

POLAR DECOMPOSITION UNDER PERTURBATIONS OF THE SCALAR PRODUCT*

GUSTAVO CORACH[†], ALEJANDRA MAESTRIPIERI[‡], AND DEMETRIO STOJANOFF[§]

*Dedicated to our friend and teacher Angel Rafael Larotonda
on his sixtieth birthday*

Abstract. Let \mathcal{A} be a unital C^* -algebra with involution $*$ represented in a Hilbert space \mathcal{H} , G the group of invertible elements of \mathcal{A} , \mathcal{U} the unitary group of \mathcal{A} , G^s the set of invertible selfadjoint elements of \mathcal{A} , $Q = \{\varepsilon \in G : \varepsilon^2 = 1\}$ the space of reflections and $P = Q \cap \mathcal{U}$. For any positive $a \in G$ consider the a -unitary group $\mathcal{U}_a = \{g \in G : a^{-1}g^*a = g^{-1}\}$, i.e., the elements which are unitary with respect to the scalar product $\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle$ for $\xi, \eta \in \mathcal{H}$. If π denotes the map that assigns to each invertible element its unitary part in the polar decomposition, it is shown that the restriction $\pi|_{\mathcal{U}_a} : \mathcal{U}_a \rightarrow \mathcal{U}$ is a diffeomorphism, that $\pi(\mathcal{U}_a \cap Q) = P$, and that $\pi(\mathcal{U}_a \cap G^s) = \mathcal{U}_a \cap G^s = \{u \in G : u = u^* = u^{-1} \text{ and } au = ua\}$.

Key words. Polar decomposition, C^* -algebras, positive operators, projections.

AMS subject classifications. Primary 47A30, 47B15

1. Introduction. If \mathcal{A} is the algebra of bounded linear operators in a Hilbert space \mathcal{H} , denote by G the group of invertible elements of \mathcal{A} . Every $T \in G$ admits two *polar decompositions*

$$T = U_1P_1 = P_2U_2$$

where U_1, U_2 are unitary operators (i.e. $U_i^* = U_i^{-1}$) and P_1, P_2 are positive operators (i.e. $\langle P_i\xi, \xi \rangle \geq 0$ for every $\xi \in \mathcal{H}$). It turns out that $U_1 = U_2$, $P_1 = (T^*T)^{1/2}$ and $P_2 = (TT^*)^{1/2}$. We shall call $U = U_1 = U_2$ *the unitary part* of T . Consider the map

$$\pi : G \rightarrow \mathcal{U} \quad \pi(T) = U$$

where \mathcal{U} is the unitary group of \mathcal{A} . If G^+ denotes the set of all positive invertible elements of \mathcal{A} , then every $A \in G^+$ defines an inner product $\langle \cdot, \cdot \rangle_A$ on \mathcal{H} which is equivalent to the original $\langle \cdot, \cdot \rangle$; namely

$$\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle \quad (\xi, \eta \in \mathcal{H}).$$

*Received by the editors on 28 September 1999. Accepted for publication on 16 January 2000. Handling editor: Chandler Davis.

[†]Instituto Argentino de Matemática, Saavedra 15, Piso 3, 1083 Buenos Aires, Argentina (gcorach@mate.dm.uba.ar). Partially supported by Fundación Antorchas, CONICET (PIP 4463/96), Universidad de Buenos Aires (UBACYT TX92 and TW49) and ANPCYT (PICT97-2259)

[‡]Instituto de Ciencias, UNGS, Roca 851, San Miguel, Buenos Aires, Argentina (amaestri@ungs.edu.ar)

[§]Departamento de Matemática, UNLP, calles 1 y 50, 1900 La Plata, Argentina (demetrio@mate.dm.uba.ar). Partially supported by CONICET (PIP 4463/96), Universidad de Buenos Aires (UBACYT TW49) and ANPCYT (PICT97-2259)

Every $X \in \mathcal{A}$ admits an A -adjoint operator X^{*A} , which is the unique $Y \in \mathcal{A}$ such that

$$\langle X\xi, \eta \rangle_A = \langle \xi, Y\eta \rangle_A \quad (\xi, \eta \in \mathcal{H}).$$

It is easy to see that $X^{*A} = A^{-1}X^*A$. Together with the definition of $*_A$ one gets the sets of A -Hermitian operators

$$\mathcal{A}_A^h = \{X \in \mathcal{A} : X^{*A} = X\} = \{X \in \mathcal{A} : AX = X^*A\},$$

A -unitary operators

$$\mathcal{U}_A = \{X \in G : X^{*A} = X^{-1}\} = \{X \in \mathcal{A} : AX^{-1} = X^*A\}$$

and A -positive operators

$$G_A^+ = \{X \in \mathcal{A}_A^h \cap G : \langle X\xi, \xi \rangle_A \geq 0 \quad \forall \xi \in \mathcal{H}\}$$

As in the “classical” case, i.e. $A = I$, we get two polar decompositions for each $T \in G$

$$T = V_1R_1 = R_2V_2$$

with $V_i \in \mathcal{U}_A$, $R_i \in G_A^+$, $i = 1, 2$ and as before $V_1 = V_2 = V$. Thus, we get a map

$$\pi_A : G \rightarrow \mathcal{U}_A \quad , \quad \pi_A(T) = V \quad , \quad T \in G.$$

This paper is devoted to a simultaneous study of the maps π_A ($A \in G^+$), the way that

$$\mathcal{U}_A \quad , \quad G_A^h \quad , \quad G_A^+$$

intersect

$$\mathcal{U}_B \quad , \quad G_B^h \quad , \quad G_B^+$$

for different $A, B \in G^+$ and the intersections of these sets with

$$Q = \{S \in \mathcal{A} : S^2 = I\} \quad \text{and} \quad P_A = \{S \in Q : S^{*A} = S\}$$

(reflections and A -Hermitian reflections of \mathcal{A}). The main result is the fact that, for every $A, B \in G^+$,

$$\pi_A|_{\mathcal{U}_B} : \mathcal{U}_B \rightarrow \mathcal{U}_A$$

is a bijection. The proof of this theorem is based on the form of the positive solutions of the operator equation $XAX = B$ for $A, B \in G^+$. This identity was first studied by G. K. Pedersen and M. Takesaki [10] in their study of the Radon-Nykodym theorems in von Neumann algebras. As a corollary we get a short proof of the equality

$$\pi_A(Q \cap \mathcal{U}_B) = P_A$$

for every $A, B \in G^+$, which was proven in [1] as a C^* -algebraic version of results of Pasternak-Winiarski [8] on the analyticity of the map $A \mapsto P_A^{\mathcal{M}}$, where $P_A^{\mathcal{M}}$ is the A -orthogonal projection on the closed subspace \mathcal{M} of \mathcal{H} . We include a parametrization of all solutions of Pedersen-Takesaki equation. The results are presented in the context of unital C^* -algebras.

2. Preliminaries. Let \mathcal{A} be a unital C^* -algebra, $G = G(\mathcal{A})$ the group of invertible elements of \mathcal{A} , $\mathcal{U} = \mathcal{U}(\mathcal{A})$ the unitary group of \mathcal{A} , $G^+ = G^+(\mathcal{A})$ the set of positive invertible elements of \mathcal{A} and $G^s = G^s(\mathcal{A})$ the set of positive selfadjoint elements of \mathcal{A} . Let $Q = Q(\mathcal{A}) = \{\varepsilon \in G : \varepsilon^2 = 1\}$ be the space of reflections and

$$P = P(\mathcal{A}) = Q \cap G^s = Q \cap \mathcal{U} = \{\rho \in G : \rho = \rho^* = \rho^{-1}\}$$

the space of orthogonal reflections, also called the Grassmann manifold of \mathcal{A} .

Each $g \in G$ admits two *polar decompositions*

$$g = \lambda u = u' \lambda' , \quad \lambda, \lambda' \in G^+ , \quad u, u' \in \mathcal{U}.$$

In fact, $\lambda = (gg^*)^{1/2}$, $u = (gg^*)^{-1/2}g$, $\lambda' = (g^*g)^{1/2}$ and $u' = g(g^*g)^{-1/2}$. A simple exercise of functional calculus shows that $u = u'$. We shall say that u is *the unitary part* of g . Observe that in the decomposition $g = \lambda u$ (resp. $g = u\lambda'$) the components λ , u (resp. u , λ') are uniquely determined, for instance, if $\lambda u = \lambda_0 u_0$, then $\lambda_0^{-1}\lambda = u_0 u^{-1}$ is a unitary element with positive spectrum: $\sigma(\lambda_0^{-1}\lambda) = \sigma(\lambda_0^{-1/2}\lambda\lambda_0^{1/2}) = \sigma(\lambda) \subseteq \mathbb{R}^+$. Then $\lambda_0^{-1}\lambda = u_0 u^{-1} = 1$. The map

$$\pi : G \rightarrow \mathcal{U} \quad \text{given by} \quad \pi(g) = u \quad (g \in G)$$

is a fibration with very rich geometric properties (see [11], [2] and the references therein). We are interested in the way that the 0 $\pi^{-1}(u) = G^+ u = uG^+$ intersect the base space of a similar fibration induced by a different involution. More precisely, each $a \in G^+$ induces a C^* involution on \mathcal{A} , namely

$$(1) \quad x^{\#_a} = a^{-1}x^*a.$$

If \mathcal{A} is represented in the Hilbert space \mathcal{H} , then $a \in G^+$ induces the inner product $\langle \cdot, \cdot \rangle_a$ given by

$$\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle , \quad \xi, \eta \in \mathcal{H}.$$

It is clear that $\langle x\xi, \eta \rangle_a = \langle \xi, x_a^{\#} \eta \rangle$ for all $x \in \mathcal{A}$ and ξ, η in \mathcal{H} . \mathcal{A} is a C^* -algebra with this involution and with the norm $\|\cdot\|_a$ associated to $\langle \cdot, \cdot \rangle_a$, $\|x\|_a = \|a^{1/2}xa^{-1/2}\|$, $x \in \mathcal{A}$. For each $a \in G^+$, consider the unitary group \mathcal{U}_a corresponding to the involution $\#_a$:

$$\mathcal{U}_a = \{g \in G : g^{\#_a} = g^{-1}\} = \{g \in G : a^{-1}g^*a = g^{-1}\}.$$

We shall study the restriction $\pi|_{\mathcal{U}_a}$ and the way that different \mathcal{U}_a , \mathcal{U}_b are set in G . Moreover, we shall also consider the *a-hermitian* part of G ,

$$G_a^s = \{g \in G : g^{\#_a} = g\} = \{g \in G : a^{-1}g^*a = g\},$$

the *a-positive* part of G_a^s

$$G_a^+ = \{g \in G_a^s : \sigma(g) \subseteq \mathbb{R}^+\}$$

and the intersections of these sets when a varies in G^+ . The reader is referred to [7] and [5] for a discussion of operators which are hermitian for some inner product.

Observe that each $a \in G^+$ induces a fibration $\pi_a : G \rightarrow \mathcal{U}_a$ with fibers homeomorphic to G_a^+ . This paper can be seen in some sense as a simultaneous study of the fibrations π_a , $a \in G^+$.

Let us mention that, from an intrinsic viewpoint, \mathcal{U}_a can be identified with \mathcal{U} . Indeed, consider the map $\varphi_a : \mathcal{A} \rightarrow \mathcal{A}$ given by

$$(2) \quad \varphi_a(b) = a^{-1/2}ba^{1/2} \quad (b \in \mathcal{A}).$$

Then $\varphi_a(\mathcal{U}) = \mathcal{U}_a$, $\varphi_a(G^s) = G_a^s$ and $\varphi_a(G^+) = G_a^+$, since $\varphi_a : (\mathcal{A}, *) \rightarrow (\mathcal{A}, \#_a)$ is an isomorphism of C^* -algebras. We are concerned with the way in which the base space and fibers of different fibrations behave with respect to each other.

3. The polar decomposition. In [10], Pedersen and Takesaki proved a technical result which was relevant for their generalization of the Sakai's Radon-Nikodym theorem for von Neumann algebras [11]. More precisely, they determined the uniqueness and existence of positive solutions of the equation $THT = K$ for H, K positive bounded operators in a Hilbert space. We need a weak version of their result, namely when H, K are positive invertible operators. In this case it is possible to give an explicit solution.

LEMMA 3.1 ([10]). *If H, K are positive invertible bounded operators in a Hilbert space, the equation*

$$(3) \quad THT = K$$

has a unique solution, namely $T = H^{-1/2}(H^{1/2}KH^{1/2})^{1/2}H^{-1/2}$.

Proof. Multiply (3) at left and right by $H^{1/2}$ and factorize

$$H^{1/2}THTH^{1/2} = (H^{1/2}TH^{1/2})^2.$$

Then we get the equation $(H^{1/2}TH^{1/2})^2 = H^{1/2}KH^{1/2}$. Taking (positive) square roots and using the invertibility of $H^{1/2}$ we get the result. \square

Returning to the map $\pi : G \rightarrow \mathcal{U}$, consider the fiber $\pi^{-1}(u) = \{\lambda u : \lambda \in G^+\}$. In order to compare the fibration π with π_a , the following is the key result

THEOREM 3.2. *Let $a \in G^+$. Then, for every $u \in \mathcal{U}$ the fiber $\pi^{-1}(u)$ intersects \mathcal{U}_a at a single point, namely $a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2} \cdot u$. In other words, the restriction $\pi|_{\mathcal{U}_a} : \mathcal{U}_a \rightarrow \mathcal{U}$ is a homeomorphism.*

Proof. If $g = \lambda u \in \mathcal{U}_a$ then $a^{-1}g^*a = g^{-1}$ is equivalent to $a^{-1}u^{-1}\lambda a = u^{-1}\lambda^{-1}$, so, after a few manipulations,

$$(4) \quad \lambda a \lambda = u a u^{-1}.$$

By Pedersen and Takesaki's result, there is a unique $\lambda \in G^+$ which satisfies equation (4) for fixed $a \in G^+$, $u \in \mathcal{U}$, namely $\lambda = a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2}$. Thus, $(\pi|_{\mathcal{U}_a})^{-1} : \mathcal{U} \rightarrow \mathcal{U}_a$ is given by

$$(5) \quad (\pi|_{\mathcal{U}_a})^{-1}(u) = a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2} \cdot u$$

which obviously is a continuous map. \square

Let $a \in G^+$ and consider the involution $\#_a$ defined in equation (1). It is natural to look at those reflections $\varepsilon \in Q$ which are $\#_a$ -orthogonal, i.e the so called $\#_a$ -Grassmann manifold of \mathcal{A} . Let us denote this space by

$$P_a = \{\varepsilon \in Q : \varepsilon = \varepsilon^{\#_a} = \varepsilon^{-1}\} = Q \cap \mathcal{U}_a = Q \cap G_a^s.$$

In [8], Pasternak-Winiarski studied the behavior of the orthogonal projection onto a closed subspace of a Hilbert space when the inner product varies continuously. Note that we can identify naturally the space of idempotents q with the reflections of Q via the affine map $q \mapsto \varepsilon = 2q - 1$, which also maps the space of orthogonal projections onto P . Based on [8], a geometric study of the space Q is made in [1], where the characterization $\pi(P_a) = P$ is given (proposition 5.1 of [1]). In the following proposition we shall give a new proof of this fact by showing that the homeomorphism $\pi|_{\mathcal{U}_a} : \mathcal{U}_a \rightarrow \mathcal{U}$ maps $P_a \subseteq \mathcal{U}_a$ onto $P \subseteq \mathcal{U}$. Therefore the formula given in equation (5) for the inverse of $\pi|_{\mathcal{U}_a}$ extends the formula given in proposition 5.1 of [1] for $(\pi|_{P_a})^{-1}$, since they must coincide on P .

PROPOSITION 3.3. *Let $a \in G^+$. Then $\pi(P_a) = \pi(Q \cap \mathcal{U}_a) = P$. Therefore $\pi|_{P_a} : P_a \rightarrow P$ is a homeomorphism.*

Proof. By the previous remarks, we just need to show that $\pi(P_a) = P$. Observe that if $\varepsilon \in Q$ then $\rho = \pi(\varepsilon) \in P$: in fact, if $\varepsilon = \lambda\rho$ then $\varepsilon = \varepsilon^{-1} = \rho^{-1}\lambda^{-1}$; but, since the unitary part of ε corresponding to both right and left polar decompositions coincide, we get $\rho^{-1} = \rho$. Then $\rho^* = \rho^{-1} = \rho$ and $\rho \in P$. Thus, $\pi(\mathcal{U}_a \cap Q) \subseteq P$.

Let $\alpha = (\pi|_{\mathcal{U}_a})^{-1} : \mathcal{U} \rightarrow \mathcal{U}_a$. Then by (5)

$$\alpha(u) = a^{-1/2}(a^{1/2}uau^{-1}a^{1/2})^{1/2}a^{-1/2} \cdot u.$$

In order to prove the result we need to show that if $\rho \in P$ then $\alpha(\rho) \in P_a = Q \cap \mathcal{U}_a$, i.e. $\alpha(\rho) \in Q$. Indeed,

$$\begin{aligned} \alpha(\rho)^2 &= a^{-1/2}(a^{1/2}\rho a\rho a^{1/2})^{1/2}a^{-1/2}\rho a^{-1/2}(a^{1/2}\rho a\rho a^{1/2})^{1/2}a^{-1/2}\rho \\ &= a^{-1/2}((a^{1/2}\rho a^{1/2})^2)^{1/2}(a^{1/2}\rho a^{1/2})^{-1}((a^{1/2}\rho a^{1/2})^2)^{1/2}a^{-1/2}\rho. \end{aligned}$$

Thus, applying the continuous functional calculus (see e.g. [9]) to the selfadjoint element $a^{1/2}\rho a^{1/2}$, if $f(t) = |t| = (t^2)^{1/2}$ and $g(t) = \frac{1}{t}$, $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} (\lambda\rho)^2 &= a^{-1/2}f(a^{1/2}\rho a^{1/2})g(a^{1/2}\rho a^{1/2})f(a^{1/2}\rho a^{1/2})a^{-1/2}\rho \\ &= a^{-1/2}[f(a^{1/2}\rho a^{1/2})]^2g(a^{1/2}\rho a^{1/2})a^{-1/2}\rho \\ &= a^{-1/2}(a^{1/2}\rho a^{1/2})a^{-1/2}\rho = 1. \quad \square \end{aligned}$$

3.1. Positive parts. In order to complete the results on the relationship between polar decomposition and inner products, consider the complementary map of the decomposition $g = \lambda u$, namely

$$\pi^+ : G \rightarrow G^+, \quad \pi^+(g) = (gg^*)^{1/2}, \quad (g \in G).$$

Of course, there is another “complementary map”, namely $g \mapsto (g^*g)^{1/2}$ corresponding to the decomposition $g = u\lambda'$. We shall see that for every $a \in G^+$, the restriction

$$\pi^+|_{G_a^+} : G_a^+ \rightarrow G^+$$

is a homeomorphism. Indeed, given $\mu \in G^+$, consider the polar decomposition $a\mu = \lambda u$, with $\lambda \in G^+$ and $u \in \mathcal{U}$. Then $a^{-1}\lambda = \mu u^*$, so $\pi^+(a^{-1}\lambda) = \mu$ and $a^{-1}\lambda \in G_a^+$, since $a^{-1}(a^{-1}\lambda)^*a = a^{-1}\lambda$ and the spectrum $\sigma(a^{-1}\lambda) \subseteq \mathbb{R}^+$. Note that

$$(\pi^+|_{G_a^+})^{-1}(\mu) = a^{-1}\pi^+(a\mu),$$

which is clearly a continuous map. An interesting rewriting of the above statement is the following:

PROPOSITION 3.4. *If \mathcal{A} is a unital C^* -algebra and $a, \lambda \in G^+$, then there exists a unique $u \in \mathcal{U}$ such that $a\lambda u \in G^+$.*

Proof. Indeed, if $g = (\pi^+|_{G_a^+})^{-1}(\lambda) \in G_a^+$ and $u = \pi(g)$, then $\lambda u = g \in G_a^+$ means exactly that $a\lambda u \in G^+$. \square

It is worth mentioning that $x \in G$ is the unique positive solution of Pedersen-Takesaki equation $axx = b$ if and only if $a^{1/2}xb^{-1/2} \in \mathcal{U}$. Changing a, b by a^2, b^{-2} respectively, we can write Lemma 3.1 as follows:

PROPOSITION 3.5. *If \mathcal{A} is a unital C^* -algebra and $a, b \in G^+$, then there exists a unique $x \in G^+$ such that $axb \in \mathcal{U}$.*

3.2. Products of positive operators. The map $\Theta : G^+ \times G^+ \rightarrow \mathcal{U}$ given by

$$\Theta(a, b) = axb = a(a^{-1}(ab^2a)^{1/2}a^{-1})b = (ab^2a)^{1/2}a^{-1}b, \quad a, b \in G^+,$$

is not surjective: in fact, the image of Θ consists of those unitary elements which can be factorized as a product of three positive elements. On one side $\Theta(a, b) = axb \in \mathcal{U}$ is the product of three elements of G^+ . On the other side, if $axb \in \mathcal{U}$ then by Pedersen-Takesaki’s result x is the unique positive solution of $xa^2x = b^{-2}$.

It is easy to show that $-1 \in \mathcal{U}$ can not be decomposed as a product of four positive elements. See [12] and [13] for a complete bibliography on these factorization problems. See [3] for more results on factorization of elements of G and characterizations of $P_n = \{a_1 \dots a_n : a_i \in G^+\}$, at least in the finite dimensional case.

3.3. Parametrization of the solutions of Pedersen-Takesaki equations.

Given $a, b \in G^+$, denote by $m = |b^{1/2}a^{1/2}| = (a^{1/2}ba^{1/2})^{1/2}$. Then the set of all solutions of the equation $axx = b$ is

$$\{a^{-1/2} m \varepsilon a^{-1/2} : \varepsilon \in Q \quad \text{and} \quad \varepsilon m = m\varepsilon\}.$$

In fact, $axx = b$ if and only if $(a^{1/2}xa^{1/2})^2 = m^2$ and the set of all solutions of $x^2 = c^2$ for $c \in G^+$ is $\{c\varepsilon : \varepsilon^2 = 1 \quad \text{and} \quad \varepsilon c = c\varepsilon\}$. The singular case, which is much more interesting, deserves a particular study that we intend to do in a forthcoming paper.

4. Intersections and unions. For any selfadjoint $c \in \mathcal{A}$ we shall consider the relative commutant subC*-algebra

$$\mathcal{A}_c = \mathcal{A} \cap \{c\}' = \{d \in \mathcal{A} : dc = cd\}$$

and denote by $\mathcal{U}(\mathcal{A}_c) = \mathcal{A}_c \cap \mathcal{U}$, the unitary group of \mathcal{A}_c and, analogously $G^s(\mathcal{A}_c)$, $G^+(\mathcal{A}_c)$, $Q(\mathcal{A}_c)$ and $P(\mathcal{A}_c)$.

The space G^s has a deep relationship with Q (in [4] there is a partial description of it). Here we only need to notice that the unitary part of any $c \in G^s$ also belongs to P . Indeed, if $\lambda\rho$ is the polar decomposition of c , then $\lambda\rho = c = c^* = \rho^*\lambda$. By the uniqueness of the unitary part, $\rho = \rho^* = \rho^{-1} \in P$. Observe also that $\rho\lambda = \lambda\rho$. Moreover, since $\lambda = |c| = (c^2)^{-1}$, then $\rho = f(c)$ where $f(t) = t|t|$. So $\rho c = c\rho$.

THEOREM 4.1. *Let \mathcal{A} a unital C*-algebra and $a \in G^+$. Then*

$$\mathcal{U}_a \cap G^s = P(\mathcal{A}_a) = \{u \in P : au = ua\}.$$

Proof. By the previous remarks, if $b \in G^s \cap \mathcal{U}_a$ and $b = \lambda\rho$ is its polar decomposition, then $\rho \in P$ and $\rho\lambda^{-1} = a^{-1}\rho\lambda a$. Using that $\rho\lambda = \lambda\rho$ we get easily

$$\lambda^{-1}a\lambda^{-1} = \rho a \rho = \lambda a \lambda.$$

By the uniqueness of the positive solution, $\lambda = \lambda^{-1}$ and, since $\lambda \in G^+$, this means that $\lambda = 1$. Thus $a = \rho a \rho$ and then $\rho \in P(\mathcal{A}_a)$. Conversely, if $\rho \in P(\mathcal{A}_a)$, then $\rho \in \mathcal{U}_a$, since $a^{-1}\rho^*a = a^{-1}\rho a = \rho = \rho^{-1}$. \square

REMARK 4.2. Let $a \in G^+$. Then easy computations show that

1. $\mathcal{U}_a \cap \mathcal{U} = \mathcal{U} \cap \mathcal{A}_a = \mathcal{U}(\mathcal{A}_a)$.
2. $G_a^s \cap G^s = G^s \cap \mathcal{A}_a = G^s(\mathcal{A}_a)$.
3. $G_a^+ \cap G^+ = G^+ \cap \mathcal{A}_a = G^+(\mathcal{A}_a)$.
4. $\mathcal{U}_a \cap G^+ = \{1\}$.

We shall give two proofs of item 4:

First proof: $\pi(G^+) = \{1\}$ but π restricted to \mathcal{U}_a is one to one.

Second proof: if $x \in \mathcal{U}_a \cap G^+$, then its spectrum $\sigma(x) \subseteq S^1 \cap \mathbb{R}^+ = \{1\}$; on the other side, x is normal with respect to the involutions $\#_a$, so x is a normal element such that $\sigma(x) = \{1\}$ and it must be $x = 1$.

Let $b \in G^+$. Recall that the map φ_b defined in (2) changes the usual involution by $\#_b$ and also all the corresponding spaces (e.g $\varphi_b(G^s) = G_b^s$).

LEMMA 4.3. *Let $b \in G^+$. Then for any $c \in G^+$, if $d = b^{-1/2}cb^{-1/2}$,*

$$\varphi_b(\mathcal{U}_d) = \mathcal{U}_c, \quad \varphi_b(G_d^s) = G_c^s, \quad \text{and} \quad \varphi_b(G_d^+) = G_c^+$$

Proof. Notice that $\mathcal{U}_d = \varphi_d(\mathcal{U})$, so $\varphi_b(\mathcal{U}_d) = \varphi_b \circ \varphi_d(\mathcal{U})$. But $\varphi_b \circ \varphi_d = \varphi_c \circ Ad(u^*)$ where $u = \pi(d^{1/2}b^{1/2})$, since $c = |d^{1/2}b^{1/2}|^2$ and $d^{1/2}b^{1/2} = uc^{1/2}$. As $Ad_{u^*}(\mathcal{U}) = \mathcal{U}$ (and the same happens for G^s and G^+), we get $\varphi_b(\mathcal{U}_d) = \mathcal{U}_c$ and the other two identities. \square

Then we can generalize the results above for any pair $a, b \in G^+$ instead of a and 1:

COROLLARY 4.4. *Let $a, b \in G^+$ and $c = b^{-1/2}ab^{-1/2}$. Then*

1. $\mathcal{U}_a \cap G_b^s = \varphi_b(\mathcal{U}_c \cap G^s) = b^{-1/2}(P(\mathcal{A}_c))b^{1/2}$.
2. $\mathcal{U}_a \cap \mathcal{U}_b = \varphi_b(\mathcal{U}_c \cap \mathcal{U}) = b^{-1/2}\mathcal{U}(\mathcal{A}_c)b^{1/2}$.
3. $G_a^s \cap G_b^s = \varphi_b(G_c^s \cap G^s) = b^{-1/2}G^s(\mathcal{A}_c)b^{1/2}$.
4. $G_a^+ \cap G_b^+ = \varphi_b(G_c^+ \cap G^+) = b^{-1/2}G^+(\mathcal{A}_a)b^{1/2}$.
5. $\mathcal{U}_a \cap G_b^+ = \varphi_b(\mathcal{U}_c \cap G^+) = \{1\}$.

Proof. Use Proposition 4.1, Remark 4.2 and Lemma 4.3. \square

In the following proposition we describe the set of elements of \mathcal{A} which are unitary (resp. positive, Hermitian) for some involution $\#_a$ ($a \in G^+$). We state the result without proof.

PROPOSITION 4.5. *If \mathcal{A} is a unital C^* -algebra, the following identities hold:*

$$\begin{aligned} \bigcup_{a \in G^+} \mathcal{U}_a &= \bigcup_{g \in G} g \mathcal{U} g^{-1} = \bigcup_{a \in G^+} a \mathcal{U} a^{-1}, \\ \bigcup_{a \in G^+} G_a^+ &= \bigcup_{g \in G} g G^+ g^{-1} = \bigcup_{a \in G^+} a G^+ a^{-1} = G^+ G^+, \end{aligned}$$

where $G^+ G^+ = \{ab : a, b \in G^+\}$ and

$$\bigcup_{a \in G^+} G_a^s = \bigcup_{g \in G} g G^s g^{-1} = \bigcup_{a \in G^+} a G^s a^{-1} = G^s G^+ = G^+ G^s.$$

The following example shows that there is no obvious spectral characterization of these subsets of G : if x is nilpotent, then $1 + x$ does not belong to any of them but $\sigma(1 + x) = \{1\} \subseteq \mathbb{R}, \mathbb{R}^+$ and the circle S^1 .

4.1. Final geometric remarks. All subsets of \mathcal{A} studied in this paper have a rich structure as differential manifolds. The reader is referred to [6] and [2] for the case of \mathcal{U} and to [4] (and the references therein) for Q, P, G^s and G^+ . The map φ_a defined in equation (2) is clearly a diffeomorphism which allows to get all the information on $\mathcal{U}_a, G_a^s, G_a^+$ from that available on \mathcal{U}, G^s, G^+ , respectively. The main results of the paper say that the map π is a diffeomorphism between \mathcal{U}_a and \mathcal{U} , P_a and P and so on.

REFERENCES

- [1] Andruchow E., Corach G. and Stojanoff D.; Geometry of oblique projections, *Studia Mathematica* 137:61–79, 1999.
- [2] Atkin C. J. , The Finsler geometry of groups of isometries of Hilbert spaces, *Journal of the Australian Mathematical Society (Series A)* 42:196–222, 1987.
- [3] Ballantine C. S., Products of positive definite matrices IV, *Linear Algebra and its Applications* 3:79–114, 1970.
- [4] Corach G.; Operator inequalities, geodesics and interpolation, in Zemanek, Jaroslav (ed.), *Functional analysis and operator theory. Proceedings of the 39th semester at the Stefan Banach International Mathematical Center in Warsaw, Poland, held March 2-May 30, 1992*. Warsaw: Polish Academy of Sciences, Banach Cent. Publ. Vol. 30, pp. 101–115, 1994.
- [5] Dieudonné J.; Quasi-hermitian operators, in *Proc. Internat. Symp. Linear Spaces*, Jerusalem, pp. 115–122, 1961.

- [6] Harpe, P. de la; *Classical Banach-Lie algebras and Banach-Lie groups in Hilbert space*, Lecture Notes in Mathematics 285, Springer Verlag, Berlin-New York, 1972.
- [7] Lax P. D.; Symmetrizable linear transformations, *Communications in Pure and Applied Mathematics* 7:633–647, 1954.
- [8] Pasternak-Winiarski Z.; On the dependence of the orthogonal projector on deformations of the scalar product, *Studia Mathematica* 128:1–17, 1998.
- [9] Pedersen G. K.; *Analysis Now*, Springer-Verlag, New York, 1989
- [10] Pedersen G. K. and Takesaki M.; The operator equation $THT = K$, *Proceedings of the American Mathematical Society* 36:311–312, 1972.
- [11] Pedersen G. and Takesaki M.; The Radon Nikodym theorem for von Neumann algebras, *Acta Mathematica* 130:53–87, 1973.
- [12] Phillips, N. C.; Every invertible Hilbert space operator is a product of seven positive operators, *Canadian Mathematical Bulletin* 38:230–236, 1995.
- [13] Wu, P. Y.; The operator factorization problem, *Linear Algebra and its Applications* 117:35–63, 1989.