = 4$ and cp-rank $A_3 = 5$. Using these results, and the same arguments, it is easy to see that $A_4 = (J_3 \oplus 0_3) + (0_1 \oplus A_3)$ is a CP matrix realization of S_6, rank $A_4 = 5$ and cp-rank $A_4 = 6$. $\square$

Combining Theorem 2.12, Example 2.1, Propositions 2.14 and 2.15, we obtain our second main result.

THEOREM 2.16. *Let G be a connected graph, then the following are equivalent:*
(a) *G is of type II.*
(b) *G contains no even cycle and no triangle free graph with more edges than vertices.*
(c) *Each block of G is an edge or an odd cycle, and at most one of G's blocks has more than 3 edges.*

A general characterization of the CP matrices A that satisfy cp-rank $A =$ rank A cannot rely on graph and rank alone. This is shown in the following concluding remark.

REMARK 2.17. Though H_4, S_5 and S_6 are not of type II, each of these graphs has a CP matrix realization with cp-rank equal to the rank. More precisely: If G is one of these graphs, denote by n the number of G's vertices ($n = 4, 5$, or 6). Then there exists a rank r CP matrix realization of G iff $c(G) \leq r \leq n$. For every such matrix with rank $\neq n - 1$, the cp-rank is equal to the rank. For $r = n - 1$ there exists a CP matrix realization A of G such that cp-rank $A =$ rank $A = n - 1$, and also a CP matrix realization of G whose cp-rank$= n$ and rank$= n - 1$.

We demonstrate the proof in the case of S_6. For the purpose of this proof assume the vertices are numbered as in the proof of Proposition 2.8. By [16], the minimal rank of a CP matrix realization of S_6 is $c(S_6) = 3$. We already know that S_6 is of type I, and that there exists a CP matrix realization of S_6 with rank 5 and cp-rank 6. It remains to show that for every CP matrix A with $G(A) = S_6$ and rank 3 or 4, cp-rank $A =$ rank A, and to give an example of a rank 5 CP realization of S_6 whose cp-rank is also 5.

We begin by proving the claim for rank 3. Let A is a CP matrix realization of S_6 with rank $A = 3$. Let (1.1) be a minimal rank 1 representation of A. Each of the vertices $1, 2, 3$ belongs to the support of at least one of the vectors b_i. Assume $i \in \operatorname{supp} b_i$, $i = 1, 2, 3$. Suppose one of these vertices belongs also to another support,

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say $1 \in \operatorname{supp} b_4$. The pattern of these 4 vectors is as follows ($+$ denotes a positive entry and $*$ a nonnegative one):

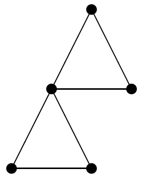
$$b_1 b_2 b_3 b_4 = \begin{matrix} + & 0 & 0 & + \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & * & * & 0 \\ * & 0 & * & * \\ * & * & 0 & * \end{matrix} .$$

By the pattern it is clear that b_1, b_2, b_3 are linearly independent. Since all four vectors are in $\operatorname{cs} A$, $\{b_1, b_2, b_3\}$ is a basis for $\operatorname{cs} A$, and b_4 is a linear combination of these three vectors. But then (again by the pattern) b_4 is a scalar multiple of b_1. In that case, $b_1 b_1^T + b_4 b_4^T$ can be replaced by one rank 1 symmetric nonnegative matrix, which contradicts the assumption of minimality of the rank 1 representation. Hence each of the vertices $i \in \{1, 2, 3\}$ belongs only to $\operatorname{supp} b_i$. If there is a b_4 in the representation, then $\operatorname{supp} b_4 \subseteq \{4, 5, 6\}$ and b_4 is a linear combination of b_1, b_2, b_3. But if $b = \sum_{i=1}^{3} \alpha_i b_i$, and the first three entries of b are zero, then by the pattern of b_1, b_2, b_3 we get $\alpha_1 = \alpha_2 = \alpha_3 = 0$, hence $b = 0$. Therefore $m = 3$.

Now for rank 4: Let A be a CP matrix with graph S_6 and rank 4, (1.1) a minimal rank 1 representation of A. Again we may assume that $i \in \operatorname{supp} b_i$, $i = 1, 2, 3$. Assume that two of these vertices belong also to one more support each. Say $1 \in \operatorname{supp} b_4$, $2 \in \operatorname{supp} b_5$. Then $b_i \in \operatorname{cs} A$ for $i = 1, \ldots, 5$, and $b_3 \notin \operatorname{Span}\{b_1, b_4, b_2, b_5\}$ (since $\operatorname{supp} b_1, \operatorname{supp} b_4 \subseteq \{1, 5, 6\}$ and $\operatorname{supp} b_2, \operatorname{supp} b_5 \subseteq \{2, 4, 6\}$). Thus

$$\dim(\mathrm{Span}\{\mathrm{b}_1, \mathrm{b}_4, \mathrm{b}_2, \mathrm{b}_5\}) \leq 3.$$

This implies that the rank of $B = b_1 b_1^T + b_4 b_4^T + b_2 b_2^T + b_5 b_5^T$ is at most 3. B's third row is zero, and the graph of $B[1, 2, 4, 5, 6]$ is subgraph of the following graph



But any subgraph of this graph is of type II, hence $\text{cp-rank}\, B = \operatorname{rank} B \leq 3$. This means that we can replace $b_1 b_1^T + b_4 b_4^T + b_2 b_2^T + b_5 b_5^T$ in the rank 1 representation by at most three summands, which contradicts the minimality of the representation. Hence at most one of the vertices $1, 2, 3$ is in more than one support. Suppose that vertices 1 and 2 are each in exactly one support, say $1 \in \operatorname{supp} b_1$ and $2 \in \operatorname{supp} b_2$. Then $A = b_1 b_1^T + b_2 b_2^T + C$, where $C = \sum_{i=3}^{m} b_i b_i^T$ is a CP matrix. The first two rows (and columns) of $b_1 b_1^T + b_2 b_2^T$ are equal to the first two rows of A. Therefore $C = 0_2 \oplus C'$, where C' is necessarily the Schur complement of $A[1, 2]$; for details on the

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Schur complement see [1]. In particular, $\operatorname{rank} C' = \operatorname{rank} A - \operatorname{rank}(b_1 b_1^T + b_2 b_2^T) = 2$, so cp-rank $C' = 2$ and cp-rank $A \leq 2 + 2 = 4$, which implies cp-rank $A = 4$.

Finally, we present a rank 5 CP matrix with graph S_6 and cp-rank 5. Let

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Then $A = RR^T$ satisfies all the requirements.

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