

## ON NONNEGATIVE OPERATORS AND FULLY CYCLIC PERIPHERAL SPECTRUM \*

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*Dedicated to Hans Schneider on the occasion of his seventieth birthday*

**Abstract.** In this note the properties of the peripheral spectrum of a nonnegative linear operator  $A$  (for which the spectral radius is a pole of its resolvent) in a complex Banach lattice are studied. It is shown, e.g., that the peripheral spectrum of a natural quotient operator is always fully cyclic. We describe when the nonnegative eigenvectors corresponding to the spectral radius  $r$  span the kernel  $N(r - A)$ . Finally, we apply our results to the case of a nonnegative matrix, and show that they sharpen earlier results by B.-S. Tam [*Tamkang J. Math.* 21:65–70, 1990] on such matrices and full cyclicity of the peripheral spectrum.

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**1. Introduction.** It follows from results of H.H. Schaefer [6, I.2.6 and V.4.6], that in the finite dimensional case  $\mathbb{C}^n$  and in certain Banach function lattices a nonnegative operator  $A$  has a fully cyclic peripheral point spectrum iff for all  $\lambda \in \mathbb{C}$  with  $|\lambda| = r(A)$

$$x \in N(\lambda \Leftrightarrow A) \text{ implies } |x| \in N(r(A) \Leftrightarrow A);$$

here  $r(A)$  denotes the spectral radius of  $A$ ; see also B.-S. Tam [8, Lemma 2.1].

In this note we consider nonnegative operators  $A$  in a Banach lattice for which the spectral radius  $r(A)$  is a pole of its resolvent. We give necessary and sufficient conditions that for a given  $\lambda \in \mathbb{C}$  with  $|\lambda| = r(A)$  the inclusion  $\{|x| : x \in N(\lambda \Leftrightarrow A)\} \subset N(r(A) \Leftrightarrow A)$  holds; in particular, we give necessary and sufficient conditions that  $N(r(A) \Leftrightarrow A)$  has a basis of nonnegative eigenvectors, and is a sublattice, respectively.

As examples show (see [8, Example 2.7]), the inclusion above is, even in the matrix case, not only a property of the spectrum and the associated directed graph of  $A$ . This will be very clear from Theorem 3.5. On the other hand, Theorem 4.2 in the matrix case and under the assumption that the nonnegative vectors in  $N(r(A) \Leftrightarrow A)$  span this kernel will show that the property of  $N(r(A) \Leftrightarrow A)$  being a sublattice can be characterized by properties of the reduced and of the singular graphs of  $A$ .

The main method of the investigation is the systematic application of an idea going back to Lotz and Schaefer (cf. [6]), and successfully developed by Greiner [2], [4]. The closed lattice ideals  $I_0$  and  $I_1$ , defined below in terms of the Laurent expansion

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of the resolvent of  $A$  around the spectral radius, their quotient  $I_h$  and the restrictions of (or the induced operator by)  $A$  are the most important technical means of the study, and a number of the results presented here find their natural formulation in this terminology.

In Section 4 we compare the results of Section 3 on nonnegative operators in Banach lattices with those of B.-S. Tam [8] on nonnegative matrices. One of the main results here is Theorem 4.2, which shows that for a nonnegative square matrix  $A$  the kernel  $N(r \Leftrightarrow A_1)$  for the restriction  $A_1$  of  $A$  to the lattice ideal  $I_1$  is a sublattice iff there does not exist a (strongly connected equivalence) class of  $A$  having access to two different distinguished basic classes (as expressed in the already standard graph theoretic terminology). The results of Section 4 sharpen the main results of Tam [8, Theorems 2.4 and 2.5] on matrices with fully cyclic peripheral spectrum.

**2. Definitions and Preliminaries.** In the following  $A$  denotes a nonnegative operator in a complex Banach lattice  $E$ . We assume that its spectral radius  $r = r(A)$  is a pole of order  $p$  ( $\geq 1$ ) of its resolvent  $R(\cdot, A)$ , i.e., we have for some  $\delta > 0$  and operators  $Q_k$  ( $k = \Leftrightarrow p, \Leftrightarrow p + 1, \dots$ ) the Laurent expansion

$$R(\lambda, A) = \sum_{k=-p}^{\infty} (\lambda \Leftrightarrow r)^k Q_k \quad \text{if } 0 < |\lambda \Leftrightarrow r| < \delta.$$

Note that  $Q_{-p}$  is nonnegative, since  $A$  is nonnegative. Following G. Greiner [2], [4, Chapters B III and C III], we define

$$I_1 = \{x \in E : (Q_{-p}|x| = \dots = Q_{-3}|x| =) Q_{-2}|x| = 0\}$$

(hence  $I_1 = E$  if  $p = 1$ ), and

$$I_0 = \{x \in E : Q_{-1}|x| = 0\}.$$

Note that  $I_0$  can be trivial. Then we have (see [4, p. 174 and p. 303]) that  $I_1$  and  $I_0$  are closed ideals of  $E$ , and they are invariant under  $A$  and  $Q_k$  ( $k = \Leftrightarrow p, \Leftrightarrow p + 1, \dots$ ). If  $A_1$  and  $A_0$  denote the restrictions of  $A$  to  $I_1$  and  $I_0$ , respectively, then

$$\begin{aligned} r(A_1) &= r(A) \quad \text{is a pole of } R(\cdot, A_1) \quad \text{of order } 1; \\ r(A_0) &< r(A), \quad \text{here we set } r(A_0) = \Leftrightarrow \infty \quad \text{if } I_0 = \{0\}. \end{aligned}$$

Since  $I_0 \subsetneq I_1$ , the quotient space  $I_h = I_1/I_0$  is well-defined and is a Banach lattice. Further,  $A_1$  induces uniquely a nonnegative operator  $A_h$  in  $I_h$  such that  $q_h A_1 = A_h q_h$ , where  $q_h$  denotes the quotient map  $I_1 \rightarrow I_1/I_0$ . Then (see [4, p. 174 and p. 303])  $r(A_h) = r(A_1) = r(A) = r$ ,  $r$  is a pole of order 1 of  $R(\cdot, A_h)$ , and the residuum of  $R(\cdot, A_h)$  at  $r$  (which is induced by  $Q_{-1}$  in  $I_h$ ) is strictly positive in the sense that the zero element is the only nonnegative element in  $I_h$  which it maps to the zero element. Since  $r$  is a pole of order 1 of  $R(\cdot, A_h)$ , the associated residuum is a projection with kernel  $R(r \Leftrightarrow A_h) = \text{range of } r \Leftrightarrow A_h$ . Therefore  $R(r \Leftrightarrow A_h)$  cannot contain any nonzero element  $y$  of  $I_h$  for which either  $y$  or  $\Leftrightarrow y$  is nonnegative.

Let  $J$  be the closed ideal of  $I_h$  generated by  $N(r \Leftrightarrow A_h)$ .  $J$  is  $A_h$ -invariant, since  $N(r \Leftrightarrow A_h)$  is  $A_h$ -invariant. Let  $A_i = A_h | J$  (i.e. the restriction of  $A_h$  to  $J$ ). Further, let  $I_j = I_h/J$  and  $A_j = A_h/J$  (i.e. the operator induced by  $A_h$  in  $I_j$ ). We set  $r(A_j) = \Leftrightarrow\infty$  if  $I_j = \{0\}$ .

The important basic connections between different kernels and spectral radii of the operators defined above will be collected in the proposition below. The following useful lemma, which is not new, will be applied in the proof of the proposition and several times at other places of this note.

**LEMMA 2.1.** *Let  $T$  be a linear map in a vector space  $V$  and let  $M$  be a  $T$ -invariant linear submanifold of  $V$ . Let  $T|M$  denote the restriction of  $T$  to  $M$ , let  $T/M$  denote the linear map induced by  $T$  in the quotient space  $V/M$ , and let  $q_M : V \rightarrow V/M$  denote the quotient map. Then the following hold.*

- (I) *If  $T|M$  is bijective, then  $q_M$  maps the kernel  $N(T)$  bijectively onto  $N(T/M)$ .*
- (II) *If  $T/M$  is injective, then  $N(T) = N(T|M)$ .*

*Proof.* (I) Since  $q_M T = (T/M)q_M$ , the quotient map  $q_M$  maps  $N(T)$  into  $N(T/M)$ . Let  $z \in N(T/M)$ . Take  $v \in V$  such that  $q_M(v) = z$ , then  $q_M(Tv) = T/M z = 0$ , i.e.,  $Tv \in M$ . Since  $T|M$  is surjective, we have  $Tv = (T|M)w$  for some  $w \in M \subset V$ . Thus  $u = v \Leftrightarrow w \in N(T)$  and  $q_M(u) = q_M(v) = z$ . If  $u \in N(T)$  satisfies  $q_M(u) = 0$ , then  $u \in M$  and  $(T|M)u = 0$ . Since  $T|M$  is injective, we obtain  $u = 0$ .

(II) Clearly we have  $N(T|M) \subset N(T)$ . Let  $u \in N(T)$ . Then  $T/M q_M(u) = q_M(Tu) = 0$ , and therefore  $q_M(u) = 0$ , since  $T/M$  is injective. Thus  $u \in M \cap N(T) = N(T|M)$ .

□

**PROPOSITION 2.2.** *Let the assumptions and notations preceding Lemma 2.1 hold. Then the following hold.*

- (I)  *$r(A) = r(A_1) = r(A_h) = r(A_i) > \max\{r(A_j), r(A_0)\}$ , and  $r(A)$  is a pole of order 1 of  $R(\cdot, A_1)$ ,  $R(\cdot, A_h)$  and  $R(\cdot, A_i)$ , respectively.*
- (II) *For all  $\lambda \in \mathbb{C}$  with  $|\lambda| = r(A) = r$  we have  $N(\lambda \Leftrightarrow A) \supset N(\lambda \Leftrightarrow A_1)$ ,  $N(\lambda \Leftrightarrow A_h) = N(\lambda \Leftrightarrow A_i)$ ,  $q_h$  maps  $N(\lambda \Leftrightarrow A_1)$  bijectively onto  $N(\lambda \Leftrightarrow A_h)$ ,  $\dim N(r \Leftrightarrow A_1) \geq \dim N(\lambda \Leftrightarrow A_1) = \dim N(\lambda \Leftrightarrow A_h) = \dim N(\lambda \Leftrightarrow A_i)$ , and  $\{|z| : z \in N(\lambda \Leftrightarrow A_h)\} \subset N(r \Leftrightarrow A_h)$ . It follows that  $N(r \Leftrightarrow A_h)$  is a sublattice of  $I_h$ .*

*Proof.* (I) As seen above, the residuum  $Q_{h,-1}$  of  $R(\cdot, A_h)$  at  $r$  is strictly positive. Then its restriction  $Q_{i,-1}$  to  $J$  is also strictly positive (note that  $\{0\} \neq N(r \Leftrightarrow A_h) \subset J$ ). This implies the three equalities. Since  $R(Q_{h,-1}) = N(r \Leftrightarrow A_h) \subset J$ , it follows that  $R(\cdot, A_j)$  is holomorphic at  $r$ . Therefore  $r(A_j) < r(A)$ , since  $A_j$  is a nonnegative operator in the Banach lattice  $I_j$  [3, Proposition 4.1.1.i]. Note that we have defined  $r(A_j) = \Leftrightarrow\infty$  if  $I_j = \{0\}$ .

(II) The first inclusion is evident. For the second statement we apply Lemma 2.1 (II) with  $V = I_h$ ,  $M = J$  and  $T = \lambda \Leftrightarrow A_h$ . Note that  $T/M = \lambda \Leftrightarrow A_j$  is injective, since  $r(A_j) < r(A) = |\lambda|$ . The third statement follows from Lemma 2.1 (I) if we set  $V = I_1$ ,  $M = I_0$  and  $T = \lambda \Leftrightarrow A_1$ . Note that  $T|M = \lambda \Leftrightarrow A_0$  is bijective, since  $r(A_0) < r(A) = |\lambda|$ . The equality  $\dim N(\lambda \Leftrightarrow A_1) = \dim N(\lambda \Leftrightarrow A_h) = \dim N(\lambda \Leftrightarrow A_i)$  is now evident. We shall prove  $\dim N(r \Leftrightarrow A_1) \geq \dim N(r \Leftrightarrow A_i)$ , which needs a proof only if  $m = \dim N(r \Leftrightarrow A_1) < \infty$ . By [4, C-III, Lemma 3.13], the ideal  $J$  is the mutually orthogonal sum of  $m$   $A_i$ -invariant ideals  $J_k$  ( $k = 1, \dots, m$ ), the restrictions

$A_{ik} = A_i|_{J_k}$  are irreducible, and  $r(A_{ik}) = r(A)$  is a pole of  $R(\cdot, A_{ik})$  ( $k = 1, \dots, m$ ). The eigenspaces  $N(r \Leftrightarrow A_{ik})$  are one-dimensional; see [6, V. § 5]. By [6, Corollary to Theorem V.5.4],  $\dim N(\lambda \Leftrightarrow A_{ik}) \leq 1$ .  $A_i$  is the direct sum of the restrictions  $A_{ik}$ . Then  $N(\lambda \Leftrightarrow A_i)$  is the direct sum of the  $N(\lambda \Leftrightarrow A_{ik})$  ( $k = 1, \dots, m$ ). Therefore  $\dim N(\lambda \Leftrightarrow A_i) \leq m = \dim N(r \Leftrightarrow A_1)$ . For the proof of the next statement take  $z \in N(\lambda \Leftrightarrow A_h)$ . Let  $y = (r \Leftrightarrow A_h)|z|$ . Then  $y \leq r|z| \Leftrightarrow |A_h z| \leq |(\lambda \Leftrightarrow A_h)z| = 0$ . By a remark preceding Lemma 2.1, then  $y = 0$ .  $\square$

**3. Results and Proofs.** We shall always assume that  $A$  is a nonnegative linear operator in a Banach lattice  $E$ , and its spectral radius is a pole of its resolvent.

We shall freely use the concepts and notations of Section 2. The next lemma is crucial for the main results of this note.

**LEMMA 3.1.** *Let  $|\lambda| = r(A) = r$ . For each  $z \in N(\lambda \Leftrightarrow A_h)$  there exists a unique nonnegative  $w$  in  $N(r \Leftrightarrow A_1)$  such that  $q_h(w) = |z|$ ; for the unique  $u \in N(\lambda \Leftrightarrow A_1)$  satisfying  $q_h(u) = z$  it follows that  $q_h(w) = q_h(|u|)$  and  $w \geq |u|$ .*

*Proof.* Let  $z \in N(\lambda \Leftrightarrow A_h)$ . By Proposition 2.2(II), there exists a unique  $u \in N(\lambda \Leftrightarrow A_1)$  with  $q_h(u) = z$ . Proposition 2.2(II) implies  $|z| \in N(r \Leftrightarrow A_h)$ . Since  $q_h$  is a lattice homomorphism, we get  $q_h((r \Leftrightarrow A_1)|u|) = (r \Leftrightarrow A_h)q_h(|u|) = (r \Leftrightarrow A_h)|z| = 0$ , i.e.  $(r \Leftrightarrow A_1)|u| \in I_0$ . Since  $(r \Leftrightarrow A_1)|u| \leq |(\lambda \Leftrightarrow A_1)u| = 0$ , there exists a unique nonnegative  $u_0$  in  $I_0$  such that  $(r \Leftrightarrow A_0)u_0 = \Leftrightarrow (r \Leftrightarrow A_1)|u|$  (notice that  $r(A_0) < r(A) = r$ , thus  $r \Leftrightarrow A_0$  has a nonnegative inverse). Then  $w = |u| + u_0$  is a vector we are looking for. If  $v \in I_1$  satisfies  $q_h(v) = q_h(|u|)$ , then  $|u| \Leftrightarrow v \in I_0$ . If further  $v \in N(r \Leftrightarrow A_1)$ , then  $|u| + u_0 \Leftrightarrow v \in I_0 \cap N(r \Leftrightarrow A_1) = N(r \Leftrightarrow A_0)$ . But  $r \Leftrightarrow A_0$  is injective, therefore  $v = |u| + u_0 = w$ , i.e.  $w$  is unique as stated.  $\square$

**PROPOSITION 3.2.** *Under our general assumptions the following hold.*

- (I)  $N(r \Leftrightarrow A) \cap E_+ \subset N(r \Leftrightarrow A_1)$ .
- (II)  $\text{span}(N(r \Leftrightarrow A) \cap E_+) = N(r \Leftrightarrow A_1)$ .
- (III) *the nonnegative eigenvectors of  $A$  corresponding to  $r$  span the eigenspace  $N(r \Leftrightarrow A)$  iff  $N(r \Leftrightarrow A) \subset I_1$  iff  $N(r \Leftrightarrow A) = N(r \Leftrightarrow A_1)$ .*
- (IV) *if  $N(r \Leftrightarrow A_1)$  is finite dimensional, then  $N(r \Leftrightarrow A_1)$  has a basis of nonnegative eigenvectors of  $A_1$  corresponding to  $r$ .*

*Proof.* (I)  $u \in N(r \Leftrightarrow A)$  is equivalent to  $R(\lambda, A)u = (\lambda \Leftrightarrow r)^{-1}u$  if  $0 < |\lambda \Leftrightarrow r| < \delta$  for some  $\delta > 0$ . Therefore  $u \in N(r \Leftrightarrow A) \cap E_+$  implies  $u \in \{x \in E : Q_{-2}|x| = 0\} = I_1$ .

(II) Let  $u \in N(r \Leftrightarrow A_1)$ . Then its real and its imaginary parts belong to  $N(r \Leftrightarrow A_1)$ .

Therefore we assume w.l.o.g. that  $u$  is real. Choose  $w$  as in Lemma 3.1. Then  $u = \frac{1}{2}(w + u) \Leftrightarrow \frac{1}{2}(w \Leftrightarrow u)$ ,  $w \pm u \in N(r \Leftrightarrow A_1)$  and  $w \pm u \in E_+$ .

(III) The first part follows from  $N(r \Leftrightarrow A_1) = N(r \Leftrightarrow A) \cap I_1$  and (II), the second part is clear.

(IV) follows simply from (II).  $\square$

**REMARK 3.3.** In connection with (III) and (IV) recall that U.G. Rothblum [5] proved that in the case  $\dim E < \infty$  the generalized eigenspace  $N((r \Leftrightarrow A)^p)$  (where  $p$  is the order of the pole  $r$ ) always has a basis of nonnegative vectors. However, [1, Example 18] shows that in the general case ( $\dim E = \infty$ ) the corresponding statement can be false even if  $p = 2$  and  $\dim N((r \Leftrightarrow A)^2) = 2$ .

**THEOREM 3.4.** *Let the general assumptions hold. The following assertions are*

equivalent.

(I)  $N(r \Leftrightarrow A_1)$  is a sublattice of  $E$ ;

(II) If  $z_1$  and  $z_2$  are disjoint vectors in  $N(r \Leftrightarrow A_h)$ , then the unique nonnegative  $w_1$  and  $w_2$  with  $w_i \in N(r \Leftrightarrow A_1)$  and  $q_h(w_i) = |z_i|$  for  $i = 1, 2$  (see Lemma 3.1) are disjoint.

*Proof.* (I)  $\Rightarrow$  (II) Let  $z_1$  and  $z_2$  be disjoint vectors in  $N(r \Leftrightarrow A_h)$ . For  $i = 1, 2$  choose  $w_i$  as in Lemma 3.1. Then  $q_h(|\alpha_1 w_1 + \alpha_2 w_2|) = |q_h(\alpha_1 w_1 + \alpha_2 w_2)| = |\alpha_1 z_1 + \alpha_2 z_2| = |\alpha_1| |z_1| + |\alpha_2| |z_2|$  for all complex  $\alpha_1$  and  $\alpha_2$ , since  $z_1$  and  $z_2$  are disjoint [3, Theorem 1.1.1.vi]. By Lemma 3.1, we get  $w_1 + w_2 = |\alpha_1 w_1 + \alpha_2 w_2|$  for all  $\alpha_1$  and  $\alpha_2$  with modulus 1, since  $w_1 + w_2$  is nonnegative and  $|\alpha_1 w_1 + \alpha_2 w_2|$  belongs to  $N(r \Leftrightarrow A_1)$ ; here we have used that, by (I),  $N(r \Leftrightarrow A_1)$  is a sublattice. This is equivalent to the disjointness of  $w_1$  and  $w_2$ .

(II)  $\Rightarrow$  (I) Let  $u \in N(r \Leftrightarrow A_1)$ . Then  $z = q_h(u) \in N(r \Leftrightarrow A_h)$ . Since  $N(r \Leftrightarrow A_h)$  is a sublattice of  $I_h$  (see Proposition 2.2(II)),  $z^+$  and  $z^-$  (the positive and the negative parts of  $z$ , respectively) belong to  $N(r \Leftrightarrow A_h)$ . Applying Lemma 3.1, we choose nonnegative  $w_1$  and  $w_2$  in  $N(r \Leftrightarrow A_1)$  such that  $q_h(w_1) = z^+$  and  $q_h(w_2) = z^-$ . Then  $q_h(u) = z = q_h(w_1 \Leftrightarrow w_2)$ . Since  $u$  and  $w_1 \Leftrightarrow w_2$  belong to  $N(r \Leftrightarrow A_1)$ , we obtain  $u = w_1 \Leftrightarrow w_2$ , by Proposition 2.2(II). Now  $z^+$  and  $z^-$  are disjoint [3, Theorem 1.1.1.iv], therefore  $w_1$  and  $w_2$  are disjoint, by assumption. But then  $|u| = w_1 + w_2 \in N(r \Leftrightarrow A_1)$ .  $\square$

For the rest of this section we assume that  $I_0$  is a projection band in  $E$ . Note that this holds in any Banach lattice with order continuous norm [3, Corollary 2.4.4.xii, Theorem 2.4.2.ii), and Theorem 1.2.9.], and a fortiori in  $\mathbb{C}^\ell$ . Then  $I_0$  is also a projection band in  $I_1$ , and we can identify  $I_h$  with the kernel of the band projection of  $I_1$  onto  $I_0$ , i.e.  $I_1 = I_h \oplus I_0$ , where  $\oplus$  means disjoint sum in the lattice sense. Since  $I_0$  is  $A_1$ -invariant, we can represent  $A_1$  with respect to this direct sum as a triangular operator matrix as follows,

$$A_1 = \begin{bmatrix} A_h & 0 \\ A_{0h} & A_0 \end{bmatrix}.$$

**THEOREM 3.5.** *In addition to the general assumptions we assume that  $I_0$  is a projection band in  $E$ . Let  $u \in N(\lambda \Leftrightarrow A_1)$  with  $|\lambda| = r(A)$  and  $u = u_h + u_0$ , where  $u_h \in I_h$  and  $u_0 \in I_0$ . The following assertions are equivalent.*

(I)  $|u| \in N(r \Leftrightarrow A_1)$ ;

(II)  $|(\lambda \Leftrightarrow A_0)u_0| = (r \Leftrightarrow A_0)|u_0|$  and  $|A_{0h}u_h| = A_{0h}|u_h|$ .

*Proof.*  $u \in N(\lambda \Leftrightarrow A_1)$  is equivalent to  $(\lambda \Leftrightarrow A_h)u_h = 0$  and  $(\lambda \Leftrightarrow A_0)u_0 = A_{0h}u_h$ . The first equality implies  $(r \Leftrightarrow A_h)|u_h| = 0$ , by Proposition 2.2(II). Therefore  $|u| \in N(r \Leftrightarrow A_1)$  is equivalent to  $(r \Leftrightarrow A_0)|u_0| = A_{0h}|u_h|$ . We also obtain

$$A_{0h}|u_h| \geq |A_{0h}u_h| = |(\lambda \Leftrightarrow A_0)u_0| \geq (r \Leftrightarrow A_0)|u_0|.$$

From this inequality it follows immediately that (I) and (II) are equivalent.  $\square$

The second equality in (II) means that  $A_{0h} : I_h \rightarrow I_0$  is a lattice homomorphism on  $N(\lambda \Leftrightarrow A_h)$ . However, even if (I) holds,  $A_{0h}$  need not be a lattice homomorphism of *all* of  $I_h$ , as the following example shows.

EXAMPLE 3.6. Let

$$A = A_1 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_h & 0 \\ A_{0h} & A_0 \end{bmatrix}.$$

For  $x_h = [\Leftrightarrow 2, 2, 0]^\top \in I_h$  we get  $A_{0h}x_h = 0$  and  $A_{0h}|x_h| = 4$ .

Since  $r(A_0) < r(A) = r$ , the first equality in (II) implies  $|u_0| = (r \Leftrightarrow A_0)^{-1} |(\lambda \Leftrightarrow A_0)u_0| \geq |(r \Leftrightarrow A_0)^{-1}(\lambda \Leftrightarrow A_0)u_0|$ . We cannot expect that we have here equality, even if statement (I) in Theorem 3.5 holds, as is shown by the following example.

EXAMPLE 3.7. Let

$$A = A_1 = \begin{bmatrix} 02 & 00 \\ 20 & 00 \\ 10 & 01 \\ 01 & 10 \end{bmatrix} = \begin{bmatrix} A_h & O \\ I & A_0 \end{bmatrix}.$$

The peripheral eigenvalues of the matrix  $A$  are 2 and  $\Leftrightarrow 2$ . We have  $u \in N(\pm 2 \Leftrightarrow A)$  iff  $u = [\alpha, \pm\alpha, \pm\alpha, \alpha]^\top$ ,  $\alpha \in \mathbb{C}$ . Therefore statement (I) of Theorem 3.5 holds for  $\lambda = \pm 2$ . Let  $u = [u_h^\top, u_0^\top]^\top \in N(\Leftrightarrow 2 \Leftrightarrow A)$ . Then  $\Leftrightarrow u_0 = u_h = (\Leftrightarrow 2 \Leftrightarrow A_0)u_0$ ,  $(2 \Leftrightarrow A_0)^{-1}(\Leftrightarrow 2 \Leftrightarrow A_0)u_0 = \Leftrightarrow \frac{1}{3}u_0$ , but  $(2 \Leftrightarrow A_0)^{-1}|(\Leftrightarrow 2 \Leftrightarrow A_0)u_0| = |u_0|$ .

In the next section we will use the following result.

PROPOSITION 3.8. *In addition to the general assumptions we assume that  $I_0$  is a projection band in  $E$  and that  $N(r \Leftrightarrow A_1) = N(r \Leftrightarrow A_h) \oplus \{0\}$ . For  $\lambda \in \mathbb{C}$  with  $|\lambda| = r(A) = r$  the following assertions are equivalent.*

- (I)  $N(\lambda \Leftrightarrow A) \subset I_1$ ;
- (II)  $\{|u| : u \in N(\lambda \Leftrightarrow A)\} \subset N(r \Leftrightarrow A)$ .

*Proof.* (I)  $\Rightarrow$  (II): From (I) we obtain  $N(\lambda \Leftrightarrow A) = N(\lambda \Leftrightarrow A_1)$ . Let  $u \in N(\lambda \Leftrightarrow A)$  and for  $z = q_h(u)$  choose  $w$  as in Lemma 3.1. Then  $w \in N(r \Leftrightarrow A_1) = N(r \Leftrightarrow A_h) \oplus \{0\} \subset I_h$ , and  $w \geq |u|$ . This implies  $u \in I_h$  and then  $|u| = w$ , since  $q_h(|u|) = q_h(w)$ , where  $q_h$  is now (identified with) the band projection of  $I_1$  onto  $I_h$ .

(II)  $\Rightarrow$  (I) follows from Proposition 3.2(I); notice that we use for this implication only the general assumptions. □

**4. The matrix case.** In the last section of this note we want to compare the results in Section 3 with those of B.-S. Tam on nonnegative matrices [8, Theorems 2.4 and 2.5]. We shall use the graph theoretic concepts defined in [7, § 2], and [8, § 1].

In the complex Banach lattice  $\mathbb{C}^\ell$  each ideal is of the form  $I_\alpha = \{x \in \mathbb{C}^\ell : x_i (= i\text{-th component of } x) = 0 \text{ if } i \notin \alpha\}$  for some  $\alpha \subset \{1, \dots, \ell\}$ ; see [6, p. 2]. Let  $\alpha$  and  $\beta$  be nonempty subsets of  $\{1, \dots, \ell\}$ . For  $x \in \mathbb{C}^\ell$  we denote by  $x_\alpha$  the subvector of  $x$  with indices from  $\alpha$ . For an  $\ell \times \ell$ -matrix  $A$  we denote by  $A_{\alpha\beta}$  the submatrix of  $A$  with row indices from  $\alpha$  and column indices from  $\beta$ . We write  $A_\alpha$  instead of  $A_{\alpha\alpha}$ . In the next lemma we collect some facts on the connection between the ideals

defined in Section 2 and the ideals  $I_\alpha$  for a class  $\alpha$  of  $A$  (i.e.,  $\alpha$  is a strongly connected component in the directed graph associated with  $A$  [8, § 1]).

LEMMA 4.1. *Let  $A$  be a nonnegative square matrix, and let  $\alpha$  be a class of  $A$ .*

- (I) *If  $I$  is an  $A$ -invariant ideal and  $I_\alpha \cap I \neq \{0\}$ , then  $I_\alpha \subset I$ .*
- (II) *If  $I_\alpha \subset I_0 \oplus I_j$ , then  $\alpha$  is nonbasic (i.e.  $r(A_\alpha) < r(A)$ ).*
- (III) *Let  $\alpha$  be a basic class. Then  $\alpha$  is distinguished iff  $I_\alpha \subset I_1$  iff  $I_\alpha \subset I_h$  iff  $I_\alpha = J_k$  for some  $k \in \{1, \dots, m\}$ .*
- (IV) *If  $\alpha$  has access to a distinguished basic class of  $A$ , then either  $\alpha$  is this distinguished class of  $A$  or  $I_\alpha \subset I_0$ .*

*Proof.* (I) The ideal  $I_\alpha \cap I$  is  $A_\alpha$ -invariant. Since  $A_\alpha$  is an irreducible matrix,  $I_\alpha \cap I \neq \{0\}$  implies  $I_\alpha \cap I = I_\alpha$ .

(II) We have  $r(A_\alpha) \leq \max\{r(A_0), r(A_j)\} < r(A)$ .

(III) Let  $\alpha$  be a distinguished basic class of  $A$ . Then there exists a nonnegative eigenvector  $x$  of  $A$  corresponding to  $r$ , such that  $x_\alpha$  is strictly positive; see [7, Theorem 3.1]. By Proposition 3.2(I), we have that  $x \in I_1$ . Therefore  $I_\alpha \subset I_1$ . If  $\alpha$  is a basic class of  $A$  with  $I_\alpha \subset I_1$ , then  $I_\alpha \subset J \subset I_h$ , by (II). Let  $\alpha$  be a basic class of  $A$  with  $I_\alpha \subset I_h$ . Then  $I_\alpha \subset J$ , since  $r(A_j) < r(A)$ .  $A_i$  is the direct sum of the irreducible operators  $A_{ik} = A_i|_{J_k}$  ( $k = 1, \dots, m$ ), so we get  $A_\alpha = A_{ik}$  for some  $k \in \{1, \dots, m\}$ . Then  $I_\alpha = J_k$  for this  $k$ . If  $I_\alpha = J_k$  for some  $k \in \{1, \dots, m\}$ , then  $r(A_\alpha) = r(A)$  and  $A_\alpha$  is irreducible. Therefore  $\alpha$  is a strongly connected set in the directed graph associated with  $A$ . From the triangular structure of  $A_1$  with respect to the direct sum  $I_1 = I_j \oplus J_1 \oplus \dots \oplus J_m \oplus I_0$  it follows that  $\alpha$  is a strongly connected component in this graph.

(IV) Let  $\alpha$  have access to a distinguished basic class  $\beta$  of  $A$ . Assume  $\alpha \neq \beta$ . Then  $\alpha$  is nonbasic. Furthermore, there exists a nonnegative eigenvector  $x$  of  $A$  corresponding to  $r$ , such that  $x_\alpha$  is strictly positive; see [7, Theorem 3.1]. Proposition 3.2(I) implies  $x \in I_1$ . Therefore  $x \in N(r \Leftrightarrow A_1)$ . From Proposition 2.2(II) it follows that  $N(r \Leftrightarrow A_1) \subset J \oplus I_0$ . Therefore  $I_\alpha \subset J \oplus I_0$ . From the last part of the proof of (III) we see that  $I_\alpha \cap J \neq \{0\}$  would imply  $r(A_\alpha) = r(A)$ . Hence we obtain  $I_\alpha \subset I_0$ , since  $\alpha$  is nonbasic.  $\square$

B.-S. Tam [8, Lemma 2.1] showed that  $A$  has fully cyclic peripheral spectrum (see [6, Definition 2.5]), iff for all  $\lambda \in \mathbb{C}$  with  $|\lambda| = r(A)$

$$\{[u] : u \in N(\lambda \Leftrightarrow A)\} \subset N(r \Leftrightarrow A).$$

Thus, if  $A$  has a fully cyclic peripheral spectrum, then  $N(r \Leftrightarrow A)$  is a sublattice of  $\mathbb{C}^\ell$ . By Proposition 3.2, the latter assertion implies statement (a) in [8, Theorems 2.4 and 2.5]. In the matrix case we have the following result as a supplement to Theorem 3.4.

THEOREM 4.2. *For a nonnegative square matrix the following assertions are equivalent.*

- (I)  *$N(r \Leftrightarrow A_1)$  is a sublattice, where  $r = r(A)$ ,*
- (II) *there does not exist a class of  $A$  which has access to two different distinguished basic classes.*

*Proof.* We prove that the assertion (II) is equivalent to Theorem 3.4(II).

(I)  $\Rightarrow$  (II) Assume there exists a class  $\alpha$  of  $A$  which has access to two different

distinguished basic classes  $\alpha_1$  and  $\alpha_2$ . Then  $\alpha$  has to be a nonbasic class, since a basic class does not have access to a distinguished basic class. By Lemma 4.1(IV) this implies  $I_\alpha \subset I_0$ . From [7, 3.1(i)] it follows that there are two nonnegative eigenvectors  $w_1$  and  $w_2$  of  $A$  corresponding to  $r$  with  $\alpha \subset \text{supp}(w_i)$ ,  $\alpha_i \subset \text{supp}(w_i)$  and  $\alpha_i = \text{supp}(q_h(w_i))$ . By Proposition 3.2(II),  $w_i \in N(r \Leftrightarrow A_1)$ ,  $i = 1, 2$ . Since  $\alpha_1$  and  $\alpha_2$  are disjoint sets, we get that  $q_h(w_1)$  and  $q_h(w_2)$  are disjoint nonnegative vectors in  $N(r \Leftrightarrow A_h)$ . But  $w_1$  and  $w_2$  are not disjoint, and this contradicts Theorem 3.4(II).

(II)  $\Rightarrow$  (I) If  $z$  is in  $N(r \Leftrightarrow A_h)$ , then  $|z|$  belongs to  $N(r \Leftrightarrow A_h)$ ; see Proposition 2.2(II). Then  $\text{supp}(z) = \text{supp}(|z|)$  is the union of some distinguished basic classes. From [7, 3.1(i) and (ii)] it follows that for the unique nonnegative  $w \in N(r \Leftrightarrow A_1)$  with  $q_h(w) = |z|$  its support is the union of all classes which have access to distinguished basic classes contained in  $\text{supp}(z)$ . Since  $w$  is a nonnegative eigenvector of  $A$ , the ideal generated by  $w$  is  $A$ -invariant; this ideal is  $I_\sigma$ , where  $\sigma = \text{supp}(w)$ . Now let  $z_1$  and  $z_2$  be disjoint vectors in  $N(r \Leftrightarrow A_h)$ , and assume that the unique nonnegative  $w_1$  and  $w_2$  in  $N(r \Leftrightarrow A_1)$  with  $q_h(w_i) = |z_i|$  for  $i = 1, 2$  are not disjoint. Then there exists a class  $\alpha$  of  $A$  with  $I_\alpha \cap I_{\sigma_1} \cap I_{\sigma_2} \neq \{0\}$ , where  $\sigma_i = \text{supp}(w_i)$  for  $i = 1, 2$ . Since  $I_{\sigma_i}$  are  $A$ -invariant for  $i = 1, 2$ , we have  $I_\alpha \subset I_{\sigma_1} \cap I_{\sigma_2}$ . Then  $\alpha$  has access to a distinguished basic class  $\alpha_1$  in  $\sigma_1$  and to one  $\alpha_2$  in  $\sigma_2$ . Now  $\alpha_i \subset \text{supp}(z_i)$ . Therefore  $I_{\alpha_i} \subset I_h$ . By Lemma 4.1(III), the  $\alpha_i$  are distinguished. Thus  $\alpha$  has access to two different distinguished basic classes of  $A$ . This contradicts (II), so  $w_1$  and  $w_2$  are disjoint. Thus Theorem 3.4(II) holds.  $\square$

Theorem 4.2 and Proposition 3.2 show that statements (a) and (b) in [8, Theorem 2.4] together are equivalent to the statement that  $N(r \Leftrightarrow A)$  is a sublattice of  $\mathbb{C}^\ell$ .

The next theorem will show, how conditions (c) in Theorems 2.4 and 2.5 of [8] are related to our results.

**THEOREM 4.3.** *Consider for a nonnegative square matrix  $A$  and a peripheral eigenvalue  $\lambda$  of  $A$  satisfying the following statements.*

- (I) *If  $\lambda$  is an eigenvalue of  $A_\alpha$  for some class  $\alpha$  of  $A$ , then  $\alpha$  is distinguished.*
- (II)  *$N(\lambda \Leftrightarrow A) \subset I_1$ .*
- (III) *If  $\lambda$  is an eigenvalue of  $A_\alpha$  for some class  $\alpha$  of  $A$ , then all initial classes of the family*

$$F(\alpha) = \{\gamma : \gamma \text{ is a class of } A, \lambda \text{ is an eigenvalue of } A_\gamma, \gamma \succ = \alpha\}$$

*are distinguished.*

- (IV) *If  $\lambda$  is an eigenvalue of  $A_\alpha$  for some class  $\alpha$  of  $A$ , then  $\lambda$  is also an eigenvalue of  $A_\beta$  for some distinguished class  $\beta$  of  $A$ , which has access to  $\alpha$ .*
- (V)  *$\lambda$  is an eigenvalue of  $A_\beta$  for some distinguished class  $\beta$  of  $A$ .*
- (VI)  *$N(\lambda \Leftrightarrow A) \cap I_1 \neq \{0\}$ .*

*Then (I)  $\Rightarrow$  (II)  $\Rightarrow$  (III)  $\Leftrightarrow$  (IV)  $\Rightarrow$  (V)  $\Leftrightarrow$  (VI).*

*Proof.* (I)  $\Rightarrow$  (II) Let  $0 \neq u \in N(\lambda \Leftrightarrow A)$ . Then for all classes  $\alpha$  of  $A$ , which are final in  $\text{supp}(u)$ ,  $\lambda$  is an eigenvalue of  $A_\alpha$ ; see [8, Lemma 2.3]. Therefore, by assumption, all (final) classes in  $\text{supp}(u)$  have access to a distinguished basic class of  $A$ . By Lemma 4.1(III) and (IV), we have  $u \in I_1$ .

(II)  $\Rightarrow$  (III) Take an initial class  $\beta$  in  $F(\alpha)$ . Notice that  $F(\alpha)$  is nonempty, since  $\alpha$



belongs to it. Then  $\beta$  is basic, since  $|\lambda| = r(A)$ . Let  $\omega$  be the union of all classes of  $A$ , which have access to  $\beta$  and are different from  $\beta$ . Set  $\delta = \{1, \dots, \ell\} \setminus (\omega \cup \beta)$ . Note that one or both of the sets  $\omega$  and  $\delta$  can be empty. Then the corresponding matrices in the decomposition below do not appear. With respect to the decomposition  $\mathbb{C}^\ell = I_\delta \oplus I_\beta \oplus I_\omega$  the matrix  $A$  has the block triangular form

$$\begin{bmatrix} A_\delta & 0 & 0 \\ A_{\beta\delta} & A_\beta & 0 \\ A_{\omega\delta} & A_{\omega\beta} & A_\omega \end{bmatrix}.$$

By the choice of  $\beta$ ,  $\lambda$  is not an eigenvalue of  $A_\omega$ . Take an eigenvector  $u_\beta$  of  $A_\beta$  corresponding to  $\lambda$ . Define  $u_\omega = (\lambda \Leftrightarrow A_\omega)^{-1} A_{\omega\beta} u_\beta$  and  $u = [o^\top, u_\beta^\top, u_\omega^\top]^\top$ . Then  $u$  is an eigenvector of  $A$  corresponding to  $\lambda$ . Now  $u \in N(\lambda \Leftrightarrow A) \subset I_1$  by assumption. Then  $o \neq [o^\top, u_\beta^\top, o^\top]^\top \in I_\beta \cap I_1$  implies  $I_\beta \cap I_1 \neq \{0\}$ . By Lemma 4.1(I), we have  $I_\beta \subset I_1$ . Since  $\beta$  is basic, we get from Lemma 4.1 (III) that it is distinguished.

(III)  $\Rightarrow$  (IV) is clear.

(IV)  $\Rightarrow$  (III) Take an initial class  $\kappa$  of  $A$  in  $F(\alpha)$ . Then  $\lambda$  is an eigenvalue of  $A_\kappa$ . By assumption, there exists a distinguished basic class  $\beta$  of  $A$  with  $\beta \succ \kappa$  and  $\lambda$  is an eigenvalue of  $A_\beta$ . Then  $\beta \succ \kappa \succ \alpha$ , thus  $\beta \in F(\alpha)$ . Therefore  $\kappa = \beta$ , since  $\kappa$  is initial in  $F(\alpha)$ .

(IV)  $\Rightarrow$  (V) is clear, since each eigenvalue of  $A$  has to be an eigenvalue of  $A_\alpha$  for at least one class  $\alpha$  of  $A$ .

(V)  $\Rightarrow$  (VI) For a distinguished basic class  $\beta$  let  $u_\beta$  be an eigenvector of  $A_\beta$  corresponding to  $\lambda$ . By Lemma 4.1(III),  $A_\beta = A_{ik}$  for some  $k \in \{1, \dots, m\}$ . Let  $u_h = [u_1^\top, \dots, u_m^\top]^\top$  with  $u_k = u_\beta$  and  $u_i = 0$  if  $i \neq k$ . The vector  $u_h$  is an eigenvector of  $A_h$  corresponding to  $\lambda$ , since  $A_h = A_{i1} \oplus \dots \oplus A_{im}$ . Define  $u_0 = (\lambda \Leftrightarrow A_0)^{-1} A_{0h} u_h$ , and  $u_1 = [u_h^\top, u_0^\top]^\top$ . Then  $u_1$  is an eigenvector of  $A_1$  corresponding to  $\lambda$ . Thus  $\{0\} \neq N(\lambda \Leftrightarrow A_1) = N(\lambda \Leftrightarrow A) \cap I_1$ .

(VI)  $\Rightarrow$  (V) Let  $u \in I_1$  be an eigenvector of  $A$  corresponding to  $\lambda$ . If  $\beta$  is a final class of  $A$  in  $\text{supp}(u)$ , then  $\beta$  is a basic class [8, Lemma 2.3]. Since  $I_\beta \subset I_1$ ,  $\beta$  is distinguished (see Lemma 4.1(III)).  $\square$

The following examples will show that the converses of the three one-way implications in Theorem 4.3 are not true in general.

EXAMPLE 4.4. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ a & b & 0 & 0 & 1 \\ c & d & 0 & 1 & 0 \end{bmatrix} \quad \text{with nonnegative numbers } a, b, c \text{ and } d.$$

Then  $\alpha = \{1, 2\}$ ,  $\beta = \{3\}$  and  $\gamma = \{4, 5\}$  are the classes of  $A$ . The matrices  $A_\alpha$ ,  $A_\gamma$  and  $A$  have eigenvalues 1 and  $\Leftrightarrow 1$ . The matrix  $A_\beta$  has the eigenvalue 1. Therefore all classes of  $A$  are basic. The classes  $\beta$  and  $\gamma$  are distinguished and  $\alpha$  is not distinguished. Further, here  $I_1 = \{x \in \mathbb{C}^5 : x_1 = x_2 = 0\} = I_\beta \oplus I_\gamma$  and  $I_0 = \{0\}$ . Now let

(1)  $a = d = 1$  and  $b = c = 0$ . Then statement (I) is not true for  $\lambda = \Leftrightarrow 1$ , but  $N(\Leftrightarrow 1 \Leftrightarrow A) = \{x \in \mathbb{C}^5 : x_1 = x_2 = x_3 = 0, x_4 + x_5 = 0\} \subset I_1$ . For  $\lambda = r(A)$  statement (I) in Theorem 4.3 is equivalent to: “All basic classes of  $A$  are distinguished.” This is equivalent to  $p = 1$ , where  $p$  is the order of the pole  $r(A)$  of the resolvent, cf. Section 2. The example also shows that even for  $\lambda = r(A)$  (II) does not imply (I), since  $N(1 \Leftrightarrow A) = \{x \in \mathbb{C}^5 : x_1 = x_2 = 0, x_4 \Leftrightarrow x_5 = 0\} \subset I_1$  and  $p = 2$ .

(2)  $a = b = c = d = 1$ . Then statement (II) is not true for  $\lambda = \Leftrightarrow 1$ , since  $N(\Leftrightarrow 1 \Leftrightarrow A) = \{x \in \mathbb{C}^5 : x_1 + x_2 = 0, x_2 + 2x_3 = 0, x_4 + x_5 = 0\}$ . But statement (III) is true for  $\lambda = \Leftrightarrow 1$ , since  $F(\alpha) = \{\gamma, \alpha\}$  with  $\gamma > \alpha$  and  $F(\gamma) = \{\gamma\}$ .

(3)  $a = b = c = d = 0$ . Then statement (IV) is not true for  $\lambda = \Leftrightarrow 1$ , since  $\gamma > \alpha$ . But statement (V) is true, since  $\Leftrightarrow 1$  is an eigenvalue of  $A_\gamma$ .

Part (b) of [8, Theorem 2.5], which states that each distinguished basic class of  $A$  is initial (in the family of all classes of  $A$ ), is equivalent to  $N(r \Leftrightarrow A_1) = N(r \Leftrightarrow A_h) \oplus \{0\}$ . This follows immediately from [7, 3.1(i) and (ii)]. Combining this observation with Proposition 3.8 we obtain the following result.

**COROLLARY 4.5.** *Let  $A$  be a nonnegative square matrix, such that each distinguished class of  $A$  is initial (in the family of all classes of  $A$ ). Then  $A$  has a fully cyclic peripheral spectrum iff  $N(\lambda \Leftrightarrow A) \subset I_1$  for all peripheral eigenvalues of  $A$ .*

This corollary is stronger than [8, Theorem 2.5] as Theorem 4.3 (I)  $\Rightarrow$  (II) and Example 4.4(1) show.

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